Chapter 3
Third Order Differential Equations

3.1. Linear Equations

3.1.1. Preliminary Comments

1. A linear homogeneous equation of the third order has the form

$$f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution of this equation. The substitution

$$y = y_0(x) \int z(x)\,dx$$

yields a linear equation of the second order:

$$f_3 y_0 z'' + (3f_3 y_0' + f_2 y_0)z' + (3f_3 y_0'' + 2f_2 y_0' + f_1 y_0)z = 0, \tag{2}$$

where prime denotes differentiation with respect to x.

2. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly-independent particular solutions of equation (1). Then, the general solution of this equation can be written in the form

$$y = C_1 y_1 + C_2 y_2 + C_3\Bigl(y_2 \int y_1 \psi\,dx - y_1 \int y_2 \psi\,dx\Bigr), \tag{3}$$

where

$$\psi = \exp\Bigl(-\int \frac{f_2}{f_3}\,dx\Bigr)\bigl(y_1 y_2' - y_1' y_2\bigr)^{-2}.$$

For specific equations described below in 3.1.2–3.1.8, often only particular solutions will be given, while the generalized solution can be obtained with formula (3).

3. A linear nonhomogeneous equation of the third order has the form:

$$f_3(x)y'''_{xxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x). \tag{4}$$

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two linearly-independent particular solutions of the corresponding homogeneous equation (1). Then, the general solution of equation (4) is defined by formula (3) with

$$\psi = \Delta^{-2} e^{-F}\Bigl(1 + \frac{1}{C_3}\int \frac{g}{f_3}\Delta e^F\,dx\Bigr), \qquad \text{where} \quad F = \int \frac{f_2}{f_3}\,dx, \quad \Delta = (y_1' y_2 - y_1 y_2').$$

4. The substitution

$$y = z\exp\Bigl(-\frac{1}{3}\int \frac{f_2}{f_3}\,dx\Bigr)$$

reduces equation (1) to the form wherein the second derivative is absent:

$$z''' + \Bigl(-\varphi_2' - \frac{1}{3}\varphi_2^2 + \varphi_1\Bigr)z' + \Bigl(-\frac{1}{3}\varphi_2'' - \frac{1}{3}\varphi_1\varphi_2 + \frac{2}{27}\varphi_2^3 + \varphi_0\Bigr)z = 0,$$

where $\varphi_k = f_k/f_3$ $(k = 0,\ 1,\ 2)$.

3.1.2. Equations Containing Power Functions

1. $\boldsymbol{y'''_{xxx} + \lambda y = 0}$.

Solution:

$$y = \begin{cases} C_1 + C_2 x + C_3 x^2 & \text{if } \lambda = 0, \\ C_1 e^{-kx} + e^{kx/2}\Bigl(C_2\cos\dfrac{kx\sqrt{3}}{2} + C_3\sin\dfrac{kx\sqrt{3}}{2}\Bigr) & \text{if } \lambda \neq 0, \end{cases}$$

where k is the real root of the equation $\lambda = k^3$.

2. $\boldsymbol{y'''_{xxx} + \lambda y = ax^2 + bx + c}$.

Solution: $y = w + \dfrac{1}{\lambda}(ax^2 + bx + c)$, where w is the general solution of the equation 3.1.2.1: $w'''_{xxx} + \lambda w = 0$.

3. $\boldsymbol{y'''_{xxx} = axy + b}$.

This is a special case of equation 5.1.2.4.

4. $\boldsymbol{y'''_{xxx} + (ax + b)y = 0}$.

For $a = 0$, this is an equation of the form 3.1.2.1.

For $a \neq 0$, the substitution $a\xi = ax + b$ leads to an equation of the form 3.1.2.3: $y'''_{\xi\xi\xi} + a\xi y = 0$.

5. $\boldsymbol{y'''_{xxx} + ax^3 y = bx}$.

The substitution $\xi = x^2$ leads to an equation of the form 3.1.2.90: $2\xi y'''_{\xi\xi\xi} + 3y''_{\xi\xi} + \frac{1}{4}a\xi y = \frac{1}{4}b$.

6. $\boldsymbol{y'''_{xxx} + (3a^2 x - a^3 x^3)y = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + axy'_x + (a^2x^2 - a)y = C\exp\Bigl(\frac{ax^2}{2}\Bigr)$$

(see 2.1.2.28 for the solution of the corresponding homogeneous equation).

7. $y'''_{xxx} = ax^n y.$

1°. For $n=-9,\ -7,\ -6,\ -9/2,\ -3,\ -3/2, 1$, and 3, see equations 3.1.2.183,3.1.2.180, 3.1.2.175, 3.1.2.185, 3.1.2.150, 3.1.2.184, 3.1.2.3, and 3.1.2.5, respectively.

2°. The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the similar form: $w'''_{xxx} = -at^{-n-6}w$.

3°. For $n \neq -3$, the transformation $\xi = x^{(n+3)/3}$, $u = x^{n/3}y$ leads to an equation of the form 3.1.2.151:

$$\xi^3 u'''_{\xi\xi\xi} + (1-\nu^2)\xi u'_\xi + (\nu^2 - 1 - a\nu^3\xi^3)u = 0, \qquad \text{where} \quad \nu = \frac{3}{n+3}.$$

8. $y'''_{xxx} + [a^3x^{3n} - 3a^2nx^{2n-1} + an(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^{n+1}}{n+1}\Bigr)$.

The substitution $y = \exp\Bigl(-\dfrac{ax^{n+1}}{n+1}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order equation of the form 2.1.2.44: $z''_{xx} - 3ax^n z'_x + (3a^2x^{2n} - 3anx^{n-1})z = 0$.

9. $y'''_{xxx} + aby'_x + a^2x(3 - b - ax^2)y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + axy'_x + (a^2x^2 + ab - a)y = C\exp\Bigl(\frac{ax^2}{2}\Bigr)$$

(see 2.1.2.28 for the solution of the corresponding homogeneous equation).

