Chapter 5
Higher Orders Differential Equations

5.1. Linear Equations

5.1.1. Preliminary Comments

In this Chapter, we denote higher derivatives by $y_x^{(n)}$ that stands for $\dfrac{d^n y}{dx^n}$.

1. The general solution of a nonhomogeneous linear equation of the n th order

$$f_n(x)y_x^{(n)} + f_{n-1}y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = 0 \tag{1}$$

has the form

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x), \tag{2}$$

where $y_1(x)$, $y_2(x)$, …, $y_n(x)$ make up a fundamental set of solutions (y_k are linearly-independent solutions; $y_k \not\equiv 0$); C_1, C_2, …, C_n are arbitrary constants.

2. Let $y_0 = y_0(x)$ be a nontrivial particular solution of equation (1). Then, the substitution $y = y_0(x)\int z(x)\,dx$ leads to a linear equation of the $(n-1)$ th order for function $z(x)$.

Given m linearly-independent solutions $y_1(x)$, $y_2(x)$, …, $y_m(x)$ of equation (1), its order can be lowered down to $(n-m)$ by the following technique. The substitution $y = y_m(x)\int z(x)\,dx$ leads to an $(n-1)$ th order equation for $z(x)$, with the following linearly-independent solutions known:

$$z_1 = \left(\frac{y_1}{y_m}\right)'_x, \quad z_2 = \left(\frac{y_2}{y_m}\right)'_x, \quad \ldots, \quad z_{m-1} = \left(\frac{y_{m-1}}{y_m}\right)'_x.$$

Furthermore, the substitution $z = z_{m-1}(x)\int w(x)\,dx$ yields an $(n-2)$ th order equation, etc. Thus, the above procedure applied m times results in an $(n-m)$ th order homogeneous linear equation.

3. A nonhomogeneous linear equation of the n th order has the form

$$f_n(x)y_x^{(n)} + f_{n-1}y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = g(x). \tag{3}$$

The general solution of equation (3) is the sum of its particular solution and the general solution of the corresponding homogeneous equation (1).

Let $\{ y_1(x),\ \dots,\ y_n(x) \}$ be a fundamental set of solutions of the homogeneous differential equation (1), and $W(x)$ is the Wronskian:

$$W(y_1,\ y_2,\ \dots,\ y_n) = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}, \tag{4}$$

where $y_k^{(m)}(x) = \dfrac{d^m y_k}{dx^m}$, $m = 1,\ 2,\ \dots,\ n-1$; $k = 1,\ 2,\ \dots,\ n$. Denote by $W_\nu(x)$ determinant (4) wherein the νth column is replaced by the column $0, 0, \dots, 0, g$ (from top to bottom). Then, the general solution of the nonhomogeneous linear equation (3) can be written as

$$y = \sum_{\nu=1}^{n} C_\nu y_\nu(x) + \sum_{\nu=1}^{n} y_\nu(x) \int \frac{W_\nu(x)\,dx}{f_n(x)W(x)}.$$

5.1.2. Equations Containing Power Functions

1. $\boldsymbol{y_x^{(6)} + ay = 0}$.

1°. For $a = 0$,

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 x^5.$$

2°. For $a = k^6 > 0$,

$$\begin{aligned} y &= C_1 \cos kx + C_2 \sin kx + \cos \frac{kx}{2}(C_3 \cosh\xi + C_4 \sinh\xi) \\ &\quad + \sin \frac{kx}{2}(C_5 \cosh\xi + C_6 \sinh\xi), \qquad \xi = \frac{kx\sqrt{3}}{2}. \end{aligned}$$

3°. For $a = -k^6 < 0$,

$$\begin{aligned} y &= C_1 \cosh kx + C_2 \sinh kx + \cosh \frac{kx}{2}(C_3 \cos\xi + C_4 \sin\xi) \\ &\quad + \sinh \frac{kx}{2}(C_5 \cos\xi + C_6 \sin\xi), \qquad \xi = \frac{kx\sqrt{3}}{2}. \end{aligned}$$

2. $\boldsymbol{y_x^{(2n)} = a^{2n} y}$.

Solution:

$$y = C_1 e^{ax} + C_2 e^{-ax} + \sum_{k=1}^{n-1} e^{\varphi_k}(A_k \cos\theta_k + B_k \sin\theta_k),$$

where $\varphi_k = ax\cos\dfrac{k\pi}{n}$, $\theta_k = ax\sin\dfrac{k\pi}{n}$; $C_1,\ C_2,\ A_k,\ B_k$ $(k = 1,\ 2,\ \dots,\ n-1)$ are arbitrary constants.

3. $\boldsymbol{y_x^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y'_x + a_0 y = 0.}$

The homogeneous constant-coefficient linear equation.

To solve this equation determine the n roots of the characteristic polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

The general solution is determined by these characteristic roots. Several cases are possible:

1. The roots are all real and different. Denote them by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the general solution of the original equation is

$$y = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) + \cdots + C_n \exp(\lambda_2 n).$$

2. There are $m \le n$ equal real roots, $\lambda_1 = \lambda_2 = \cdots = \lambda_m$, while the other roots are real and different. In this case, the general solution is

$$\begin{aligned} y = {}& \exp(\lambda_1 x)(C_1 + C_2 x + \cdots + C_m x^{m-1}) \\ &+ C_{m+1}\exp(\lambda_{m+1}x) + C_{m+2}\exp(\lambda_{m+2}x) + \cdots + C_n \exp(\lambda_2 n). \end{aligned}$$

3. There are m equal pairs ($2m \le n$) of complex conjugate roots, $\lambda = \alpha \pm i\beta$, while the other roots are real and different. Then, the general solution has the form

$$\begin{aligned} y &= \exp(\alpha x)\cos(\beta x)(A_1 + A_2 x + \cdots + A_m x^{m-1}) \\ &\quad + \exp(\alpha x)\sin(\beta x)(B_1 + B_2 x + \cdots + B_m x^{m-1}) \\ &\quad + C_{2m+1}\exp(\lambda_{2m+1}x) + C_{2m+2}\exp(\lambda_{2m+2}x) + \cdots + C_n\exp(\lambda_2 n), \end{aligned}$$

where $A_1, \ldots, A_m, B_1, \ldots, B_m$ are arbitrary constants.

4. In the general case, there are r different roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ of multiplicities $m_1, m_2, \ldots, m_r$, respectively. Hence, the characteristic polynomial can be factorized:

$$P(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\ldots(\lambda - \lambda_r)^{m_r},$$

where $m_1 + m_2 + \cdots + m_r = n$. Then, the general solution of the original equation is given by the formula

$$y = \sum_{k=1}^{r} \exp(\lambda_k x)(C_{k,0} + C_{k,1}x + \cdots + C_{k,m_k-1}x^{m_k-1}),$$

where $C_{k,l}$ are arbitrary constants.

If $P(\lambda)$ has complex conjugate roots, in the above solution the real and imaginary parts should be taken, in view of the formula: $\exp(\alpha \pm i\beta) = e^{\alpha}(\cos\beta \pm i\sin\beta)$.

4. $\boldsymbol{y_x^{(n)} = axy + b, \qquad a > 0.}$

Solution:

$$y = \sum_{\nu=0}^{n} C_\nu \varepsilon_\nu \int_0^\infty \exp\left[\varepsilon_\nu x t - \frac{t^{n+1}}{a(n+1)}\right] dt,$$

where $\varepsilon_\nu = \exp\Bigl(\dfrac{2\pi\nu i}{n+1}\Bigr)$, $\displaystyle\sum_{\nu=0}^{n} C_\nu = \frac{b}{a}$, $i^2 = -1$.

5. $\boldsymbol{y_x^{(n)} + ax^\nu y'_x + a\nu x^{\nu-1} y = 0.}$

This equation can be reduced to an $(n-1)$ th order equation: $y_x^{(n-1)} + ax^\nu y = C$, where C is an arbitrary costant.

6. $\boldsymbol{y_x^{(n)} + ax^{k+1} y'_x - a(n-1)x^k y = 0.}$

The substitution $z = xy'_x - (n-1)y$ leads to an $(n-1)$ th order equation: $z_x^{(n-1)} + ax^{k+1}z = 0$.

7. $\boldsymbol{y_x^{(n)} + ax^{k+1} y'_x + a(k+n)x^k y = 0.}$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of the form 5.1.2.5: $w_t^{(n)} + bt^\nu w'_t + b\nu t^{\nu-1} w = 0$, where $b = a(-1)^{n+1}$, $\nu = 1 - k - 2n$.

8. $\boldsymbol{y_x^{(n)} + ax^k y_x^{(m)} - (ab^m x^k + b^n)y = 0.}$

Particular solution: $y_0 = e^{bx}$.

9. $\boldsymbol{y_x^{(n)} + (ax^k - b^{n-m}) y_x^{(m)} - ab^m x^k y = 0.}$

Particular solution: $y_0 = e^{bx}$.

10. $\boldsymbol{y_x^{(n)} + (ax^{m+1} + bx^m) y'_x - ax^m y = 0.}$

Particular solution: $y_0 = ax + b$.

11. $\boldsymbol{y_x^{(n)} + ay_x^{(n-1)} + bx^m y'_x + abx^m y = 0.}$

Particular solution: $y_0 = e^{-ax}$.

12. $\boldsymbol{xy_x^{(n)} - nmy_x^{(n-1)} + axy = 0, \qquad n = 2,\ 3,\ 4,\ \dots, \qquad m = 1,\ 2,\ 3,\ \dots}$

Solution:

$$y = x^{(m+1)n-1}\Big(x^{1-n}\frac{d}{dx}\Big)^m (x^{1-n}w),$$

where w is the general solution of the constant coefficient equation $w_x^{(n)} + aw = 0$.

13. $\boldsymbol{xy_x^{(n)} + ny_x^{(n-1)} = axy + b.}$

The substitution $w = xy$ leads to a constant coefficient equation: $w_x^{(n)} = aw + b$.

14. $\boldsymbol{xy_x^{(n)} + ny_x^{(n-1)} = ax^2 y + b.}$

The substitution $w = xy$ leads to an equation of the form 5.1.2.4: $w_x^{(n)} = axw + b$.

15. $\boldsymbol{xy_x^{(n)} + (n-m-1)y_x^{(n-1)} + ax^k y'_x - amx^{k-1} y = 0.}$

Particular solution: $y_0 = x^m$.

16. $\boldsymbol{xy_x^{(n)} + ax^k y_x^{(m)} - (ax^k + amx^{k-1} + x + n)y = 0.}$

Particular solution: $y_0 = xe^x$.

17. $\boldsymbol{xy_x^{(n)} = \sum_{\nu=0}^{n-1} [(aA_{\nu+1} - A_\nu)x + A_{\nu+1}] y_x^{(\nu)}},$

where $A_n = 1$, $A_0 = 0$; a and A_ν are arbitrary numbers ($\nu = 1, 2, \ldots, n-1$).

Denote $f(\lambda) = \sum_{\nu=0}^{n-1} A_{\nu+1}\lambda^\nu$. Let all the roots $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of the equation $f(\lambda) = 0$ be different, and $f(a) \neq 0$. Then, the solution is as follows:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_{n-1} e^{\lambda_{n-1} x} + C_n e^{ax} \left[x - \frac{f'_a(a)}{f(a)} \right].$$

18. $\boldsymbol{\sum_{\nu=0}^{n} (a_\nu x + b_\nu) y_x^{(\nu)} = 0}.$

The Laplace equation.

