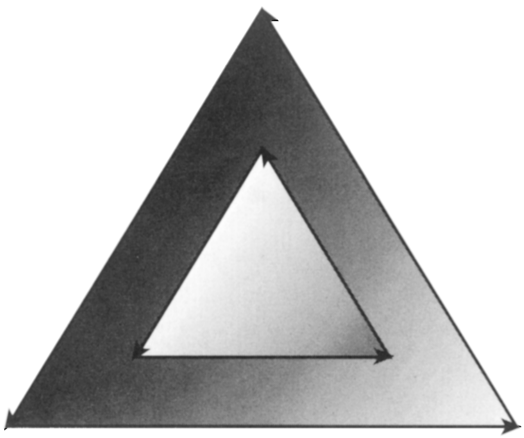
Handbook of Exact Solutions for Ordinary Differential Equations



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FOREWORD

Exact solutions have always played and still play an important role in properly understanding the qualitative features of many phenomena and processes in various fields of natural science.

Equations of applied and theoretical physics often contain parameters or functions which are found experimentally and therefore are not stringently fixed. At the same time, equations that model real phenomena and processes must be sufficiently simple to make possible their analysis and solution. It is natural to adopt, as one of feasible criterions of simplicity, the requirement that the model equation admits a solution in a closed form.

It should be noted that even exact solutions of nonlinear equations (including those without a clear physical sense and which do not correspond to real phenomena and processes) play an important role of "test" problems for verifying the correctness and assessment of accuracy of various numerical, asymptotic, and approximate methods. Moreover, the model equations and problems admitting exact solutions serve as the basis for the development of new numerical, asymptotic, and approximate methods, which, in turn, enable us to study more complicated problems having no analytical solution.

This book contains nearly 5000 ordinary differential equations and their solutions. The total number of linear and nonlinear equations is several times greater than those found in any other text. The table below compares data presented in this book with those of currently available handbooks concerning the general number of concrete second- and higher-order nonlinear ordinary differential equations analyzed.

The order of equations	E. Kamke (1976)	M. Murphy (1960)	This book
Second order	249	315	1228
Third order	13	22	587
Fourth order	3	3	75
Higher order	3	9	160
Total number of equations	268	349	2045

When selecting the material, the authors gave preference to the following two types of equations:

1. Equations that traditionally attracted the attention of many researchers: those of the simplest appearance but involving the most difficulties for integration (Abel equations, Emden—Fowler equations, Painlevé equations, etc.).
2. Equations that encountered in various applications (in the theory of heat and mass transfer, nonlinear mechanics, hydrodynamics, the theory of nonlinear oscillations, the theory of combustion, chemical engineering science, etc.).

Special attention is paid to equations containing arbitrary functions. The other equations contain one or more arbitrary parameters (i.e., actually, this book deals with whole *families* of ordinary differential equations) which can be fixed by a reader at will. Many solutions have been obtained just recently with the aid of new (discrete group) methods described in other books by the authors (1993, 1994).

When compiling this book, the handbooks by E. Kamke (1976), M. Murphy (1960), and D. Zwillinger (1989) were partly used in which one can find basic notions and definitions of the theory of ordinary differential equations, apart from concrete equations. In these handbooks, classical and some new methods of solving differential equations are described as well—see also the books by E.L. Ince (1964), P.J. Olver (1986), and N.H. Ibragimov

(1993). The latter books give a great number of references to the original papers and books by other authors, which are devoted to exact solutions and methods of the theory of ordinary differential equations.

In addition, when describing solutions of linear ordinary differential equations, which are connected to higher transcendental functions (Bessel, Legendre, Mathieu, hypergeometric, etc.), the handbooks by G. Beitmen and A. Erdeii (1953–1955), M. Abramowitz and I.A. Stegun (1964) were used.

In some sections of this book, asymptotic solutions of some classical equations of nonlinear mechanics and theoretical physics are also given, which are discussed in the books by J.D. Cole (1968), M.V. Fedoryuk (1983), and A.H. Nayfeh (1973, 1971) in detail.

The detailed table of contents enables a reader to quickly navigate through this book in searching for desired equations.

The authors hope that this book will be helpful for a wide range of scientists, lectures, engineers, and students engaged in the fields of mathematics, physics, mechanics, and chemical engineering science.

Andrei D. Polyanin
Valentin F. Zaitsev

Some Remarks and Notation

1. In this book, in the original equations the independent variable is denoted by x, and the dependent one is denoted by y. In the given solutions, the symbols C, C_0, C_1, C_2, ... stand for arbitrary integration constants.