10. $y'''_{xxx} + axy'_x + any = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = w_x^{(n-1)}$, where w is the solution of the second order equation $w''_{xx} + axw = C$.

11. $y'''_{xxx} + axy'_x - 2ay = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.2: $w''_{xx} + axw = 0$.

12. $y'''_{xxx} + axy'_x + b(ax + b^2)y = 0.$

Particular solution: $y_0 = e^{-bx}$.

The substitution $w = y'_x + by$ leads to an equation of the form 2.1.2.12: $w''_{xx} - bw'_x + (ax + b^2)w = 0$.

13. $y'''_{xxx} + axy'_x + (abx + a + b^3)y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} - by'_x + (ax + b^2)y = Ce^{-bx}$$

(see 2.1.2.103 for the solution of the corresponding homogeneous equation).

14. $\boldsymbol{y'''_{xxx} + (ax+b)y'_x + ay = 0.}$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (ax+b)y = C$$

(see 2.1.2.7 for the solution of the corresponding homogeneous equation).

15. $\boldsymbol{y'''_{xxx} + (ax+b)y'_x - ay = 0.}$

Particular solution: $y_0 = ax + b$.

The transformation

$$\xi = ax + b, \quad z = \frac{y'_x}{ax+b} - \frac{ay}{(ax+b)^2}$$

leads to a second order equation of the form 2.1.2.62: $\xi z''_{\xi\xi} + 3z'_\xi + a^{-2}\xi^2 z = 0$.

16. $\boldsymbol{y'''_{xxx} + (ax+b)y'_x + 3ay = 0.}$

The substitution $a\xi = ax + b$ leads to an equation of the form 3.1.2.35 with $m = 0$: $y'''_{\xi\xi\xi} + a\xi y'_\xi + 3ay = 0$.

17. $\boldsymbol{y'''_{xxx} + (2ax+b)y'_x + ay = 0.}$

The substitution $a\xi = ax + \frac{1}{2}b$ leads to an equation of the form 3.1.2.37 with $n = 1$: $y'''_{\xi\xi\xi} + 2a\xi y'_\xi + ay = 0$.

18. $\boldsymbol{y'''_{xxx} + (ax-b^2)y'_x + abxy = 0.}$

The substitution $w = y'_x + by$ leads to an equation of the form 2.1.2.103: $w''_{xx} - bw'_x + axw = 0$.

19. $\boldsymbol{y'''_{xxx} + (ax-b^2)y'_x + a(bx+1)y = 0.}$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} - by'_x + axy = Ce^{-bx}$$

(see 2.1.2.103 for the solution of the corresponding homogeneous equation).

20. $\boldsymbol{y'''_{xxx} + (ax+b)y'_x + c(ax+b+c^2)y = 0.}$

The substitution $w = y'_x + cy$ leads to a second order linear equation of the form 2.1.2.12: $w''_{xx} - cw'_x + (ax+b+c^2)w = 0$.

21. $\boldsymbol{y'''_{xxx} + (ax+b)y'_x + cx(c^2x^2+ax+b-3c)y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{cx^2}{2}\Bigr)$.

The substitution $y = \exp\Bigl(-\dfrac{cx^2}{2}\Bigr)\displaystyle\int z(x)\,dx$ leads to an equation of the form 2.1.2.28: $z''_{xx} - 3cxz'_x + (3c^2x^2 + ax + b - 3c)z = 0$.

22. $\boldsymbol{y'''_{xxx} + ax^2 y'_x + axy = 0}$.

This is a special case of equation 3.1.2.36 with $n = 1$.

23. $\boldsymbol{y'''_{xxx} + ax^2 y'_x - 2axy = 0}$.

The substitution $w = xy'_x - 2y$ leads to a second order linear equation of the form 2.1.2.7: $w''_{xx} + ax^2 w = 0$.

24. $\boldsymbol{y'''_{xxx} - a^2x^2 y'_x + a^2xy = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + axy'_x - ay = C\exp\Bigl(\frac{ax^2}{2}\Bigr)$$

(see 2.1.2.103 for the solution of the corresponding homogeneous equation).

25. $\boldsymbol{y'''_{xxx} + ax^2 y'_x + b(ax^2 + b^2)y = 0}$.

The substitution $w = y'_x + by$ leads to an equation of the form 2.1.2.28: $w''_{xx} - bw'_x + (ax^2 + b^2)w = 0$.

26. $\boldsymbol{y'''_{xxx} + (a-1)b^2x^2 y'_x + b^2x(abx^2 + 2a + 1)y = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} - bxy'_x + (ab^2x^2 + b)y = C\exp\Bigl(-\frac{bx^2}{2}\Bigr)$$

(see 2.1.2.28 for the solution of the corresponding homogeneous equation).

27. $\boldsymbol{y'''_{xxx} + (ax^2 + b)y'_x + 2axy = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (ax^2 + b)y = C$$

(see 2.1.2.4 for the solution of the corresponding homogeneous equation).

28. $\boldsymbol{y'''_{xxx} + (ax^2 - b^2)y'_x + ax(2 - bx)y = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + by'_x + ax^2 y = Ce^{bx}$$

(see 2.1.2.13 for the solution of the corresponding homogeneous equation).

29. $\boldsymbol{y'''_{xxx} + (ax^2 + b)y'_x + c(ax^2 + b + c^2)y = 0}$.

The substitution $w = y'_x + cy$ leads to a second order equation of the form 2.1.2.13: $w''_{xx} - cw'_x + (ax^2 + b + c^2)w = 0$.