Particular solutions:

$$y_k = \int_{\alpha_k}^{\beta_k} \frac{1}{P(t)} \exp\left[xt + \int \frac{Q(t)}{P(t)}\, dt \right] dt,$$

where $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$, $Q(t) = \sum_{\nu=0}^{n} b_\nu t^\nu$; α_k and β_k are found from the condition

$$\exp\left(xt + \int \frac{Q(t)}{P(t)}\, dt \right) \bigg|_{\alpha_k}^{\beta_k} = 0.$$

In many cases, the path of integration should be chosen on the complex plane.

19. $\boldsymbol{x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1) y_x^{(n-2)} = ax^2 y + b}.$

The substitution $w = x^2 y$ leads to a constant coefficient equation: $w_x^{(n)} = aw + b$.

20. $\boldsymbol{x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1) y_x^{(n-2)} = ax^3 y + b}.$

The substitution $w = x^2 y$ leads to a equation of the form 5.1.2.4: $w_x^{(n)} = axw + b$.

21. $\boldsymbol{x(x+m) y_x^{(n)} + x(ax^k - x - n) y_x^{(m)} - a(x+m)x^k y = 0}.$

Particular solution: $y_0 = xe^x$.

22. $\boldsymbol{x^{2n} y_x^{(n)} = ay}.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to a constant coefficient equation: $w_t^{(n)} = (-1)^n aw$.

23. $\boldsymbol{x^n y_x^{(2n)} = ay}.$

Solution:

$$y = x^{n/2} \sum_{k=1}^{n} \left[C_{k1} I_n\left(2\beta_k \sqrt{x}\right) + C_{k2} K_n\left(2\beta_k \sqrt{x}\right) \right],$$

where I_n and K_n are modified Bessel functions; $\beta_1, \beta_2, \ldots, \beta_n$ are the roots of the equation $\beta^n = \sqrt{a}$.

24. $\boldsymbol{x^{3n}y_x^{(2n)} = ay}$.

The transformation $x = t^{-1}$, $y = wt^{1-2n}$ leads to an equation of the form 5.1.2.23: $t^n w_t^{(2n)} = aw$.

25. $\boldsymbol{x^{n+1/2}y_x^{(2n+1)} = ay}$.

Solution:

$$y = x^{(2n+1)/4} \sum_{k=0}^{2n} C_k \bigl[J_{-n-1/2}\bigl(2\beta_k\sqrt{x}\,\bigr) + iJ_{n+1/2}\bigl(2\beta_k\sqrt{x}\,\bigr)\bigr],$$

where β_0, β_1, …, β_{2n} are the roots of the equation $\beta^{2n+1} = -ai$; $i^2 = -1$.

26. $\boldsymbol{x^{3n+3/2}y_x^{(2n+1)} = ay}$.

The transformation $x = t^{-1}$, $y = wt^{-2n}$ leads to an equation of the form 5.1.2.25: $t^{n+1/2}w_t^{(2n+1)} = -aw$.

27. $\boldsymbol{a_n x^n y_x^{(n)} + a_{n-1}x^{n-1}y_x^{(n-1)} + \dots + a_1 x y'_x + a_0 y = 0}$.

The Euler equation.

If all the roots λ_k ($k = 1, 2, \dots, n$) of the algebraic equation

$$\sum_{\nu=1}^{n} a_\nu \lambda(\lambda-1)\dots(\lambda-\nu+1) = -a_0$$

are different, the general solution of the original differential equation has the form

$$y = C_1|x|^{\lambda_1} + C_2|x|^{\lambda_2} + \dots + C_n|x|^{\lambda_n}.$$

In the general case, the substitution $t = \ln|x|$ leads to a constant coefficient equation of the form 5.1.2.3:

$$\sum_{\nu=1}^{n} a_\nu D(D-1)\dots(D-\nu+1)y = -a_0 y, \qquad \text{where} \quad D = \frac{d}{dx}.$$

28. $\boldsymbol{x^{2n+1}y_x^{(n)} = ay + bx^n}$.

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of the form 5.1.2.4: $w_t^{(n)} = (-1)^n(atw + b)$.

29. $\boldsymbol{x^{2n+1}y_x^{(n)} + nx^{2n}y_x^{(n-1)} = axy}$.

The substitution $w = xy$ leads to an equation of the form 5.1.2.22: $x^{2n}w_x^{(n)} = aw$.

30. $\boldsymbol{x^{2n+1}y_x^{(n)} + nx^{2n}y_x^{(n-1)} = ay}$.

The substitution $w = xy$ leads to an equation of the form 5.1.2.28: $x^{2n+1}w_x^{(n)} = aw$.

31. $\boldsymbol{x^n y_x^{(2n)} + 2nx^{n-1}y_x^{(2n-1)} = ay}$.

The substitution $w = xy$ leads to an equation of the form 5.1.2.23: $x^n w_x^{(2n)} = aw$.

32. $x^{3n}y_x^{(2n)} + 2nx^{3n-1}y_x^{(2n-1)} = ay.$

The substitution $w = xy$ leads to an equation of the form 5.1.2.24: $x^{3n}w_x^{(2n)} = aw$.

33. $x^{n+1}y_x^{(2n+1)} + (2n+1)x^n y_x^{(2n)} = a\sqrt{x}y.$

The substitution $w = xy$ leads to an equation of the form 5.1.2.25: $x^{n+1/2}w_x^{(2n+1)} = aw$.

34. $x^{3n+3/2}y_x^{(2n+1)} + (2n+1)x^{3n+1/2}y_x^{(2n)} = ay.$

The substitution $w = xy$ leads to an equation of the form 5.1.2.26: $x^{3n+3/2}w_x^{(2n+1)} = aw$.

35. $(ax+b)^{2n+1}y_x^{(n)} = (cx+d)y.$

The transformation

$$\xi = \frac{cx+d}{ax+b}, \qquad w = \frac{y}{(ax+b)^{n-1}}$$

leads to an equation of the form 5.1.2.4: $w_\xi^{(n)} = \Delta^{-n}\xi w$, where $\Delta = bc - ad$.

36. $(ax+b)^n(cx+d)^n y_x^{(n)} = ky.$

1°. The transformation

$$\xi = \ln\frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^{n-1}}$$

leads to a constant coefficient equation.

2°. The transformation

$$\zeta = \frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^{n-1}}$$

leads to the Euler equation 5.1.2.27: $\zeta^n w_\zeta^{(n)} = k\Delta^{-n}w$, where $\Delta = ad - bc$.

37. $(ax^2+bx+c)^n y_x^{(n)} = ky.$

The transformation

$$\xi = \int\frac{dx}{ax^2+bx+c}, \qquad w = y(ax^2+bx+c)^{\frac{1-n}{2}}$$

leads to a constant coefficient equation.

38. $(ax+b)^n(cx+d)^{3n}y_x^{(2n)} = ky.$

The transformation

$$\xi = \frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^{2n-1}}$$

leads to an equation of the form 5.1.2.23: $\xi^n w_\xi^{(2n)} = k\Delta^{-2n}w$, where $\Delta = ad - bc$.

39. $(ax+b)^{n+1/2}(cx+d)^{3n+3/2}y_x^{(2n+1)} = ky.$

The transformation
$$\xi = \frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^{2n}}$$
leads to an equation of the form 5.1.2.25: $\xi^{n+1/2}w_\xi^{(2n+1)} = k\Delta^{-2n-1}w$, where $\Delta = ad-bc$.

40. $P_{n-1}(x)y_x^{(n)} + P_{n-2}(x)y_x^{(n-1)} + \cdots + P_1(x)y''_{xx} + (a_1x+b_1)y'_x - ma_1y = 0,$
where P_ν are polynomials of the degree $\le \nu$, m is a positive integer, $a_1 \ne 0$.

A particular solution of this equation is the polynomial of degree m which can be written as
$$y_0 = \sum_{k=0}^{m}\Big(-\frac{1}{a_1}\Big)^k [x^m I x^{-m-1}(P_{n-1}D^n + \cdots + P_1D^2 + b_1D)]^k x^m,$$
where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu+1}$ with $\nu \ne -1$.

41. $[a_nx^n + P_{n-1}(x)]y_x^{(n)} + \cdots + [a_1x + P_0(x)]y'_x + a_0y = 0,$
where P_ν are polynomials of the degree $\le \nu$.

Assume that for some integer $m \ge 0$,
$$\sum_{\nu=0}^{n} C_m^\nu \nu!\, a_\nu = 0, \qquad \text{where} \quad C_m^\nu = \frac{m!}{\nu!\,(m-\nu)!},$$
and m is the least of the numbers satisfying this condition. Then, there exists a solution in the form of a polynomial of degree m such that no polynomial of a smaller degree satisfies the original equation.

5.1.3. Equations Containing Exponential Functions

1. $y_x^{(n)} + (ax+b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solution: $y_0 = ax+b$.

2. $y_x^{(n)} + (ae^{\lambda x} - b^{n-m})y_x^{(m)} - ab^m e^{\lambda x}y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + ay_x^{(n-1)} + be^{\lambda x}y'_x + abe^{\lambda x}y = 0.$

Particular solution: $y_0 = e^{-ax}$.

4. $y_x^{(n)} + ae^{\lambda x}y_x^{(m)} - (ab^m e^{\lambda x} + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

5. $\quad y_x^{(n)} = \sum_{k=0}^{n-1} (A_{k+1} e^{\lambda x} + bA_{k+1} - A_k) y_x^{(k)},$

where $A_n = 1$, $A_0 = 0$; b and A_k are arbitrary numbers ($k = 1, 2, \ldots, n-1$).

Particular solutions: $y_m = e^{\mu_m x}$, where μ_m are the roots of the polynomial equation $\sum_{k=0}^{n-1} A_{k+1} \mu^k = 0$.

6. $\quad x y_x^{(n)} + axe^{\lambda x} y_x^{(m)} - [a(x+m)e^{\lambda x} + x + n] y = 0.$

Particular solution: $\quad y_0 = xe^x$.

7. $\quad x y_x^{(n)} + (n - m - 1) y_x^{(n-1)} + axe^{\lambda x} y_x' - ame^{\lambda x} y = 0.$

Particular solution: $\quad y_0 = x^m$.

8. $\quad x(x+m) y_x^{(n)} + x(ae^{\lambda x} - x - n) y_x^{(m)} - a(x+m)e^{\lambda x} y = 0.$

Particular solution: $\quad y_0 = xe^x$.

9. $\quad (ax^m + be^x + c) y_x^{(n)} = be^x y, \qquad m = 0,\ 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $\quad y_0 = ax^m + be^x + c$.

10. $\quad (ax^m e^x + b) y_x^{(n)} = (-1)^n by, \qquad m = 0,\ 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $\quad y_0 = ax^m + be^{-x}$.

11. $\quad \left(ae^x + \sum_{k=0}^{n-1} b_k x^k \right) y_x^{(n)} = ae^x y.$

Particular solution: $\quad y_0 = ae^x + \sum_{k=0}^{n-1} b_k x^k$.

12. $\quad y_x^{(n)} + a \cosh^k x\, y_x^{(m)} - (ab^m \cosh^k x + b^n) y = 0.$

Particular solution: $\quad y_0 = e^{bx}$.