2. The following notation is used for derivatives: $y'_x = \dfrac{dy}{dx}$, $y''_{xx} = \dfrac{d^2y}{dx^2}$, $y'''_{xxx} = \dfrac{d^3y}{dx^3}$, $y''''_{xxxx} = \dfrac{d^4y}{dx^4}$, and $y_x^{(n)} = \dfrac{d^ny}{dx^n}$ with $n \geq 5$.

3. In some cases, we use the operator notation $\left(f\dfrac{d}{dx}\right)^n g$ which is defined by the recurrence relation

$$\left(f(x)\frac{d}{dx}\right)^n g(x) = f(x)\frac{d}{dx}\left[\left(f(x)\frac{d}{dx}\right)^{n-1} g(x)\right].$$

4. In some sections of the book (see, for example, 1.3, 2.3–2.6, 3.2–3.4), for the sake of brevity, solutions are represented as several formulae containing the terms with the signs "$\pm$" and "$\mp$." By this is meant two formulae—one correspond to the upper signs, and another corresponds to the lower signs. For example, the solution of equation 1.3.1.6 can be written in the parametric form

$$x = af^{-1}\exp(\mp\tau^2),\quad y = af^{-1}\bigl[\exp(\mp\tau^2) \pm 2\tau f\bigr],\quad \text{where } f = \int \exp(\mp\tau^2)\,d\tau - C,\quad A = \mp 2a^2.$$

This is equivalent to that the solutions of equation 1.3.1.6 are given by the formulae

$$x = af^{-1}\exp(-\tau^2),\quad y = af^{-1}\bigl[\exp(-\tau^2) + 2\tau f\bigr],\quad \text{where } f = \int \exp(-\tau^2)\,d\tau - C,\quad A = -2a^2$$

and

$$x = af^{-1}\exp(\tau^2),\quad y = af^{-1}\bigl[\exp(\tau^2) - 2\tau f\bigr],\quad \text{where } f = \int \exp(\tau^2)\,d\tau - C,\quad A = 2a^2.$$

5. When referencing to a particular equation, the notation like "4.1.2.5" stands for "equation 5 in Subsection 4.1.2."

6. The book includes two supplements that provide a reader with useful information on some elementary and special functions which appear in solutions of the differential equation outlined.

7. References that may be helpful for a reader are given at the end of the book.

THE AUTHORS

Andrei D. Polyanin, Ph.D., D.Sc., is a noted scientist in the fields of ordinary differential equations, engineering and applied mathematics, heat and mass transfer, nonlinear mechanics, and chemical engineering science.

Professor Polyanin graduated from the Faculty of Mechanics and Mathematics of the Moscow State University in 1974 and received his Candidate of Sciences (Ph.D.) degree in 1981 (at the Institute for Problems in Mechanics of the U.S.S.R. Academy of Sciences). His Ph.D. thesis was devoted to the asymptotic analysis of the problems of heat and mass transfer. In 1986, Professor Polyanin received his Doctor of Sciences degree; his D.Sc. thesis was dedicated to the mass and heat exchange between reacting particles and flow.

Since 1975, Professor Polyanin is a member of staff of the Institute for Problems in Mechanics of the Russian Academy of Sciences.

Professor Polyanin has made important contributions to new approximate analitical methods in the theory of heat and mass transfer, hydrodynamics, and chemical engineering science, as well as to new methods of the theory of ordinary differential equations. In 1991 he was awarded a Chaplygin Prize of the U.S.S.R. Academy of Sciences for his research in mechanics.

Professor Polyanin has published more than 100 research papers and seven books. He is also an author of three patents.

Valentin F. Zaitsev, Ph.D., D.Sc., is a noted scientist in the fields of ordinary differential equations, mathematical physics, and nonlinear mechanics.

Professor Zaitsev graduated from the Radio Electronics Faculty of the Leningrad Politechnical Institute (now Saint-Petersburg Technical University) in 1969 and received his Candidate of Sciences (Ph.D.) degree in 1983 (at the Leningrad State University). His Ph.D. thesis was devoted to the group approach to the study of some classes of ordinary differential equations. In 1992, Professor Zaitsev received his Doctor of Sciences degree; his D.Sc. thesis was dedicated to the discrete-group analysis of the ordinary differential equations.

Since 1971, Professor Zaitsev is in the Research Institute for Computational Mathematics and Control Processes of the St.-Petersburg State University. He is also a Professor at the Russian State Pedagogical University (St.-Petersburg), the Orel State Pedagogical Institute, and Orel State Politechnical Institute.

Professor Zaitsev has made important contributions to new methods in the theory of ordinary differential equations. He is an author of more than 70 scientific publications, including five monographs and the patent.

Zaitsev V.F. also read the theoretical course at the Leningrad Conservatory, later participating in development of mathematical methods in musical sciences.

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