30. $\boldsymbol{y'''_{xxx} - (3b^2x^2 + a + 3b)y'_x + 2bx(b^2x^2 - a)y = 0.}$

1°. Particular solutions with $a > 0$:

$$y_1 = \exp\Bigl(\frac{bx^2}{2} + x\sqrt{a}\,\Bigr), \qquad y_2 = \exp\Bigl(\frac{bx^2}{2} - x\sqrt{a}\,\Bigr).$$

2°. Particular solutions with $a < 0$:

$$y_1 = \cos\bigl(x\sqrt{-a}\,\bigr)\exp\Bigl(\frac{bx^2}{2}\Bigr), \qquad y_2 = \sin\bigl(x\sqrt{-a}\,\bigr)\exp\Bigl(\frac{bx^2}{2}\Bigr).$$

3°. Particular solutions with $a = 0$:

$$y_1 = \exp\Bigl(\frac{bx^2}{2}\Bigr), \qquad y_2 = x\exp\Bigl(\frac{bx^2}{2}\Bigr).$$

31. $\boldsymbol{y'''_{xxx} + (ax^2 + bx + c)y'_x + kx[(a + k^2)x^2 + bx + c - 3k]y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{kx^2}{2}\Bigr)$.

The substitution $y = \exp\Bigl(-\dfrac{kx^2}{2}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order linear equation of the form 2.1.2.28:

$$z''_{xx} - 3kxz'_x + [(a + 3k^2)x^2 + bx + c - 3k]z = 0.$$

32. $\boldsymbol{y'''_{xxx} + (ax^4 + bx)y'_x - 2(ax^3 + b)y = 0.}$

This is a special case of equation 3.1.2.38 with $n = 2$.

33. $\boldsymbol{y'''_{xxx} + ax^n y'_x - 2ax^{n-1}y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.7: $w''_{xx} + ax^n w = 0$.

34. $\boldsymbol{y'''_{xxx} + ax^n y'_x + anx^{n-1}y = 0.}$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + ax^n y = C$$

(see 2.1.2.7 for the solution of the corresponding homogeneous equation).

35. $\boldsymbol{y'''_{xxx} + ax^{m+1}y'_x + a(m + 3)x^m y = 0.}$

The substitution $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 3.1.2.34 with $n = -m - 5$:

$$w'''_{ttt} + at^{-m-5}w'_t - a(m + 5)t^{-m-6}w = 0.$$

36. $\boldsymbol{y'''_{xxx} + ax^{2n}y'_x + anx^{2n-1}y = 0.}$

Solution:

$$y = C_1 x J^2_\nu(u) + C_2 x J_\nu(u)Y_\nu(u) + C_3 x Y^2_\nu(u),$$

where $\nu = \dfrac{1}{2(n + 1)}$, $u = \dfrac{\sqrt{a}}{2(n + 1)}x^{n+1}$; J_ν and Y_ν are Bessel functions.

37. $y'''_{xxx} + 2ax^n y'_x + anx^{n-1} y = 0.$

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 form a fundamental set of solutions of a second order equation of the form 2.1.2.7: $2w''_{xx} + ax^n w = 0$.

38. $y'''_{xxx} + (ax^{2n} + bx^{n-1})y'_x - 2(ax^{2n-1} + bx^{n-2})y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.10: $w''_{xx} + (ax^{2n} + bx^{n-1})w = 0$.

39. $y'''_{xxx} + (ax^{2n} + bx^{n-1})y'_x - c[(a+c^2)x^{3n} + (b+3cn)x^{2n-1} + n(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\left(\dfrac{cx^{n+1}}{n+1}\right)$.

The substitution $y = \exp\left(\dfrac{cx^{n+1}}{n+1}\right) \displaystyle\int z(x)\,dx$ leads to a second order linear equation of the form 2.1.2.44:

$$z''_{xx} + 3cx^n z'_x + [(a+3c^2)x^{2n} + (b+3cn)x^{n-1}]z = 0.$$

40. $y'''_{xxx} + 3ay''_{xx} + 3a^2 y'_x + a^3 y = 0.$

Solution: $y = e^{-ax}(C_1 + C_2 x + C_3 x^2)$.

41. $y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = 0.$

Denote $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.

1°. Let the characteristic polynomial $P(\lambda)$ be factorizable:

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

where λ_1, λ_2, and λ_3 are real numbers. The following cases may take place:

a) all the roots λ_k are different:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x}$$

b) $\lambda_1 = \lambda_2 \neq \lambda_3$:

$$y = (C_1 + C_2 x)e^{\lambda_1 x} + C_3 e^{\lambda_3 x}.$$

c) $\lambda_1 = \lambda_2 = \lambda_3$: see 3.1.2.40 for this case.

2°. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda^2 + 2b_1\lambda + b_0)$, where $b_1^2 < b_0$. Then

$$y = C_1 e^{\lambda_1 x} + e^{-b_1 x}(C_2 \cos\mu x + C_3 \sin\mu x), \quad \mu = \sqrt{b_0 - b_1^2}.$$

42. $y'''_{xxx} + ay''_{xx} + (bx + c)y'_x + (abx + ac + b)y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (bx + c)y = Ce^{-ax}$$

(see 2.1.2.2 for the solution of the corresponding homogeneous equation).

43. $y'''_{xxx} + 3ay''_{xx} + 2(bx + a^2)y'_x + b(2ax + 1)y = 0.$

This is a special case of equation 3.1.2.71 with $n = 0$, $m = 1$.

44. $y'''_{xxx} + ay''_{xx} + (bx^2 + cx + d)y'_x + a(bx^2 + cx + d)y = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.2.6: $w''_{xx} + (bx^2 + cx + d)w = 0$.

45. $y'''_{xxx} + ay''_{xx} + bx^n y'_x + abx^n y = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.2.7: $w''_{xx} + bx^n w = 0$.

46. $y'''_{xxx} + 3axy''_{xx} + 3a^2x^2y'_x + (a^3x^3 + b)y = 0.$

The substitution $y = w\exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$ leads to a constant coefficient equation: $w'''_{xxx} - 3aw'_x + bw = 0$.

47. $y'''_{xxx} + axy''_{xx} + (abx + a - b^2)y'_x + aby = 0.$

Particular solutions: $\quad y_1 = e^{-bx}, \quad y_2 = e^{-bx}\displaystyle\int \exp\Bigl(2bx - \frac{ax^2}{2}\Bigr)\,dx.$

48. $y'''_{xxx} + 3axy''_{xx} + (2a^2x^2 + a + b)y'_x + abxy = 0.$

Solution:

$$y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2,$$

where w_1 and w_2 are linearly-independent solutions of a second order equation of the form 2.1.2.25: $w''_{xx} + axw'_x + \frac{1}{4}bw = 0$.