13. $\quad y_x^{(n)} + (a \cosh^k x - b^{n-m}) y_x^{(m)} - ab^m \cosh^k x\, y = 0.$

Particular solution: $\quad y_0 = e^{bx}$.

14. $\quad y_x^{(n)} + (ax+b) \cosh^m(\lambda x) y_x' - a \cosh^m(\lambda x) y = 0.$

Particular solution: $\quad y_0 = ax + b$.

15. $\quad x y_x^{(n)} + ax \cosh^k x\, y_x^{(m)} - [a(x+m) \cosh^k x + x + n] y = 0.$

Particular solution: $\quad y_0 = xe^x$.

16. $\quad y_x^{(n)} + a \sinh^k x\, y_x^{(m)} - (ab^m \sinh^k x + b^n) y = 0.$

Particular solution: $\quad y_0 = e^{bx}$.

17. $y_x^{(n)} + (a\sinh^k x - b^{n-m})y_x^{(m)} - ab^m \sinh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

18. $y_x^{(n)} + (ax+b)\sinh^m(\lambda x)y'_x - a\sinh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

19. $xy_x^{(n)} + ax\sinh^k x\, y_x^{(m)} - [a(x+m)\sinh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

20. $y_x^{(n)} + a\tanh^k x\, y_x^{(m)} - (ab^m \tanh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

21. $y_x^{(n)} + (a\tanh^k x - b^{n-m})y_x^{(m)} - ab^m \tanh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

22. $y_x^{(n)} + (ax+b)\tanh^m(\lambda x)y'_x - a\tanh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

23. $xy_x^{(n)} + ax\tanh^k x\, y_x^{(m)} - [a(x+m)\tanh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

24. $y_x^{(n)} + a\coth^k x\, y_x^{(m)} - (ab^m \coth^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

25. $y_x^{(n)} + (a\coth^k x - b^{n-m})y_x^{(m)} - ab^m \coth^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

26. $y_x^{(n)} + (ax+b)\coth^m(\lambda x)\, y'_x - a\coth^m(\lambda x)\, y = 0.$

Particular solution: $y_0 = ax + b$.

27. $xy_x^{(n)} + ax\coth^k x\, y_x^{(m)} - [a(x+m)\coth^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

5.1.4. Equations Containing Trigonometric Functions

1. $y_x^{(n)} + a\sin^k x\, y_x^{(m)} - (ab^m \sin^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

2. $y_x^{(n)} + (a\sin^k x - b^{n-m})y_x^{(m)} - ab^m \sin^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + a y_x^{(n-1)} + b \sin^m(\lambda x) y'_x + ab \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

4. $y_x^{(n)} + (ax+b) \sin^m(\lambda x) y'_x - a \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

5. $y_x^{(n)} + a \cos^k x\, y_x^{(m)} - (ab^m \cos^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

6. $y_x^{(n)} + (a \cos^k x - b^{n-m}) y_x^{(m)} - ab^m \cos^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

7. $y_x^{(n)} + a y_x^{(n-1)} + b \cos^m(\lambda x) y'_x + ab \cos^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

8. $y_x^{(n)} + (ax+b) \cos^m(\lambda x) y'_x - a \cos^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

9. $y_x^{(n)} + a \tan^k x\, y_x^{(m)} - (ab^m \tan^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

10. $y_x^{(n)} + (a \tan^k x - b^{n-m}) y_x^{(m)} - ab^m \tan^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

11. $y_x^{(n)} + a y_x^{(n-1)} + b \tan^m(\lambda x) y'_x + ab \tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

12. $y_x^{(n)} + (ax+b) \tan^m(\lambda x) y'_x - a \tan^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

13. $x y_x^{(n)} + ax \sin^k(\lambda x) y_x^{(m)} - [a(x+m) \sin^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = xe^x$.

14. $x y_x^{(n)} + ax \cos^k(\lambda x) y_x^{(m)} - [a(x+m) \cos^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = xe^x$.

15. $x y_x^{(n)} + ax \tan^k(\lambda x) y_x^{(m)} - [a(x+m) \tan^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = xe^x$.

16. $(ax^m + b \sin x) y_x^{(n)} = b \sin\left(x + \dfrac{\pi n}{2}\right) y, \qquad m = 0,\ 1,\ 2,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + b \sin x$.

17. $\displaystyle \left(a\sin x+\sum_{k=0}^{n-1} b_k x^k\right) y_x^{(n)} = a\sin\left(x+\frac{\pi n}{2}\right) y.$

Particular solution: $\displaystyle y_0 = a\sin x+\sum_{k=0}^{n-1} b_k x^k.$

18. $\displaystyle (ax^m + b\cos x) y_x^{(n)} = b\cos\left(x+\frac{\pi n}{2}\right) y, \qquad m = 0,\ 1,\ 2,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + b\cos x.$

19. $\displaystyle \left(a\cos x+\sum_{k=0}^{n-1} b_k x^k\right) y_x^{(n)} = a\cos\left(x+\frac{\pi n}{2}\right) y.$

Particular solution: $\displaystyle y_0 = a\sin x+\sum_{k=0}^{n-1} b_k x^k.$

5.1.5. Equations Containing Arbitrary Functions

1. $y_x^{(n)} = f(x).$

Solution:

$$y = \sum_{\nu=0}^{n-1} C_\nu x^\nu + \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt,$$

where x_0 may be chosen arbitrarily.

2. $y_x^{(n)} + xf(x)y'_x - mf(x)y = 0.$

If $m = 0,\ 1,\ 2,\ \dots,$ or $n-1$, the equation has a particular solution $y_0 = x^m$, and the substitution $z = xy'_x - my$ leads to an $(n-1)$ th order equation:

$$D^{n-m-1}\left(\frac{z_x^{(m)}}{x}\right) + f(x)z = 0, \qquad \text{where} \quad D = \frac{d}{dx}.$$

In particular, for $m = n-1$, we have $z_x^{(n-1)} + xf(x)z = 0.$

3. $y_x^{(n)} + (ax+b)f(x)y'_x - af(x)y = 0.$

Particular solution: $y_0 = ax + b.$

4. $y_x^{(n)} + f(x)(x^2 y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2.$

The substitution $z = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a linear equation of the $(n-2)$ th order.

5. $y_x^{(n)} + f(x)y_x^{(m)} - [a^n + a^m f(x)]y = 0.$

Particular solution: $y_0 = e^{ax}.$

6. $\boldsymbol{y_x^{(n)} + (f - a^{n-m}) y_x^{(m)} - a^m f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = e^{ax}$.

7. $\boldsymbol{y_x^{(n)} + a y_x^{(n-1)} + f y'_x + a f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = e^{-ax}$.

8. $\boldsymbol{y_x^{(n)} + f(x) y_x^{(n-1)} + g(x) y_x^{(n-2)} + h(x) = 0.}$

The substitution $w(x) = y_x^{(n-2)}$ leads to a linear equation of the second order: $w''_{xx} + f(x) w'_x + g(x) w + h(x) = 0$.

9. $\boldsymbol{y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y'_x + a_0 y = f(x).}$

The nonhomogeneous constant-coefficient linear equation.

The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation (see 5.1.2.3) and any particular solution of the nonhomogeneous equation.

If all the roots of the polynomial

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

are different, the original equation has the general solution

$$y = \sum_{\nu=1}^{n} C_\nu e^{\lambda_\nu x} + \sum_{\nu=1}^{n} \frac{e^{\lambda_\nu x}}{P'_\lambda(\lambda_\nu)} \int f(x) e^{-\lambda_\nu x}\, dx$$

(with complex roots, the real part should be taken).

In Table 5.1 are listed the forms of particular solutions corresponding to some special forms of functions on the right-hand side of the linear nonhomogeneous equation.

10. $\boldsymbol{x^n y_x^{(n)} + b_{n-1} x^{n-1} y_x^{(n-1)} + \cdots + b_1 x y'_x + b_0 y = f(x).}$

The substitution $x = ae^t$ $(a \neq 0)$ leads to an equation of the form 5.1.5.9.

11. $\boldsymbol{y_x^{(n)} + f(x) \sum_{k=0}^{n-1} (-1)^k k! C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)} = 0,}$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_m = x^m$, where $m = 1,\ 2,\ \ldots,\ n-1$.

The substitution $z = \sum_{k=0}^{n-1} (-1)^k k! C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)}$ leads to a first order linear equation: $z'_x + x^{n-1} f(x) z = 0$. Having solved this equation, we obtain an $(n-1)$ th order equation of the form 5.1.5.10 for function $y(x)$.

12. $\boldsymbol{y_x^{(n)} = \sum_{k=0}^{n-1} (a_{k+1} f - a_k) y_x^{(k)},}$

where $f = f(x)$; $a_n = 1$, $a_0 = 0$; a_k are arbitrary numbers $(k = 1,\ 2,\ \ldots,\ n-1)$.

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ \ldots,\ n-1)$, where λ_k are the roots of the polynomial equation $\sum_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

TABLE 5.1

The forms of particular solutions of the nonhomogeneous constant-coefficient linear equation $y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = f(x)$ which correspond to some special forms of function $f(x)$.

The form of function $f(x)$	Roots of the characteristic equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$	The form of a particular solution $y = \widetilde{y}(x)$
$P_m(x)$	Zero is not a root of the characteristic equation (i.e., $a_0 \neq 0$)	$\widetilde{P}_m(x)$
	Zero is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)$
$P_m(x)e^{\alpha x}$ (α is a real number)	α is not a root of the characteristic equation	$\widetilde{P}_m(x)e^{\alpha x}$
	α is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)e^{\alpha x}$
$P_m(x)\cos\beta x + Q_n(x)\sin\beta x$	$i\beta$ is not a root of the characteristic equation	$\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x$
	$i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]$
$[P_m(x)\cos\beta x + Q_n(x)\sin\beta x]e^{\alpha x}$	$\alpha + i\beta$ is not a root of the characteristic equation	$[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$
	$\alpha + i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$

Notation: P_m and Q_n are polynomials of the degrees m and n with given coefficients; $\widetilde{P}_m$, $\widetilde{P}_\nu$, and $\widetilde{Q}_\nu$ are polynomials of the degrees m and ν whose coefficients are determined as a result of substituting the particular solution into the basic equation; $\nu = \max(m, n)$; $i^2 = -1$.

13. $\boldsymbol{x y_x^{(n)} + x f y_x^{(m)} - [(x+m)f + x + n]y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = xe^x$.

14. $\boldsymbol{x(x+m) y_x^{(n)} + x(f - x - n) y_x^{(m)} - (x+m) f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = xe^x$.

15. $\boldsymbol{x^k y_x^{(n)} + x f(x) y_x' - m f(x) y = 0, \qquad m = 0,\ 1,\ 2,\ \dots,\ n-1.}$

Particular solution: $y_0 = x^m$.

16. $\boldsymbol{x^n y_x^{(n)} + (n - m - 1) x^{n-1} y_x^{(n-1)} + x f y_x' - m f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = x^m$.

17. $\boldsymbol{x^n y_x^{(n)} + x^m f y_x^{(m)} - (n!C_a^n + m!C_a^m f)y = 0,}$

where $f = f(x)$, $C_a^n = \dfrac{\Gamma(a+1)}{n!\,\Gamma(a-n+1)}$ are binomial coefficients, $\Gamma(a)$ is the gamma-function.

Particular solution: $y_0 = x^a$.

18. $\displaystyle \boldsymbol{x^m y_x^{(n)} = \sum_{k=0}^{n-1} [x^m(a_{k+1} f - a_k) + a_{k+1}] y_x^{(k)},}$

where $f = f(x)$; $a_n = 1$, $a_0 = 0$; m and a_k are arbitrary numbers ($k = 1,\, 2,\, \dots,\, n-1$).

Particular solutions: $y_k = e^{\lambda_k x}$ ($k = 1,\, 2,\, \dots,\, n-1$), where λ_k are the roots of the polynomial equation $\sum_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

19. $\displaystyle \boldsymbol{\sum_{k=2}^{n} f_k(x) y_x^{(k)} = g(x)(x y'_x - y).}$

Particular solution: $y_0 = x$.

The substitution $w(x) = x y'_x - y$ leads to an $(n-1)$ th order equation.

20. $\displaystyle \boldsymbol{\sum_{k=3}^{n} f_k(x) y_x^{(k)} = g(x)(x^2 y''_{xx} - 2x y'_x + 2y).}$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

The substitution $w(x) = x^2 y''_{xx} - 2x y'_x + 2y$ leads to an $(n-2)$ th order equation.

21. $\displaystyle \boldsymbol{\sum_{k=4}^{n} f_k(x) y_x^{(k)} = g(x)(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6x y'_x - 6y).}$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$.

The substitution $w(x) = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6x y'_x - 6y$ leads to an $(n-3)$ th order equation.

22. $\displaystyle \boldsymbol{\sum_{k=m+1}^{n} f_k(x) y_x^{(k)} + g(x) \sum_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)} = 0,}$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_s = x^s$, where $s = 1,\, 2,\, \dots,\, m$.

The substitution $z = \sum_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$ th order equation:

$$\sum_{k=m+1}^{n} f_k(x) D^{(k-m-1)}\left(\frac{z'_x}{x^m}\right) + g(x) z = 0, \qquad \text{where} \quad D = \frac{d}{dx}.$$

23. $\displaystyle \boldsymbol{\sum_{k=0}^{n} (f_k - a f_{k+1}) y_x^{(k)} = 0,}$

where $f_k = f_k(x)$ ($k = 1,\, 2,\, \dots,\, n$), $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = e^{ax}$.

24. $\displaystyle\sum_{k=0}^{n} x^k[f_k + (k-m)f_{k+1}]y_x^{(k)} = 0,$

where $f_k = f_k(x)$ $(k = 1,\ 2,\ \ldots,\ n)$, $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = x^m$.

25. $fy_x^{(n)} - f_x^{(n)}y = 0, \qquad f = f(x).$

Particular solution: $y_0 = f(x)$.

26. $fy_x^{(2n+1)} + f_x^{(2n+1)}y = 0, \qquad f = f(x).$

The first integral has the form

$$\sum_{k=0}^{2n}(-1)^k f_x^{(2n-k)} y_x^{(k)} = \int g(x)\,dx + C.$$

27. $\sin x\, y_x^{(n)} + \sin x\, f(x) y_x^{(m)} - \Bigl[\sin\Bigl(x + \dfrac{\pi n}{2}\Bigr) + f(x)\sin\Bigl(x + \dfrac{\pi m}{2}\Bigr)\Bigr]y = 0.$

Particular solution: $y_0 = \sin x$.

28. $\cos x\, y_x^{(n)} + \cos x f(x) y_x^{(m)} - \Bigl[\cos\Bigl(x + \dfrac{\pi n}{2}\Bigr) + f(x)\cos\Bigl(x + \dfrac{\pi m}{2}\Bigr)\Bigr]y = 0.$

Particular solution: $y_0 = \cos x$.

29. $y_x^{(n)} = f(x)y.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of the similar form: $w_t^{(n)} = (-1)^n t^{-2n} f\Bigl(\dfrac{1}{t}\Bigr)w$.

30. $y_x^{(n)} = (cx+d)^{-2n} f\Bigl(\dfrac{ax+b}{cx+d}\Bigr)y.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{n-1}}$ leads to the equation $w_\xi^{(n)} = \Delta^{-n} f(\xi)w$, where $\Delta = ad - bc$.

31. $y_x^{(n)} + f(x)y'_x + g(x)y + h(x) = 0.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of the similar form:

$$w_t^{(n)} + (-1)^n t^{-2n}\biggl\{-t^2 f\Bigl(\frac{1}{t}\Bigr)w'_t + \Bigl[(n-1)tf\Bigl(\frac{1}{t}\Bigr) + g\Bigl(\frac{1}{t}\Bigr)\Bigr]w + t^{n-1}h\Bigl(\frac{1}{t}\Bigr)\biggr\} = 0.$$

32. $y_x^{(n+2)} + f(x)[x^2 y''_{xx} - 2nxy'_x + n(n+1)y] = 0.$

The substitution $w(x) = x^2 y''_{xx} - 2nxy'_x + n(n+1)y$ leads to an nth order equation: $w_x^{(n)} + x^2 f(x)w = 0$.

33. $\boldsymbol{xy_x^{(n)} + ny_x^{(n-1)} = x^{1-2n} f\Big(\frac{1}{x}\Big) y + x^{-n-1} g\Big(\frac{1}{x}\Big).}$

The transformation $t = x^{-1}$, $w = yx^{2-n}$ leads to an nth order equation: $w_t^{(n)} = (-1)^n [f(t)w + g(t)]$.

34. $\boldsymbol{x^2 y_x^{(n+2)} + \alpha x y_x^{(n+1)} + \beta y_x^{(n)}}$

$$\boldsymbol{+ f(x)[x^2 y''_{xx} + (\alpha - 2n) x y'_x + (\beta - \alpha n + n^2 + n) y] = 0.}$$

The substitution

$$w(x) = x^2 y''_{xx} + (\alpha - 2n) x y'_x + (\beta - \alpha n + n^2 + n) y$$

leads to an nth order equation: $w_x^{(n)} + f(x) w = 0$.

5.1.6. Asymptotic Solutions

This subsection presents asymptotic solutions, as $\varepsilon \to 0$ ($\varepsilon > 0$), of some fifth-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being a real number.

1. Consider an equation of the form

$$\varepsilon^n y_x^{(n)} - f(x) y = 0. \qquad (1)$$

on a closed interval $a \le x \le b$. Assume that $f \ne 0$. Then, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to +0$, has the form

$$y_m = \big[f(x)\big]^{-\frac{1}{2} + \frac{1}{2n}} \exp\biggl\{\frac{\omega_m}{\varepsilon} \int \big[f(x)\big]^{\frac{1}{n}}\, dx\biggr\} \big[1 + O(\varepsilon)\big],$$

where ω_1, ω_2, $\dots$, ω_n are the roots of the equation $\omega^n = 1$:

$$\omega_m = \cos\Big(\frac{2\pi m}{n}\Big) + i \sin\Big(\frac{2\pi m}{n}\Big), \qquad m = 1,\ 2,\ \dots,\ n.$$

2. Consider an equation of the form

$$\varepsilon^n y_x^{(n)} + \varepsilon^{n-1} f_{n-1}(x) y_x^{(n-1)} + \dots + \varepsilon f_1(x) y'_x + f_0(x) y = 0. \qquad (2)$$

on a closed interval $a \le x \le b$. Let $\lambda_m = \lambda_m(x)$ ($m = 1, 2, \dots, n$) be the roots of the characteristic equation

$$P(x, \lambda) \equiv \lambda^n + f_{n-1}(x) \lambda^{n-1} + \dots + f_1(x) \lambda + f_0(x) = 0.$$

Let all the roots of the characteristic equation be different on the interval $a \le x \le b$, i.e., the conditions $\lambda_m(x) \ne \lambda_k(x)$ if $m \ne k$ are satisfied, which is equivalent to the fulfillment of the conditions $P_\lambda(x, \lambda_m) \ne 0$. Then, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to +0$, are given by the formulae

$$y_m = \exp\biggl\{\frac{1}{\varepsilon} \int \lambda_m(x)\, dx - \frac{1}{2} \int [\lambda_m(x)]'_x \frac{P_{\lambda\lambda}\big(x, \lambda_m(x)\big)}{P_\lambda\big(x, \lambda_m(x)\big)}\, dx\biggr\},$$

where

$$P_\lambda(x, \lambda) \equiv \frac{\partial P}{\partial \lambda} = n\lambda^{n-1} + (n-1) f^{n-1} \lambda^{n-2} + \ldots + 2\lambda f_2(x) + f_1(x),$$

$$P_{\lambda\lambda}(x, \lambda) \equiv \frac{\partial^2 P}{\partial \lambda^2} = n(n-1)\lambda^{n-2} + (n-1)(n-2) f^{n-1} \lambda^{n-3} + \ldots + 6\lambda f_3(x) + 2 f_2(x).$$

5.2. Nonlinear Equations

5.2.1. Equations Containing Power Functions

1. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = ax^n.$

This is a special case of equation 5.2.6.1 with $f(x) = ax^n$.

2. $y_x^{(6)} = Ay^{-7/5}.$

Multiplying both sides by $y^{7/5}$ and differentiating with respect to x, we obtain the equation $5yy_x^{(7)} + 7y_x' y_x^{(6)} = 0$. Having integrated the latter three times, we arrive at a chain of equalities:

$$5yy_x^{(6)} + 2y_x' y_x^{(5)} - 2y_{xx}'' y_{xxxx}'''' + (y_{xxx}''')^2 = 2C_2, \qquad (1)$$

$$5yy_x^{(5)} - 3y_x' y_{xxxx}'''' + y_{xx}'' y_{xxx}''' = 2C_2 x + C_1, \qquad (2)$$

$$5yy_{xxxx}'''' - 8y_x' y_{xxx}''' + \tfrac{9}{2}(y_{xx}'')^2 = C_2 x^2 + C_1 x + C_0, \qquad (3)$$

where C_0, C_1, and C_2 are arbitrary constants. Eliminating the highest derivatives from (1)–(3), with the aid of the original equation, we can obtain a third order equation which can be reduced to a second order equation (see equation 5.2.1.4 with $n = 3$).

3. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = ax^n.$

This is a special case of equation 5.2.6.4 with $f(x) = ax^n$.

4. $y_x^{(2n)} = Ay^{\frac{1+2n}{1-2n}}.$

Multiply both sides by $y^{\frac{2n+1}{2n-1}}$ and differentiate with respect to x. As a result we obtain

$$(2n-1)yy_x^{(2n+1)} + (2n+1)y_x' y_x^{(2n)} = 0.$$

Three integrals containing arbitrary constants C_0, C_1, and C_2 are presented in 5.2.6.22 wherein we should let $f \equiv 0$. Eliminating the highest derivatives from those integrals and the original equation, we may always obtain a $(2n-3)$ th order equation. With the aid of the transformation

$$t = \int \frac{dx}{P}, \quad w = yP^{\frac{1-2n}{2}}, \qquad \text{where} \quad P = C_2 x^2 + C_1 x + C_0,$$

the latter equation can be reduced to the autonomous form 5.2.6.40. Therefore, the substitution $z(w) = w_t'$ finally leads to a $(2n-4)$ th order equation with respect to $z = z(w)$.

5. $y_x^{(2n)} = Ay^k, \qquad k \neq -1.$

Having integrated, we arrive at

$$\sum_{m=1}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + \frac{1}{2}(-1)^n \left[y_x^{(n)}\right]^2 = -\frac{A}{k+1} y^{k+1} + C,$$

where C is an arbitrary number. Further, the order of the obtained autonomous equation can be lowered by the substitution $w(y) = y_x'$.