49. $y'''_{xxx} + 3axy''_{xx} + (2a^2x^2 + 2bx + a)y'_x + b(2ax^2 + 1)y = 0.$

This is a special case of equation 3.1.2.71 with $n = 1$, $m = 1$.

50. $y'''_{xxx} + 3axy''_{xx} + 3(a^2x^2 + a)y'_x + (a^3x^3 + bx + c)y = 0.$

The substitution $y = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)w$ leads to an equation of the form 3.1.2.4: $w'''_{xxx} + [(b - 3a^2)x + c]w = 0$.

51. $y'''_{xxx} + 3axy''_{xx} + [2(a^2 + b)x^2 + a]y'_x + 2bx(ax^2 + 1)y = 0.$

This is a special case of equation 3.1.2.71 with $n = 1$, $m = 2$.

52. $y'''_{xxx} + (ax + b)y''_{xx} + (abx + a + c)y'_x + bcy = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + axy'_x + cy = Ce^{-bx}$$

(see 2.1.2.25 for the solution of the corresponding homogeneous equation).

53. $y'''_{xxx} + (abx + a + b)y''_{xx} + ab^2 xy'_x - ab^2 y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-bx}$.

54. $y'''_{xxx} + (ax + b)y''_{xx} + [(ab + c)x + a]y'_x + c(bx + 1)y = 0.$

By integrating, we obtain a nonhomogenous second-order linear equation:

$$y''_{xx} + axy'_x + cxy = Ce^{-bx}$$

(see 2.1.2.25 for the solution of the corresponding homogeneous equation).

55. $y'''_{xxx} + (ax + b + c)y''_{xx} + (acx + bc + s)y'_x + s(ax + b)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + c\lambda + s = 0$.

56. $y'''_{xxx} + (ax + b)y''_{xx} + (cx + 2a)y'_x + a[(c - ab)x^2 + b]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$.

The subbstitution $y = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order linear equation of the form 2.1.2.28:

$$z''_{xx} + (b - 2ax)z'_x + [a^2x^2 + (c - 2ab)x - a]z = 0.$$

57. $y'''_{xxx} + (ax + b)y''_{xx} + (cx + d)y'_x + [acx^2 + (ad + bc)x + c + bd]y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (cx + d)y = C\exp\Bigl(-\frac{ax^2}{2} - bx\Bigr)$$

(see 2.1.2.2 for the solution of the corresponding homogeneous equation).

58. $y'''_{xxx} + (ax + b)y''_{xx} + (\alpha x^2 + \beta x + \gamma)y'_x - k[\alpha x^2 + (ak + \beta)x + k^2 + bk + \gamma]y = 0.$

The substitution $w = y'_x - ky$ leads to a second order equation of the form 2.1.2.28:

$$w''_{xx} + (ax + b + k)w'_x + [\alpha x^2 + (ak + \beta)x + k^2 + bk + \gamma]w = 0.$$

59. $y'''_{xxx} - x^2 y''_{xx} + (a + b - 1)xy'_x - aby = 0.$

The following three series, converging for any x, make up a fundamental set of solutions:

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{ab(a-3)(b-3)\ldots(a-3n+3)(b-3n+3)}{(3n)!}x^{3n},$$

$$y_2 = x + \sum_{n=1}^{\infty} \frac{(a-1)(b-1)(a-4)(b-4)\ldots(a-3n+2)(b-3n+2)}{(3n+1)!}x^{3n+1},$$

$$y_3 = \frac{x^2}{2} + \sum_{n=1}^{\infty} \frac{(a-2)(b-2)(a-5)(b-5)\ldots(a-3n+1)(b-3n+1)}{(3n+2)!}x^{3n+2}.$$

60. $y'''_{xxx} + ax^n y''_{xx} - 2ax^{n-2}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.42: $w''_{xx} + ax^n w'_x + ax^{n-1}w = 0$.

61. $y'''_{xxx} + ax^n y''_{xx} - by'_x - abx^n y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp(-x\sqrt{b})$, $y_2 = \exp(x\sqrt{b})$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\bigr)$.

62. $y'''_{xxx} + ax^n y''_{xx} - 2ax^{n-1}y'_x + 2ax^{n-2}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

63. $y'''_{xxx} + ax^n y''_{xx} + bx^{n-1}y'_x - 2(a+b)x^{n-2}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.42: $w''_{xx} + ax^n w'_x + (a+b)x^{n-1}w = 0$.

64. $y'''_{xxx} + ax^n y''_{xx} - (ax^{n-1} - bx^2)y'_x + bx(ax^{n+1} + 3)y = 0.$

Particular solutions: $y_1 = \cos\left(\dfrac{x^2\sqrt{b}}{2}\right)$, $y_2 = \sin\left(\dfrac{x^2\sqrt{b}}{2}\right)$.

65. $y'''_{xxx} + ax^n y''_{xx} + (abx^n + anx^{n-1} - b^2)y'_x + abnx^{n-1}y = 0.$

Particular solutions: $y_1 = e^{-bx}$, $y_2 = e^{-bx}\displaystyle\int \exp\left(2bx - \frac{a}{n+1}x^{n+1}\right)dx$.

66. $y'''_{xxx} + ax^n y''_{xx} + bx^m y'_x - bx^{m-1}y = 0.$

Particular solution: $y_0 = x$.

67. $y'''_{xxx} + ax^n y''_{xx} + bx^m y'_x + bx^{m-1}(ax^{n+1} + m)y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first order linear equation: $w'_x + ax^n w = 0$.

68. $y'''_{xxx} + ax^n y''_{xx} - b(2ax^n + 3b)y'_x + b^2(ax^n + 2b)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

69. $y'''_{xxx} + ax^n y''_{xx} + (abx^n - b^2 + c)y'_x + c(ax^n - b)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

70. $y'''_{xxx} + ax^n y''_{xx} + (bx^m - c^2)y'_x - c(acx^n + bx^m)y = 0.$

Particular solution: $y_0 = e^{cx}$.

71. $y'''_{xxx} + 3ax^n y''_{xx} + (2a^2x^{2n} + 2bx^m + anx^{n-1})y'_x + b(2ax^{n+m} + mx^{m-1})y = 0.$

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 form a fundamental set of solutions of the second order linear equation $w''_{xx} + ax^n w'_x + \frac{1}{2}bx^m w = 0$.

72. $y'''_{xxx} = (x^n - a)y''_{xx} + (ax^n - b)y'_x + bx^n y.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

73. $y'''_{xxx} + (ax^n + b)y''_{xx} + (acx^n + bc + m)y'_x + (m + c^2)(ax^n + b - c)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + c\lambda + m + c^2 = 0$.

74. $y'''_{xxx} + (ax^n - b)y''_{xx} + cx^m y'_x - b(abx^n + cx^m)y = 0.$

Particular solution: $y_0 = e^{bx}$.

75. $y'''_{xxx} + (x^n + a)y''_{xx} + (ax^n + bx^m)y'_x + abx^m y = 0.$

Particular solution: $y_0 = e^{-ax}$.

76. $y'''_{xxx} + (ax^n + c)y''_{xx} + [acx^n + (an + b)x^{n-1})y'_x + b[cx^{n-1} + (n-1)x^{n-2}]y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + ax^n y'_x + bx^{n-1}y = Ce^{-cx}$$

(see 2.1.2.42 for the solution of the corresponding homogeneous equation).

77. $y'''_{xxx} + (ax^n + bx)y''_{xx} + b(ax^{n+1} + 2)y'_x + abx^n y = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{bx^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{bx^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\frac{bx^2}{2}\Bigr)\,dx$.

78. $y'''_{xxx} + (ax^n + bx)y''_{xx} + (abx^{n+1} + bcx + b - c^2)y'_x + c(abx^{n+1} - acx^n + b)y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $y_2 = e^{-cx}\displaystyle\int \exp\Bigl(2cx - \frac{bx^2}{2}\Bigr)\,dx$.

79. $y'''_{xxx} + (abx^n + ax^{n-1} + b)y''_{xx} + ab^2x^n y'_x - ab^2x^{n-1}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

80. $y'''_{xxx} + (ax^n + bx^m)y''_{xx} + cy'_x + c(ax^n + bx^m)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

81. $y'''_{xxx} + (ax^n + bx^m)y''_{xx} + (abx^{n+m} + bcx^m + bmx^{m-1} - c^2)y'_x$
$\qquad + c(abx^{n+m} - acx^n + bmx^{m-1})y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $y_2 = e^{-cx}\displaystyle\int \exp\Bigl(2cx - \frac{bx^{m+1}}{m+1}\Bigr)\,dx$.

82. $xy'''_{xxx} - a^2(ax + 3)y = 0.$

Particular solution: $y_0 = xe^{ax}$.

The substitution $y = xe^{ax}\int z(x)\,dx$ leads to a second order equation of the form 2.1.2.103: $xz''_{xx} + 3(ax + 1)z'_x + 3a(ax + 2)z = 0$.

83. $xy'''_{xxx} + ay'_x + b(b^2x + a)y = 0.$

The substitution $w = y'_x + by$ leads to a second order equation of the form 2.1.2.103: $xw''_{xx} - bxw'_x + (b^2x + a)w = 0$.

84. $xy'''_{xxx} + axy'_x - [b(a + b^2)x + a + 3b^2]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

The substitution $y = xe^{bx}\int z(x)\,dx$ leads to a second order equation of the form 2.1.2.103: $xz''_{xx} + 3(bx + 1)z'_x + [(a + 3b^2)x + 6b]z = 0$.

85. $xy'''_{xxx} + (b - a^2x)y'_x + aby = 0.$

The substitution $w = y'_x + ay$ leads to an equation of the form 2.1.2.103: $xw''_{xx} - axw'_x + bw = 0$.

86. $xy'''_{xxx} + (ax^2 + bx)y'_x - 2(ax + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.2: $w''_{xx} + (ax + b)w = 0$.

87. $xy'''_{xxx} + (ax^3 + bx)y'_x - 2(ax^2 + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.4: $w''_{xx} + (ax^2 + b)w = 0$.

88. $xy'''_{xxx} + 3y''_{xx} + axy = 0.$

The substitution $w = xy$ leads to a constant coefficient equation of the form 3.1.2.1: $w'''_{xxx} + aw = 0$.

89. $xy'''_{xxx} - 3ny''_{xx} + axy = 0, \qquad n = 0,\ 1,\ 2,\ \ldots$

Solution:

$$y = x^{3n+2}\left(\frac{1}{x^2}\frac{d}{dx}\right)^n\left(\frac{w}{x^2}\right),$$

where w is the solution of the equation 3.1.2.1: $w'''_{xxx} + aw = 0$.

90. $2xy'''_{xxx} + 3y''_{xx} + axy = b, \qquad a \neq 0.$

Solution:

$$y = \sum_{\nu=1}^{4} C_\nu \int_0^{\lambda_\nu} \frac{e^{xz}\,dz}{\sqrt{2z^3 + a}},$$

where λ_1, λ_2, and λ_3 are the roots of the cubic equation $2\lambda^3 + a = 0$; $\lambda_4 = -\infty$ with $x > 0$ and $\lambda_4 = +\infty$ with $x < 0$. In addition, constants C_ν are related by the constraint $\sqrt{a}\,(C_1 + C_2 + C_3 + C_4) + b = 0$, and the integrals are taken along straight lines.

91. $xy'''_{xxx} + 3y''_{xx} + ax^2y = b.$

The substitution $w = xy$ leads to an equation of the form 3.1.2.3: $w'''_{xxx} + axw = b$.

92. $xy'''_{xxx} + 3y''_{xx} + ax^4y = bx.$

The substitution $w = xy$ leads to an equation of the form 3.1.2.5: $w'''_{xxx} + ax^3w = bx$.