6. $\boldsymbol{y_x^{(n)} = ax^{-n}y^m}$.

This is a special case of equation 5.2.6.10 with $f(y) = ay^m$.

7. $\boldsymbol{y_x^{(n)} = ax^k y^m}$.

1°. The transformation $x = t^{-1}$, $y = t^{1-n}w(t)$ leads to an equation of the similar form: $w_t^{(n)} = (-1)^n A t^{-k-(n-1)m-n-1} w^m$.

2°. The transformation $\xi = x^{n+k}y^{m-1}$, $z = xy'_x/y$ leads to an $(n-1)$ th order equation.

8. $\boldsymbol{yy_x^{(2n+1)} = ax^n + b}$.

This is a special case of equation 5.2.6.16 with $f(x) = ax^n + b$.

9. $\boldsymbol{y_x^{(n)} = x^{m-nm-n-1}(ay + bx^{n-1})^m}$.

This is a special case of equation 5.2.6.11 with $f(w) = (aw + b)^m$.

10. $\boldsymbol{y_x^{(2n)} = x^{\frac{m-2nm-2n-1}{2}}\Big(ay + bx^{\frac{2n-1}{2}}\Big)^m}$.

This is a special case of equation 5.2.6.12 with $f(w) = (aw + b)^m$.

11. $\boldsymbol{y_x^{(n)} = (ay + bx^k)^m; \qquad k = 1,\ 2,\ \dots,\ n-1}$.

The substitution $aw = ay + bx^k$ leads to an autonomous equation: $w_x^{(n)} = a^m w^m$ (see 5.2.1.4, 5.2.1.5, and 5.2.6.40).

12. $\boldsymbol{y_x^{(n)} = (ax^2 + bx + c)^{\frac{m-nm-n-1}{2}} y^m}$.

This is a special case of equation 5.2.6.21 with $f(w) = w^m$.

13. $\boldsymbol{y_x^{(n)} = (ax + b)^{-n}(cx + d)^{m-nm-1}y^m}$.

This is a special case of equation 5.2.6.20 with $f(w) = w^m$.

14. $\boldsymbol{yy_x^{(2n+1)} = ay'_x y_x^{(2n)}}$.

The equation admits two different (with $a \neq -1$) first integrals:

$$y_x^{(2n)} = \widetilde{C}_1 y^a,$$

$$yy_x^{(2n)} + (a+1)\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n(a+1)\big[y_x^{(n)}\big]^2 = \widetilde{C}_2,$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. Eliminating the highest derivative from the integrals, we arrive at a $(2n-1)$ th order autonomous equation:

$$\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n\big[y_x^{(n)}\big]^2 = C_1 y^{a+1} + C_2,$$

where $C_1 = -\dfrac{\widetilde{C}_1}{a+1}$, $C_2 = \dfrac{\widetilde{C}_2}{a+1}$. The order of the obtained equation next can be lowered by the standard substitution $w(y) = y'_x$.

15. $y_x^{(n-2)} y_x^{(n)} = a\left(y_x^{(n-1)}\right)^2.$

Solution:

$$y = \begin{cases} C_0 + C_1 x + \cdots + C_{n-3}x^{n-3} + (C_{n-2} + C_{n-1}x)^{n-2+\frac{1}{1-a}} & \text{if } a \neq 1, \\ C_0 + C_1 x + \cdots + C_{n-3}x^{n-3} + C_{n-2}\exp(C_{n-1}x) & \text{if } a = 1. \end{cases}$$

16. $y_x^{(n)} = ax^{m-n} y^{1-m} (y_x')^m.$

This is a special case of equation 5.2.6.15 with $f(w) = aw^m$.

17. $y_x^{(n+1)} = ay^k y_x' \left(y_x^{(n)}\right)^m.$

This is a special case of equation 5.2.6.17 with $f(y) = y^{-k}$, $g(w) = aw^m$.

18. $y_x^{(n)} = ax^m (xy_x' - y)^k (y_{xx}'')^l.$

The substitution $w(x) = xy_x' - y$ leads to an $(n-1)$ th order equation:

$$\frac{d^{n-2}}{dx^{n-2}}\left(\frac{w_x'}{x}\right) = ax^{m-l} w^k (w_x')^l.$$

19. $y_x^{(n)} = ax^{m_1} y^{m_2} (y_x')^{m_3} \cdots (y_x^{(n-1)})^{m_{n+1}}.$

Homogeneous equation in the extended sense.

The transformation $\xi = x^\lambda y^\mu$, $w = \dfrac{xy_x'}{y}$, where

$$\lambda = n + m_1 - m_3 - 2m_4 - \cdots - (n-1)m_{n+1}, \quad \mu = m_2 + m_3 + \cdots + m_{n+1} - 1,$$

leads to an $(n-1)$ th order equation.

20. $xy_x^{(n)} + ny_x^{(n-1)} = ax^m y^m.$

This is a special case of equation 5.2.6.23 with $f(w) = aw^m$.

21. $x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^{2m} y^m.$

This is a special case of equation 5.2.6.24 with $f(w) = aw^m$.

22. $(2n-1)yy_x^{(2n+1)} + (2n+1)y_x' y_x^{(2n)} = ax^m.$

This is a special case of equation 5.2.6.22 with $f(x) = ax^m$.

23. $\left(\sqrt{y}\dfrac{d}{dx}\right)^{n-1}(y_x') = ax + b.$

The transformation $x = x(t)$, $y = (x_t')^2$ leads to a constant coefficient linear equation: $2x_t^{(n+1)} = ax + b$.

24. $2\displaystyle\sum_{m=1}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \left[y_x^{(n)}\right]^2 + \lambda(y_x')^2 = ay^2 + by + c.$

Differentiating both sides with respect to x and dividing by y_x', we arrive at a constant coefficient equation: $2y_x^{(2n)} - 2\lambda y_{xx}'' + 2ay + b = 0$.

25. $\displaystyle \mathbf{2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \big[y_x^{(n)}\big]^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma.}$

Differentiating both sides of the equation with respect to x, we have

$$y''_{xx}\big[2y_x^{(2n-1)} - \alpha x - \beta\big] = 0. \tag{1}$$

Equating the second factor to zero, we find

$$y = \frac{1}{2}\frac{\alpha x^{2n}}{(2n)!} + \frac{1}{2}\frac{\beta x^{2n-1}}{(2n-1)!} + \sum_{k=0}^{2n-2} C_k x^k.$$

Integration constants C_k and parameters α, β, and γ are related by the equality

$$2\sum_{m=2}^{n-1}(-1)^m m!\,(2n-m)!\,C_m C_{2n-m} + (-1)^n (n!)^2 C_n^2 = \beta C_1 - \alpha C_0 + \gamma,$$

which is obtained as a result of substituting the above solution into the original equation.

In addition, there is the solution corresponding to equating the first factor in (1) to zero:

$$y = \widetilde{C}_1 x + \widetilde{C}_0, \qquad \text{where} \quad \beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0.$$

26. $\displaystyle \mathbf{2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \big[y_x^{(n)}\big]^2 + s(y''_{xx})^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma,}$

where n is an integer greater than or equal to 3.

With $s = 0$ see 5.2.1.25. Let $s \neq 0$. Differentiating the equation with respect to x, we have

$$y''_{xx}\big[2y_x^{(2n-1)} + 2sy'''_{xxx} - \alpha x - \beta\big] = 0.$$

Equating the second factor to zero and integrating, we obtain

$$y = \frac{\alpha x^4}{48s} + \frac{\beta x^3}{12s} + C_2 x^2 + C_1 x + C_0 + \iiint w\,dx\,dx\,dx,$$

where $w = w(x)$ is the general solution of a constant coefficient equation of the form 5.1.2.2: $w_x^{(2n-4)} + sw = 0$. The constants of integration are related by an equality which is found as a result of substituting the obtained solution into the original equation.

In addition, there is the solution $y = \widetilde{C}_1 x + \widetilde{C}_0$, where the constants of integration are related by $\beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0$.

27. $\displaystyle \mathbf{\sum_{m=1}^{n} a_m \bigg\{2\sum_{\nu=1}^{m-1}(-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m \big[y_x^{(m)}\big]^2\bigg\} = \alpha y^2 + 2\beta y + \gamma.}$

Differentiating with respect to x, we arrive at a constant coefficient linear equation: $\sum_{m=1}^{n} a_m y_x^{(2m)} + \alpha y + \beta = 0$.

5.2.2. Equations Containing Exponential Functions

1. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = ae^{\lambda x}.$

This is a special case of equation 5.2.6.1 with $f(x) = ae^{\lambda x}$.

2. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = ae^{\lambda x}.$

This is a special case of equation 5.2.6.4 with $f(x) = ae^{\lambda x}$.

3. $y_x^{(2n)} = ae^{\lambda y}.$

This is a special case of equation 5.2.6.6 with $f(y) = ae^{\lambda y}$.

4. $y_x^{(n)} = ax^{-n} e^{\lambda y}.$

This is a special case of equation 5.2.6.10 with $f(y) = ae^{\lambda y}$.

5. $y_x^{(n)} = ax^k e^{\alpha y}.$

This is a special case of equation 5.2.6.31 with $f(w) = aw$, $m = k + n$.

6. $y_x^{(n)} = Ae^{\alpha x} y^m.$

This is a special case of equation 5.2.6.11 with $m = m_1$ and $m_2 = m_3 = \cdots = m_n = 0$.

7. $yy_x^{(2n+1)} = ae^{\lambda x} + b.$

This is a special case of equation 5.2.6.16 with $f(x) = ae^{\lambda x} + b$.

8. $y_x^{(n)} = ae^{by} e^{cx^m}, \qquad m = 1,\ 2,\ \ldots,\ n-1.$

The substitution $bw = by + cx^m$ leads to an autonomous equation: $w_x^{(n)} = ae^{bw}$, which, for even n, admits lowering of its order by two (see 5.2.2.3).

9. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = ae^{\lambda x}.$

This is a special case of equation 5.2.6.22 with $f(x) = ae^{\lambda x}$.

10. $y_x^{(n+1)} = ae^{\lambda y} y'_x \big(y_x^{(n)}\big)^m.$

This is a special case of equation 5.2.6.17 with $f(y) = e^{-\lambda y}$, $g(w) = aw^m$.

11. $y_x^{(n)} = Ae^{\alpha x} y^{m_1} (y'_x)^{m_2} \ldots \big(y_x^{(n-1)}\big)^{m_n}.$

The substitution $w(x) = ye^{\beta x}$, where $\beta = \dfrac{\alpha}{m_1 + m_2 + \cdots + m_n - 1}$, leads to an autonomous equation of the form 5.2.6.40.

12. $y_x^{(n)} = Ae^{\alpha y} x^{m_1} (y'_x)^{m_2} (y''_{xx})^{m_3} \ldots \big(y_x^{(n-1)}\big)^{m_n}.$

The transformation $z = x^\sigma e^{\alpha y}$, $w = xy'_x$, where $\sigma = n + m_1 - m_2 - 2m_3 - 3m_4 - \cdots - (n-1)m_n$, leads to an $(n-1)$ th order equation.

5.2.3. Equations Containing Hyperbolic Functions

1. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\cosh^m(\lambda x)$.

This is a special case of equation 5.2.6.1 with $f(x) = a\cosh^m(\lambda x)$.

2. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\sinh^m(\lambda x)$.

This is a special case of equation 5.2.6.1 with $f(x) = a\sinh^m(\lambda x)$.

3. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\tanh^m(\lambda x)$.

This is a special case of equation 5.2.6.1 with $f(x) = a\tanh^m(\lambda x)$.

4. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\coth^m(\lambda x)$.

This is a special case of equation 5.2.6.1 with $f(x) = a\coth^m(\lambda x)$.

5. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a\cosh^m(\lambda x)$.

This is a special case of equation 5.2.6.4 with $f(x) = a\cosh^m(\lambda x)$.

6. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a\sinh^m(\lambda x)$.

This is a special case of equation 5.2.6.4 with $f(x) = a\sinh^m(\lambda x)$.

7. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a\tanh^m(\lambda x)$.

This is a special case of equation 5.2.6.4 with $f(x) = a\tanh^m(\lambda x)$.

8. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a\coth^m(\lambda x)$.

This is a special case of equation 5.2.6.4 with $f(x) = a\coth^m(\lambda x)$.

9. $y_x^{(2n)} = a\cosh^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a\cosh^m(\lambda y)$.

10. $y_x^{(2n)} = a\sinh^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a\sinh^m(\lambda y)$.

11. $y_x^{(2n)} = a\tanh^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a\tanh^m(\lambda y)$.

12. $y_x^{(2n)} = a\coth^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a\coth^m(\lambda y)$.

13. $y_x^{(n)} = ax^{-n}\cosh^m(\lambda y)$.

This is a special case of equation 5.2.6.10 with $f(y) = a\cosh^m(\lambda y)$.

14. $\boldsymbol{y_x^{(n)} = ax^{-n}\sinh^m(\lambda y).}$

This is a special case of equation 5.2.6.10 with $f(y) = a\sinh^m(\lambda y)$.

15. $\boldsymbol{y_x^{(n)} = ax^{-n}\tanh^m(\lambda y).}$

This is a special case of equation 5.2.6.10 with $f(y) = a\tanh^m(\lambda y)$.

16. $\boldsymbol{y_x^{(n)} = ax^{-n}\coth^m(\lambda y).}$

This is a special case of equation 5.2.6.10 with $f(y) = a\coth^m(\lambda y)$.

17. $\boldsymbol{yy_x^{(2n+1)} = a\cosh^m(\lambda x).}$

This is a special case of equation 5.2.6.16 with $f(x) = a\cosh^m(\lambda x)$.

18. $\boldsymbol{yy_x^{(2n+1)} = a\sinh^m(\lambda x).}$

This is a special case of equation 5.2.6.16 with $f(x) = a\sinh^m(\lambda x)$.

19. $\boldsymbol{yy_x^{(2n+1)} = a\tanh^m(\lambda x).}$

This is a special case of equation 5.2.6.16 with $f(x) = a\tanh^m(\lambda x)$.

20. $\boldsymbol{yy_x^{(2n+1)} = a\coth^m(\lambda x).}$

This is a special case of equation 5.2.6.16 with $f(x) = a\coth^m(\lambda x)$.

21. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cosh^m(\lambda x).}$

This is a special case of equation 5.2.6.22 with $f(x) = a\cosh^m(\lambda x)$.

22. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\sinh^m(\lambda x).}$

This is a special case of equation 5.2.6.22 with $f(x) = a\sinh^m(\lambda x)$.

23. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\tanh^m(\lambda x).}$

This is a special case of equation 5.2.6.22 with $f(x) = a\tanh^m(\lambda x)$.

24. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\coth^m(\lambda x).}$

This is a special case of equation 5.2.6.22 with $f(x) = a\coth^m(\lambda x)$.

25. $\boldsymbol{y_x^{(n+1)} = a\cosh^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m.}$

This is a special case of equation 5.2.6.17 with $f(y) = \cosh^{-k}(\lambda y)$, $g(w) = aw^m$.

26. $\boldsymbol{y_x^{(n+1)} = a\sinh^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m.}$

This is a special case of equation 5.2.6.17 with $f(y) = \sinh^{-k}(\lambda y)$, $g(w) = aw^m$.

27. $\boldsymbol{y_x^{(n+1)} = a\tanh^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m.}$

This is a special case of equation 5.2.6.17 with $f(y) = \tanh^{-k}(\lambda y)$, $g(w) = aw^m$.

28. $\boldsymbol{y_x^{(n+1)} = a\coth^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m.}$

This is a special case of equation 5.2.6.17 with $f(y) = \coth^{-k}(\lambda y)$, $g(w) = aw^m$.

5.2.4. Equations Containing Logarithmic Functions

1. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a \ln^m(bx).$

This is a special case of equation 5.2.6.1 with $f(x) = a \ln^m(bx)$.

2. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})_x^2 = a \ln^m(bx).$

This is a special case of equation 5.2.6.4 with $f(x) = a \ln^m(bx)$.

3. $y_x^{(2n)} = a \ln^m(by).$

This is a special case of equation 5.2.6.6 with $f(y) = a \ln^m(by)$.

4. $yy_x^{(2n+1)} = a \ln^m(bx).$

This is a special case of equation 5.2.6.16 with $f(x) = a \ln^m(bx)$.

5. $y_x^{(n)} = y(\alpha x + m \ln y + b).$

This is a special case of equation 5.2.6.30 with $f(w) = \ln w + b$.

6. $y_x^{(n)} = x^{-n}(\alpha y + m \ln x + b).$

This is a special case of equation 5.2.6.31 with $f(w) = \ln w + b$.

7. $y_x^{(n)} = ax^{-n} \ln^m(by).$

This is a special case of equation 5.2.6.10 with $f(y) = a \ln^m(by)$.

8. $y_x^{(n)} = ax^{-n-1}[\ln y + (1-n) \ln x].$

This is a special case of equation 5.2.6.11 with $f(w) = a \ln w$.

9. $y_x^{(n)} = ax^{-n-k}(\ln y + k \ln x).$

This is a special case of equation 5.2.6.13 with $f(w) = a \ln w$.

10. $y_x^{(n)} = ayx^{-n}(m \ln y + k \ln x).$

This is a special case of equation 5.2.6.14 with $f(w) = a \ln w$.

11. $y_x^{(2n)} = ax^{-\frac{2n+1}{2}}[2 \ln y + (1-2n) \ln x].$

This is a special case of equation 5.2.6.12 with $f(w) = 2a \ln w$.

12. $y_x^{(n)} = (ax^2 + c)^{-\frac{n+1}{2}}[2 \ln y + (1-n) \ln(ax^2 + c)].$

This is a special case of equation 5.2.6.21 with $b = 0,\ f(w) = 2 \ln w$.

13. $y_x^{(n)} = be^{\alpha x}(\ln y - \alpha x).$

This is a special case of equation 5.2.6.29 with $f(w) = b \ln w$.

14. $(2n-1)yy_x^{(2n+1)}+(2n+1)y'_x y_x^{(2n)}=a\ln^m(bx).$

This is a special case of equation 5.2.6.22 with $f(x)=a\ln^m(bx)$.

15. $y_x^{(n+1)}=a\ln^k(by)\,y'_x\bigl(y_x^{(n)}\bigr)^m.$

This is a special case of equation 5.2.6.17 with $f(y)=\ln^{-k}(by)$, $g(w)=aw^m$.

16. $y_x^{(n+1)}=ay^m yy'_x\ln y_x^{(n)}.$

This is a special case of equation 5.2.6.17 with $f(y)=y^{-m}$, $g(w)=a\ln w$.

5.2.5. Equations Containing Trigonometric Functions

1. $yy_x^{(5)}+5y'_x y''''_{xxxx}+10y''_{xx}y'''_{xxx}=a\cos^m(\lambda x).$

This is a special case of equation 5.2.6.1 with $f(x)=a\cos^m(\lambda x)$.

2. $yy_x^{(5)}+5y'_x y''''_{xxxx}+10y''_{xx}y'''_{xxx}=a\sin^m(\lambda x).$

This is a special case of equation 5.2.6.1 with $f(x)=a\sin^m(\lambda x)$.

3. $yy_x^{(5)}+5y'_x y''''_{xxxx}+10y''_{xx}y'''_{xxx}=a\tan^m(\lambda x).$

This is a special case of equation 5.2.6.1 with $f(x)=a\tan^m(\lambda x)$.

4. $yy_x^{(5)}+5y'_x y''''_{xxxx}+10y''_{xx}y'''_{xxx}=a\cot^m(\lambda x).$

This is a special case of equation 5.2.6.1 with $f(x)=a\cot^m(\lambda x)$.

5. $yy_x^{(6)}+6y'_x y_x^{(5)}+15y''_{xx}y''''_{xxxx}+10(y'''_{xxx})^2=a\cos^m(\lambda x).$

This is a special case of equation 5.2.6.4 with $f(x)=a\cos^m(\lambda x)$.

6. $yy_x^{(6)}+6y'_x y_x^{(5)}+15y''_{xx}y''''_{xxxx}+10(y'''_{xxx})^2=a\sin^m(\lambda x).$

This is a special case of equation 5.2.6.4 with $f(x)=a\sin^m(\lambda x)$.

7. $yy_x^{(6)}+6y'_x y_x^{(5)}+15y''_{xx}y''''_{xxxx}+10(y'''_{xxx})^2=a\tan^m(\lambda x).$

This is a special case of equation 5.2.6.4 with $f(x)=a\tan^m(\lambda x)$.

8. $yy_x^{(6)}+6y'_x y_x^{(5)}+15y''_{xx}y''''_{xxxx}+10(y'''_{xxx})^2=a\cot^m(\lambda x).$

This is a special case of equation 5.2.6.4 with $f(x)=a\cot^m(\lambda x)$.

9. $y_x^{(2n)}=a\cos^m(\lambda y).$

This is a special case of equation 5.2.6.6 with $f(y)=a\cos^m(\lambda y)$.

10. $y_x^{(2n)}=a\sin^m(\lambda y).$

This is a special case of equation 5.2.6.6 with $f(y)=a\sin^m(\lambda y)$.

11. $y_x^{(2n)} = a \tan^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a \tan^m(\lambda y)$.

12. $y_x^{(2n)} = a \cot^m(\lambda y)$.

This is a special case of equation 5.2.6.6 with $f(y) = a \cot^m(\lambda y)$.

13. $y_x^{(n)} = a x^{-n} \cos^m(\lambda y)$.

This is a special case of equation 5.2.6.10 with $f(y) = a \cos^m(\lambda y)$.

14. $y_x^{(n)} = a x^{-n} \sin^m(\lambda y)$.

This is a special case of equation 5.2.6.10 with $f(y) = a \sin^m(\lambda y)$.

15. $y_x^{(n)} = a x^{-n} \tan^m(\lambda y)$.

This is a special case of equation 5.2.6.10 with $f(y) = a \tan^m(\lambda y)$.

16. $y_x^{(n)} = a x^{-n} \cot^m(\lambda y)$.

This is a special case of equation 5.2.6.10 with $f(y) = a \cot^m(\lambda y)$.

17. $y y_x^{(2n+1)} = a \cos^m(\lambda x)$.

This is a special case of equation 5.2.6.16 with $f(x) = a \cos^m(\lambda x)$.

18. $y y_x^{(2n+1)} = a \sin^m(\lambda x)$.

This is a special case of equation 5.2.6.16 with $f(x) = a \sin^m(\lambda x)$.

19. $y y_x^{(2n+1)} = a \tan^m(\lambda x)$.

This is a special case of equation 5.2.6.16 with $f(x) = a \tan^m(\lambda x)$.

20. $y y_x^{(2n+1)} = a \cot^m(\lambda x)$.

This is a special case of equation 5.2.6.16 with $f(x) = a \cot^m(\lambda x)$.

21. $(2n-1) y y_x^{(2n+1)} + (2n+1) y'_x y_x^{(2n)} = a \cos^m(\lambda x)$.

This is a special case of equation 5.2.6.22 with $f(x) = a \cos^m(\lambda x)$.

22. $(2n-1) y y_x^{(2n+1)} + (2n+1) y'_x y_x^{(2n)} = a \sin^m(\lambda x)$.

This is a special case of equation 5.2.6.22 with $f(x) = a \sin^m(\lambda x)$.

23. $(2n-1) y y_x^{(2n+1)} + (2n+1) y'_x y_x^{(2n)} = a \tan^m(\lambda x)$.

This is a special case of equation 5.2.6.22 with $f(x) = a \tan^m(\lambda x)$.

24. $(2n-1) y y_x^{(2n+1)} + (2n+1) y'_x y_x^{(2n)} = a \cot^m(\lambda x)$.

This is a special case of equation 5.2.6.22 with $f(x) = a \cot^m(\lambda x)$.

25. $\boldsymbol{y_x^{(n+1)} = a\cos^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m}$.

This is a special case of equation 5.2.6.17 with $f(y) = \cos^{-k}(\lambda y)$, $g(w) = aw^m$.

26. $\boldsymbol{y_x^{(n+1)} = a\sin^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m}$.

This is a special case of equation 5.2.6.17 with $f(y) = \sin^{-k}(\lambda y)$, $g(w) = aw^m$.

27. $\boldsymbol{y_x^{(n+1)} = a\tan^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m}$.

This is a special case of equation 5.2.6.17 with $f(y) = \tan^{-k}(\lambda y)$, $g(w) = aw^m$.

28. $\boldsymbol{y_x^{(n+1)} = a\cot^k(\lambda y)\, y'_x \big(y_x^{(n)}\big)^m}$.

This is a special case of equation 5.2.6.17 with $f(y) = \cot^{-k}(\lambda y)$, $g(w) = aw^m$.

5.2.6. Equations Containing Arbitrary Functions

1. $\boldsymbol{yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = f(x)}$.

Solution:

$$y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \frac{1}{12}\int_{x_0}^{x} (x-t)^4 f(t)\,dt,$$

where x_0 is an arbitrary number.

2. $\boldsymbol{yy_x^{(5)} + ay'_x y''''_{xxxx} + (3a-5)y''_{xx} y'''_{xxx} = f(x)}$.

Integrating the equation three times, we obtain

$$yy''_{xx} + \frac{a-3}{2}(y'_x)^2 = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f(t)\,dt,$$

where x_0 is an arbitrary number.

3. $\boldsymbol{(a+y)y_x^{(5)} + by'_x y''''_{xxxx} + cy''_{xx} y'''_{xxx} = f(x)}$.

Integrating, we obtain

$$(a+y)y''''_{xxxx} + (b-1)y'_x y'''_{xxx} + \frac{1}{2}(1-b+c)\left(y''_{xx}\right)^2 = \int f(x)\,dx + C.$$

4. $\boldsymbol{yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = f(x)}$.

Solution: $\quad y^2 = C_5x^5 + C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \dfrac{1}{60}\displaystyle\int_{x_0}^{x} (x-t)^5 f(t)\,dt.$

5. $\boldsymbol{y_x^{(6)} = (ax^2+bx+c)^{-7/2} f\big((ax^2+bx+c)^{-5/2}\big)}$.

This is a special case of equation 5.2.6.21 with $n = 6$.

6. $\boldsymbol{y_x^{(2n)} = f(y)}$.

The first integral of the equation is

$$\sum_{m=1}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n \big[y_x^{(n)}\big]^2 + \int f(y)\,dy = C.$$

Next, the order of the obtained equation can be lowered by the substitution $w(y) = y'_x$.

7. $\boldsymbol{y_x^{(n)} = f\big(y_x^{(n-1)}\big)}$.

Having set $u(x) = y_x^{(n-1)}$, we obtain $u'_x = f(u)$. Further, find u from the relation $x = \displaystyle\int \frac{du}{f(u)} + C_1$. Then, the $(n-1)$-fold integration yields y.

The solution in the parametric form is written as

$$x = \int_{C_1}^{u} \frac{du}{f(u)}, \quad y = \int_{C_2}^{u} \frac{du_1}{f(u_1)} \int_{C_3}^{u_1} \frac{du_2}{f(u_2)} \dots \int_{C_{n-1}}^{u_{n-3}} \frac{du_{n-2}}{f(u_{n-2})} \int_{C_n}^{u_{n-2}} \frac{u_{n-1}\,du_{n-1}}{f(u_{n-1})}.$$

8. $\boldsymbol{y_x^{(n)} = f\big(y_x^{(n-2)}\big)}$.

Setting $u(x) = y_x^{(n-2)}$, we obtain the equation $u''_{xx} = f(u)$ whose solution has the form

$$x = \int \frac{du}{\varphi(u)} + C_2, \qquad \text{where} \quad \varphi(u) = \pm\Big[C_1 + 2\int f(u)\,du\Big]^{1/2}.$$

Expressing u in terms of x and integrating the resulting relation $(n-2)$ times, we find y.

The solution in the parametric form is written as

$$x = \int_{C_2}^{u} \frac{du}{\varphi(u)}, \quad y = \int_{C_3}^{u} \frac{du_1}{\varphi(u_1)} \int_{C_4}^{u_1} \frac{du_2}{\varphi(u_2)} \dots \int_{C_{n-1}}^{u_{n-3}} \frac{du_{n-2}}{\varphi(u_{n-2})} \int_{C_n}^{u_{n-3}} \frac{u_{n-2}\,du_{n-2}}{\varphi(u_{n-2})}.$$

9. $\boldsymbol{y_x^{(n)} = f(y + ax^m), \qquad m = 0,\ 1,\ 2,\ \dots,\ n-1}$.

The substitution $w = y + ax^m$ lead to an autonomous equation: $w_x^{(n)} = f(w)$, which, for even n, admits lowering of its order by two (see 5.2.6.6).

10. $\boldsymbol{y_x^{(n)} = x^{-n} f(y)}$.

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 5.2.6.40.

11. $\boldsymbol{y_x^{(n)} = x^{-n-1} f(x^{1-n} y)}$.

The transformation $x = t^{-1}$, $y = t^{1-n} w$ leads to an autonomous equation: $w_t^{(n)} = (-1)^n f(w)$, whose order, for even n, can be lowered by two (see 5.2.6.6).

12. $\boldsymbol{y_x^{(2n)} = x^{-\frac{2n+1}{2}} f\Big(x^{\frac{1-2n}{2}} y\Big)}$.

The transformation $x = e^t$, $y = x^{\frac{2n-1}{2}} w(t)$ leads to an autonomous equation of the form 5.2.6.25, whose order can be lowered by two.

13. $\boldsymbol{y_x^{(n)} = x^{-n-k} f(yx^k)}$.

This is a special case of equation 5.2.6.44.

The transformation $t = \ln x$, $w = x^k y$ leads to an autonomous equation of the form 5.2.6.40.

14. $\boldsymbol{y_x^{(n)} = yx^{-n} f(x^k y^m)}$.

The transformation $t = x^k y^m$, $w = \dfrac{xy'_x}{y}$ leads to an $(n-1)$ th order equation.

15. $\boldsymbol{y_x^{(n)} = yx^{-n} f\left(\dfrac{xy'_x}{y}\right)}$.

The transformation $z = \dfrac{xy'_x}{y}$, $w = \dfrac{x^2 y''_{xx}}{y}$ leads to an $(n-2)$ th order equation.

16. $\boldsymbol{yy_x^{(2n+1)} = f(x)}$.

Having integrated the equation, we obtain

$$2\sum_{m=0}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \left[y_x^{(n)}\right]^2 = 2\int f(x)\,dx + C,$$

where the notation $y_x^{(0)} \equiv y$ is used.

17. $\boldsymbol{f(y) y_x^{(n+1)} = y'_x g(y_x^{(n)})}$.

Having integrated the equation, we obtain

$$\int \frac{dw}{g(w)} = \int \frac{dy}{f(y)} + C, \qquad \text{where} \quad w = y_x^{(n)}.$$

Next, the order of this equation can be lowered by the substitution $z(y) = y'_x$.

18. $\boldsymbol{y_x^{(n)} = f(x,\ y)}$.

The transformation $x = t^{-1}$, $y = t^{1-n} w(t)$ leads to an equation of the similar form: $w_t^{(n)} = (-1)^n t^{-n-1} f(t^{-1},\ t^{1-n} w)$.

19. $\boldsymbol{y_x^{(n)} = f(x,\ y_x^{(n-2)},\ y_x^{(n-1)})}$.

The substitution $w(x) = y_x^{(n-2)}$ leads to a second order equation: $w''_{xx} = f(x,\ w,\ w'_x)$.

20. $\boldsymbol{(ax+b)^n (cx+d) y_x^{(n)} = f\left(\dfrac{y}{(cx+d)^{n-1}}\right)}$.

The transformation $\xi = \ln \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{n-1}}$ leads to an autonomous equation of the form 5.2.6.40.

21. $\boldsymbol{y_x^{(n)} = (ax^2+bx+c)^{-\frac{1+n}{2}} f\Big(y(ax^2+bx+c)^{\frac{1-n}{2}}\Big).}$

1°. The transformation

$$t = \int \frac{dx}{ax^2+bx+c}, \qquad w = y(ax^2+bx+c)^{\frac{1-n}{2}} \tag{1}$$

leads to an autonomous equation with respect to $w = w(t)$, which admits lowering of its order by the substitution $z(w) = w'_t$.

2°. Let $n = 2m$ be an even integer ($m = 1, 2, 3, \dots$). In this case, transformation (1) yields an equation of the form 5.2.6.25, whose order can be lowered by two.