93. $\boldsymbol{xy'''_{xxx} + ay''_{xx} + aby'_x + b^3xy = 0}$.

The substitution $w = y'_x + by$ leads to a second order linear equation of the form 2.1.2.103: $xw''_{xx} + (a - bx)w'_x + b^2xw = 0$.

94. $\boldsymbol{xy'''_{xxx} + (a+b)y''_{xx} - xy'_x - ay = 0, \qquad a > 0, \quad b > 0}$.

Solution:

$$y = \sum_{\nu=1}^{3} C_\nu \int_{\gamma_\nu}^{\beta_\nu} |t|^{a-1}|t^2 - 1|^{(b-2)/2} e^{-tx}\,dt,$$

where $\gamma_1 = -1$, $\beta_1 = \gamma_2 = 0$, $\beta_2 = 1$; for $x > 0$, $\gamma_3 = 1$ and $\beta_3 = +\infty$; for $x < 0$, $\gamma_3 = -\infty$ and $\beta_3 = -1$.

95. $\boldsymbol{xy'''_{xxx} + ay''_{xx} + (b - c^2)xy'_x - c(ac + bx)y = 0}$.

The substitution $w = y'_x - cy$ leads to a second order equation of the form 2.1.2.103: $xw''_{xx} + (cx + a)w'_x + (bx + ac)w = 0$.

96. $\boldsymbol{xy'''_{xxx} + ay''_{xx} + [(c - b^2)x + ab]y'_x + c(a - bx)y = 0}$.

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

97. $\boldsymbol{xy'''_{xxx} + ay''_{xx} + bx^n y'_x + b(a + n - 1)x^{n-1}y = 0}$.

The substitution $w = y''_{xx} + bx^{n-1}y$ leads to a first order linear equation: $xw'_x + aw = 0$.

98. $\boldsymbol{xy'''_{xxx} + (ax + b)y''_{xx} - a^2by = 0}$.

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.2.103: $xw''_{xx} + bw'_x - abw = 0$.

99. $\boldsymbol{xy'''_{xxx} + (ax + b)y''_{xx} + cxy'_x - cy = 0}$.

The substitution $w = xy'_x - y$ leads to a second order equation of the form 2.1.2.103: $xw''_{xx} + (ax + b - 1)w'_x + cxw = 0$.

100. $\boldsymbol{xy'''_{xxx} + (ax + 3)y''_{xx} + (bx + 2a)y'_x + (cx + b)y = 0}$.

The substitution $w = xy$ leads to a constant coefficient equation: $w'''_{xxx} + aw''_{xx} + bw'_x + cw = 0$.

101. $\boldsymbol{xy'''_{xxx} + (ax + 3)y''_{xx} + a(bx + 2)y'_x + b[b(a - b)x + a]y = 0}$.

Solution:

$$y = \frac{1}{x}\left[C_1 e^{(b-a)x} + C_2 e^{-bx/2}\cos\left(\frac{bx\sqrt{3}}{2}\right) + C_3 e^{-bx/2}\sin\left(\frac{bx\sqrt{3}}{2}\right)\right].$$

102. $\boldsymbol{xy'''_{xxx} + [a(b + 1)x + b]y''_{xx} + a^2bxy'_x - a^2by = 0}$.

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

103. $\boldsymbol{xy'''_{xxx} - (x+2a)y''_{xx} - (x-2a-1)y'_x + (x-1)y = 0}$.

Solution:

$$y = C_1 e^x + x^{a+1}[C_2 I_{a+1}(x) + C_3 K_{a+1}(x)],$$

where I_a and K_a are modified Bessel functions.

104. $\boldsymbol{2xy'''_{xxx} - 4(x+a-1)y''_{xx} + (2x+6a-5)y'_x + (1-2a)y = 0}$.

Solution:

$$y = C_1 e^x + x^a e^{x/2}[C_2 I_a(x/2) + C_3 K_a(x/2)],$$

where I_a and K_a are modified Bessel functions.

105. $\boldsymbol{2xy'''_{xxx} + 3(2ax+k)y''_{xx} + 6(bx+ak)y'_x + (2cx+3bk)y = 0, \qquad k > 0}$.

Solution:

$$y = \sum_{\nu=1}^{4} C_\nu \int_0^{\lambda_\nu} e^{xz}[P(z)]^{(k-2)/2}\,dz,$$

where $P(z) = z^3 + 3az^2 + 3bz + c$; λ_1, λ_2, and λ_3 are the roots of this polynomial that are assumed to be different; $\lambda_4 = -\infty$ with $x > 0$ and $\lambda_4 = +\infty$ with $x < 0$; $C_4 = -C_1 - C_2 - C_3$.

106. $\boldsymbol{xy'''_{xxx} + (ax+b)y''_{xx} + [(ac+s-c^2)x + bc]y'_x + s[(a-c)x+b]y = 0}$.

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + c\lambda + s = 0$.

107. $\boldsymbol{xy'''_{xxx} + (ax^2+b+2)y''_{xx} - ab(b+1)y = 0}$.

This is a special case of equation 3.1.2.109 with $n = 2$.

108. $\boldsymbol{xy'''_{xxx} + (ax^2+b)y''_{xx} + 4axy'_x + 2ay = 0}$.

Integrating the equation twice, we arrive at a first order linear equation: $xy'_x + (ax^2 + b - 2)y = C_1 + C_2 x$.

109. $\boldsymbol{xy'''_{xxx} + (ax^n+b+2)y''_{xx} - ab(b+1)x^{n-2}y = 0}$.

The substitution $w = xy'_x + by$ leads to a second order equation of the form 2.1.2.42: $w''_{xx} + ax^{n-1}w'_x - a(b+1)x^{n-2}w = 0$.

110. $\boldsymbol{xy'''_{xxx} + (ax^n+3)y''_{xx} + (2ax^{n-1}+bx)y'_x + b(ax^n+1)y = 0}$.

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\bigr)$.

111. $\boldsymbol{xy'''_{xxx} + (ax^{n+1}+3)y''_{xx} + a(bx+2)x^n y'_x + b(abx^{n+1}+ax^n-b^2x)y = 0}$.

Particular solutions:

$$y_1 = \frac{1}{x}\exp\Bigl(-\frac{bx}{2}\Bigr)\cos\biggl(\frac{bx\sqrt{3}}{2}\biggr), \qquad y_2 = \frac{1}{x}\exp\Bigl(-\frac{bx}{2}\Bigr)\sin\biggl(\frac{bx\sqrt{3}}{2}\biggr).$$

112. $xy'''_{xxx} + (ax^n + 3)y''_{xx} + (abx^n + 2ax^{n-1} - b^2x)y'_x + b(ax^{n-1} - b)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}, \quad y_2 = \dfrac{1}{x}e^{-bx}.$

113. $(ax + b)y'''_{xxx} + cy'_x + k(ak^2x + bk^2 + c)y = 0.$

The substitution $w = y'_x + ky$ leads to a second order equation of the form 2.1.2.103:

$$(ax + b)w''_{xx} - k(ax + b)w'_x + (ak^2x + bk^2 + c)w = 0.$$

114. $(ax + 2)y'''_{xxx} - a^3xy'_x + 2a^3y = 0.$

Particular solutions: $y_1 = x^2, \quad y_2 = e^{-ax}.$

115. $(acx + bc - a)y'''_{xxx} - c^3(ax + b)y'_x + ac^3y = 0.$

Particular solutions: $y_1 = ax + b, \quad y_2 = e^{cx}.$

116. $(ax + b)y'''_{xxx} + (cx + d)y'_x + s[(as^2 + c)x + bs^2 + d]y = 0.$

The substitution $w = y'_x + sy$ leads to a second order equation of the form 2.1.2.103:

$$(ax + b)w''_{xx} - s(ax + b)w'_x + [(as^2 + c)x + bs^2 + d]w = 0.$$

117. $(ax + b)y'''_{xxx} + [(c - ak^2)x + d - bk^2]y'_x + k(cx + d)y = 0.$

The substitution $w = y'_x + ky$ leads to a second order equation of the form 2.1.2.103:

$$(ax + b)w''_{xx} - k(ax + b)w'_x + (cx + d)w = 0.$$

118. $(ax + b)y'''_{xxx} + [b(a + 1)x + b^2 + 1]y''_{xx} + b^2xy'_x - b^2y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-bx}.$

119. $(ax + b)y'''_{xxx} + k(ax + b)y''_{xx} + (cx + d)y'_x + k(cx + d)y = 0.$

The substitution $w = y'_x + ky$ leads to a second order equation of the form 2.1.2.103: $(ax + b)w''_{xx} + (cx + d)w = 0.$

120. $(ax + b)y'''_{xxx} + (cx + d)y''_{xx} + [(a\lambda + c\mu)x + b\lambda + d\mu]y'_x$
$+ (\lambda + \mu^2)[(c - a\mu)x + d - b\mu]y = 0.$

Particular solutions: $y_1 = \exp(s_1x), \quad y_2 = \exp(s_2x),$ where s_1 and s_2 are the roots of the quadratic equation $s^2 + \mu s + \lambda + \mu^2 = 0.$

121. $(ax + b)y'''_{xxx} + (cx + d)y''_{xx} - k[(3ak + 2c)x + 3bk + 2d]y'_x$
$+ k^2[(2ak + c)x + 2bk + d]y = 0.$

Particular solutions: $y_1 = e^{kx}, \quad y_2 = xe^{kx}.$

122. $(ax + b)y'''_{xxx} + (cx + d)y''_{xx} + sx(ax + b)y'_x + s[cx^2 + (a + d)x + b]y = 0.$

The substitution $w = y''_{xx} + sxy$ leads to a first order linear equation: $(ax + b)w'_x + (cx + d)w = 0.$

123. $(1-x)y'''_{xxx} + x(ax-2a+1)y''_{xx} + (-ax^2+2a-1)y'_x + 2a(x-1)y = 0.$

Particular solutions: $y_1 = x^2, \quad y_2 = e^x.$

124. $(ax+b)y'''_{xxx} - (a^3x^3 - 3a^2x + b^3)y'_x + abx(a^2x^2 - 3a - b^2)y = 0.$

Particular solutions: $y_1 = e^{bx}, \quad y_2 = \exp\left(-\dfrac{ax^2}{2}\right).$

125. $(ax+b)y'''_{xxx} + (cx+d)y''_{xx} + sx^n(ax+b)y'_x + sx^{n-1}[cx^2+(an+d)x+bn]y = 0.$

The substitution $w = y''_{xx} + sx^n y$ leads to a first order linear equation: $(ax+b)w'_x + (cx+d)w = 0$.

126. $(ax-1)y'''_{xxx} + x(abx^{n+1} - 2bx^n - a^2)y''_{xx}$
$+ (2bx^n - a^2bx^{n+2} + a^2)y'_x + 2ab(ax-1)x^n y = 0.$

Particular solutions: $y_1 = x^2, \quad y_2 = e^{ax}.$

127. $x^2y'''_{xxx} - 6y'_x + ax^2y + 2bx = 0.$

The substitution $y = x^2w$ leads to the equation 3.1.2.159 wherein w should be substituted for y.

128. $x^2y'''_{xxx} + (ax^2 + bx - m^2 - m)y'_x + (m-1)(ax+b)y = 0.$

The substitution $w = xy'_x + (m-1)y$ leads to a second order equation of the form 2.1.2.103: $xw''_{xx} - (m+1)w'_x + (ax+b)w = 0$.

129. $x^2y'''_{xxx} + (ax^2 + bx + c)y'_x - k[(a+k^2)x^2 + bx + c]y = 0.$

The substitution $w = y'_x - ky$ leads to a second order equation of the form 2.1.2.130:

$$x^2w''_{xx} + kx^2w'_x + [(a+k^2)x^2 + bx + c]w = 0.$$

130. $x^2y'''_{xxx} + (ax^n - b^2 - b)y'_x + a(b-1)x^{n-1}y = 0.$

The substitution $w = xy'_x + (b-1)y$ leads to a second order equation of the form 2.1.2.62: $xw''_{xx} - (b+1)w'_x + ax^{n-1}w = 0$.