Setting $P = ax^2+bx+c$, $y = wP^{\frac{2m-1}{2}}$ and multiplying both sides of the original equation by $w'_x = P^{-\frac{1+2m}{2}}\Big(Py'_x + \dfrac{1-2m}{2}P'_x y\Big)$, we obtain

$$\Big(Py'_x + \frac{1-2m}{2}P'_x y\Big) y_x^{(2m)} = f(w)w'_x.$$

Integrating both sides of this equality with respect to x (the left-hand side is integrated by parts), we have

$$\sum_{k=0}^{m-2}(-1)^k \psi_x^{(k)} y_x^{(2m-1-k)} + (-1)^{m-1}\int \psi_x^{(m-1)} y_x^{(m+1)}\,dx = \int f(w)\,dw + C, \tag{2}$$

where

$$\psi_x^{(k)} = \frac{d^k}{dx^k}\Big(Py'_x + \frac{1-2m}{2}P'_x y\Big) = Py_x^{(k+1)} + \Big(k-m+\frac{1}{2}\Big)P'_x y_x^{(k)} + ak(k-2m)y_x^{(k-1)}.$$

(remind that $n = 2m$). It can be shown that the integrand on the left-hand side of (2) is the total differential. Finally, we arrive at the first integral

$$\begin{aligned}
&\sum_{k=0}^{m-2}(-1)^k\Big[Py_x^{(k+1)} + \Big(k-m+\frac{1}{2}\Big)P'_x y_x^{(k)} + ak(k-2m)y_x^{(k-1)}\Big] y_x^{(2m-1-k)} \\
&\quad + (-1)^{m-1}\bigg\{\frac{1}{2}P\big[y_x^{(m)}\big]^2 - \frac{1}{2}P'_x y_x^{(m-1)} y_x^{(m)} \\
&\quad + a(1-m^2)y_x^{(m-2)} y_x^{(m)} + \frac{am^2}{2}\big[y_x^{(m-1)}\big]^2\bigg\} \\
&\quad = \int f(w)\,dw + C.
\end{aligned}$$

22. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = f(x).}$

Having integrated the equation, we have

$$(2n-1)yy_x^{(2n)} + 2\sum_{i=1}^{n-1}(-1)^{i+1}y_x^{(i)}y_x^{(2n-i)} + (-1)^{n+1}\big[y_x^{(n)}\big]^2 = \int f(x)\,dx + 2C_2.$$

The second integration yields

$$\sum_{i=0}^{n-1}(2n-1-2i)(-1)^i y_x^{(i)} y_x^{(2n-1-i)} = 2C_2 x + C_1 + \int_{x_0}^{x}(x-t)f(t)\,dt.$$

The third integration leads to a $(2n-2)$ th order equation:

$$\begin{aligned}
&\sum_{i=0}^{n-2}(i+1)(2n-i-1)(-1)^i y_x^{(i)} y_x^{(2n-2-i)} + \frac{1}{2}(-1)^{n-1}n^2\big[y_x^{(n-1)}\big]^2 \\
&\qquad = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x}(x-t)^2 f(t)\,dt.
\end{aligned}$$

23. $\boldsymbol{x y_x^{(n)} + n y_x^{(n-1)} = f(xy).}$

The substitution $w(x) = xy$ leads to the autonomous equation $w_x^{(n)} = f(w)$ (see 5.2.6.6 and 5.2.6.40).

24. $\boldsymbol{x^2 y_x^{(n)} + 2nx y_x^{(n-1)} + n(n-1) y_x^{(n-2)} = f(x^2 y).}$

The substitution $w(x) = x^2 y$ leads to the autonomous equation $w_x^{(n)} = f(w)$ (see 5.2.6.6 and 5.2.6.40).

25. $\boldsymbol{\sum_{m=1}^{n} a_m y_x^{(2m)} = f(y).}$

The first integral has the form

$$\sum_{m=1}^{n} a_m \left\{ \sum_{\nu=1}^{m-1} (-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + \frac{1}{2}(-1)^m \left[y_x^{(m)}\right]^2 \right\} + \int f(y)\,dy = C,$$

where C is an arbitrary constant. Further, the order of the obtained equation next be lowered by the substitution $w(y) = y'_x$.

26. $\boldsymbol{\sum_{m=1}^{n} a_m x^m y_x^{(m)} = f(y).}$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 5.2.6.40.

27. $\boldsymbol{y \sum_{m=0}^{n} a_m y_x^{(2m+1)} = f(x).}$

Having integrated the equation, we obtain

$$\sum_{m=0}^{n} a_m \left\{ 2\sum_{\nu=0}^{m-1} (-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m \left[y_x^{(m)}\right]^2 \right\} = 2\int f(x)\,dx + C,$$

where $y_x^{(0)}$ stands for y.

28. $\boldsymbol{\sum_{m=0}^{n} a_m y_x^{(m)} y_x^{(2n+1-m)} = f(x).}$

The first integral has the form

$$2\sum_{m=0}^{n-1} A_m y_x^{(m)} y_x^{(2n-m)} + A_n \left[y_x^{(n)}\right]^2 = 2\int f(x)\,dx + C,$$

where

$$A_m = \sum_{k=0}^{m} (-1)^{m+k} a_k = a_m - a_{m-1} + a_{m-2} - a_{m-3} + \cdots.$$

If the condition $A_n = 2\sum_{m=0}^{n-1} (-1)^{n-1+m} A_m$ is satisfied, the obtained equation can be integrated two times more (see, in particular, equation 5.2.6.22).

29. $\boldsymbol{y_x^{(n)} = e^{\alpha x} f(y e^{-\alpha x})}$.

The substitution $w(x) = y e^{-\alpha x}$ leads to an autonomous equation of the form 5.2.6.40.

30. $\boldsymbol{y_x^{(n)} = y f(e^{\alpha x} y^m)}$.

The transformation $z = e^{\alpha x} y^m$, $w(z) = y'_x / y$ leads to an $(n-1)$ th order equation.

31. $\boldsymbol{y_x^{(n)} = x^{-n} f(x^m e^{\alpha y})}$.

The transformation $z = x^m e^{\alpha y}$, $w(z) = x y'_x$ leads to an $(n-1)$ th order equation.

32. $\boldsymbol{y_x^{(n)} = f(y + a e^{\lambda x}) - a \lambda^n e^{\lambda x}}$.

The substitution $w(x) = y + a e^{\lambda x}$ leads to an autonomous equation: $w_x^{(n)} = f(w)$ (see 5.2.6.6 and 5.2.6.40).

33. $\boldsymbol{y_x^{(2n)} = f(y + a \cosh x) - a \cosh x}$.

The substitution $w(x) = y + a \cosh x$ leads to an autonomous equation: $w_x^{(2n)} = f(w)$ (see 5.2.6.6).

34. $\boldsymbol{y_x^{(2n)} = f(y + a \sinh x) - a \sinh x}$.

The substitution $w(x) = y + a \sinh x$ leads to an autonomous equation of the form 5.2.6.6: $w_x^{(2n)} = f(w)$.

35. $\boldsymbol{y_x^{(2n+1)} = f(y + a \cosh x) - a \sinh x}$.

The substitution $w(x) = y + a \cosh x$ leads to an autonomous equation of the form 5.2.6.40: $w_x^{(2n+1)} = f(w)$.

36. $\boldsymbol{y_x^{(2n+1)} = f(y + a \sinh x) - a \cosh x}$.

The substitution $w(x) = y + a \sinh x$ leads to an autonomous equation of the form 5.2.6.40: $w_x^{(2n+1)} = f(w)$.

37. $\boldsymbol{y_x^{(n)} = f(y + a \cos x) - a \cos\left(x + \frac{\pi n}{2}\right)}$.

The substitution $w(x) = y + a \cos x$ leads to an autonomous equation: $w_x^{(n)} = f(w)$ (see 5.2.6.6 and 5.2.6.40).

38. $\boldsymbol{y_x^{(n)} = f(y + a \sin x) - a \sin\left(x + \frac{\pi n}{2}\right)}$.

The substitution $w(x) = y + a \sin x$ leads to an autonomous equation: $w_x^{(n)} = f(w)$ (see 5.2.6.6 and 5.2.6.40).

39. $\boldsymbol{F\left(x,\ y'_x,\ y''_{xx},\ \ldots,\ y_x^{(n)}\right) = 0}$.

The substitution $w(x) = y'_x$ leads to an $(n-1)$ th order equation:

$$F\left(x,\ w,\ w'_x,\ \ldots,\ w_x^{(n-1)}\right) = 0.$$

40. $\boldsymbol{F\big(y,\ y'_x,\ y''_{xx},\ \ldots,\ y_x^{(n)}\big) = 0.}$

Autonomous equation.

The substitution $w(y) = y'_x$ leads to an $(n-1)$ th order equation. The derivatives of the original equation and the transformed one are related by the formulae

$$y''_{xx} = ww'_y, \quad y'''_{xxx} = w^2 w''_{yy} + w(w'_y)^2, \quad \ldots, \quad y_x^{(n)} = w\big(y_x^{(n-1)}\big)'_y.$$

41. $\boldsymbol{xy'_x - y = F\big(x,\ y''_{xx},\ y'''_{xxx},\ \ldots,\ y_x^{(n)}\big).}$

The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$ th order equation:

$$w = F\left(x,\ \frac{w'_x}{x},\ \frac{d}{dx}\left(\frac{w'_x}{x}\right), \ldots,\ \frac{d^{n-2}}{dx^{n-2}}\left(\frac{w'_x}{x}\right)\right) = 0.$$

42. $\boldsymbol{x^2 y''_{xx} - 2xy'_x + 2y = F\big(x,\ y'''_{xxx},\ \ldots,\ y_x^{(n)}\big) = 0.}$

The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$ th order equation:

$$w = F\left(x,\ \frac{w'_x}{x^2},\ \ldots,\ \frac{d^{n-3}}{dx^{n-3}}\left(\frac{w'_x}{x^2}\right)\right) = 0.$$

43. $\boldsymbol{\sum_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)} = F\big(x,\ y_x^{(m+1)},\ \ldots,\ y_x^{(n)}\big),}$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

The substitution $w(x) = \sum_{k=0}^{m} (-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$ th order equation; the derivatives on the right-hand side are calculated in consecutive manner using the formula $y_x^{(m+1)} = x^{-m} w'_x$.

44. $\boldsymbol{F\big(x^k y,\ x^{k+1} y'_x,\ \ldots,\ x^{k+n} y_x^{(n)}\big) = 0.}$

Homogeneous equation in the extended sense.

The transformation $t = \ln x$, $w = x^k y$ leads to an autonomous equation of the form 5.2.6.40.

45. $\boldsymbol{F\left(\frac{xy'_x}{y},\ \frac{x^2 y''_{xx}}{y},\ \ldots,\ \frac{x^n y_x^{(n)}}{y}\right) = 0.}$

Homogeneous equation in the extended sense.

The transformation $z = \dfrac{xy'_x}{y}$, $w = \dfrac{x^2 y''_{xx}}{y}$ leads to an $(n-2)$ th order equation.

46. $\boldsymbol{F\left(x^k y^m,\ \frac{xy'_x}{y},\ \frac{x^2 y''_{xx}}{y},\ \ldots,\ \frac{x^n y_x^{(n)}}{y}\right) = 0.}$

Homogeneous equation in the extended sense.

The transformation $t = x^k y^m$, $z = \dfrac{xy'_x}{y}$ leads to an $(n-1)$ th order equation.

47. $\boldsymbol{F\big(e^{\alpha x}y,\ e^{\alpha x}y'_x,\ e^{\alpha x}y''_{xx},\ \ldots,\ e^{\alpha x}y^{(n)}_x\big) = 0.}$

Exponential homogeneous equation.

The substitution $w(x) = e^{\alpha x}y$ leads to an autonomous equation of the form 5.2.6.40.

48. $\boldsymbol{F\left(e^{\alpha x}y^m,\ \frac{y'_x}{y},\ \frac{y''_{xx}}{y},\ \ldots,\ \frac{y^{(n)}_x}{y}\right) = 0.}$

Exponential homogeneous equation.

The transformation $z = e^{\alpha x}y^m$, $w = \dfrac{y'_x}{y}$ leads to an $(n-1)$ th order equation.

49. $\boldsymbol{F\big(x^m e^{\alpha y},\ xy'_x,\ x^2y''_{xx},\ \ldots,\ x^n y^{(n)}_x\big) = 0.}$

Exponential homogeneous equation.

The transformation $z = x^m e^{\alpha y}$, $w = xy'_x$ leads to an $(n-1)$ th order equation.