131. $x^2y'''_{xxx} + 3xy''_{xx} - 3y'_x + ax^2y + b = 0.$

Solution: $y = \dfrac{d}{dx}\left(\dfrac{w}{x}\right)$, where function $w = w(x)$ satisfies a constant coefficient equation of the form 3.1.2.2: $w'''_{xxx} + aw = b$.

132. $x^2y'''_{xxx} + 6xy''_{xx} + 6y'_x + ax^2y = b.$

The substitution $w = x^2y$ leads to a constant coefficient equation of the form 3.1.2.2: $w'''_{xxx} + aw = b$.

133. $x^2y'''_{xxx} - 3(n+m)xy''_{xx} + 3n(3m+1)y'_x - x^2y = 0, \qquad m, n = 1,\ 2,\ 3,\ \ldots$

Solution:

$$y = \prod_{\mu=0}^{n-1}(\delta - 3\mu - 1)\prod_{\nu=0}^{m-1}(\delta - 3\nu - 2)\sum_{k=1}^{3} C_k e^{\omega_k x},$$

where $\delta = x\dfrac{d}{dx}$, ω_k are three roots of the cubic equation $\omega^3 = 1$.

134. $x^2 y'''_{xxx} + 6xy''_{xx} + 6y'_x + ax^3 y = b.$

The substitution $w = x^2 y$ leads to an equation of the form 3.1.2.3: $w'''_{xxx} + axw = b$.

135. $x^2 y'''_{xxx} - 2(n+1)xy''_{xx} - (ax^2 - 6n)y'_x + 2axy = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution:

$$y = \begin{cases} C_1 + C_2 x^4 + C_3 x^{2n+1} & \text{if } a = 0, \\ C_1(ax^2 - 4n + 2) + C_2 e^{x\sqrt{a}} P(x) + C_3 e^{-x\sqrt{a}} Q(x) & \text{if } a \neq 0, \end{cases}$$

where P and Q are some polynomials of the degree $\leq 2n + 2$.

136. $x^2 y'''_{xxx} + 3xy''_{xx} + (4a^2 x^{2a} + 1 - 4a^2 b^2)y'_x + 4a^3 x^{2a-1} y = 0.$

Solution:

$$y = C_1 J_b^2(x^a) + C_2 J_b(x^a) Y_b(x^a) + C_3 Y_b^2(x^a),$$

where J_b and Y_b are Bessel functions.

137. $x^2 y'''_{xxx} + ax^2 y''_{xx} + (bx + c)y'_x + a(bx + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.2.106: $x^2 w''_{xx} + (bx + c)w = 0$.

138. $x^2 y'''_{xxx} + ax^2 y''_{xx} + (bx^n + c)y'_x + a(bx^n + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.2.113: $x^2 w''_{xx} + (bx^n + c)w = 0$.

139. $x^2 y'''_{xxx} - (x + a)xy''_{xx} + a(2x + 1)y'_x - a(x + 1)y = 0.$

Solution:

$$y = C_1 e^x + x^{(a+1)/2} \big[C_2 J_{a+1}\big(2\sqrt{ax}\,\big) + C_3 Y_{a+1}\big(2\sqrt{ax}\,\big)\big],$$

where J_a and Y_a are Bessel functions.

140. $x^2 y'''_{xxx} - (x^2 - 2x)y''_{xx} - (x^2 + a^2 - \tfrac{1}{4})y'_x + (x^2 - 2x + a^2 - \tfrac{1}{4})y = 0.$

Solution:

$$y = C_1 e^x + \sqrt{x}\,\big[C_2 I_a(x) + C_3 K_a(x)\big],$$

where I_a and K_a are modified Bessel functions.

141. $x^2 y'''_{xxx} - 2x(x - 1)y''_{xx} + (x^2 - 2x + \tfrac{1}{4} - a^2)y'_x + (a^2 - \tfrac{1}{4})y = 0.$

Solution:

$$y = C_1 e^x + \sqrt{x}\, e^{x/2} \big[C_2 I_a(x/2) + C_3 K_a(x/2)\big],$$

where I_a and K_a are modified Bessel functions.

142. $x^2 y'''_{xxx} - 3(x - a)xy''_{xx} + [2x^2 + 4(b - a)x + a(2a - 1)]y'_x - 2b(2x - 2a + 1)y = 0.$

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 form a fundamental set of solutions of a second order equation of the form 2.1.2.103: $xw''_{xx} + (a - x)w'_x + bw = 0$.

143. $\boldsymbol{x^2 y'''_{xxx} + x[(a+c)x+b]y''_{xx} + [(ac+\alpha)x^2 + (bc+\beta)x+\gamma]y'_x + c(\alpha x^2+\beta x+\gamma)y = 0.}$

The substitution $w = y'_x + cy$ leads to an equation of the form 2.1.2.141 with $n = 1$: $x^2 w''_{xx} + x(ax+b)w'_x + (\alpha x^2 + \beta x + \gamma)w = 0$.

144. $\boldsymbol{x^2 y'''_{xxx} + (ax^{n+1} - b^2 - b)y'_x + a(b-1)x^n y = 0.}$

The substitution $w = xy'_x + (b-1)y$ leads to a second order equation of the form 2.1.2.62: $xw''_{xx} - (b+1)w'_x + ax^n w = 0$.

145. $\boldsymbol{x^2 y'''_{xxx} - 3ax^{n+1}(n + ax^{n+1})y'_x + ax^n(n - n^2 + 2a^2 x^{2n+2})y = 0.}$

1°. Particular solutions with $n \neq -1$: $\quad y_1 = \exp\Bigl(\dfrac{ax^{n+1}}{n+1}\Bigr), \quad y_2 = x\exp\Bigl(\dfrac{ax^{n+1}}{n+1}\Bigr).$

2°. Particular solutions with $n = -1$: $\quad y_1 = x^a, \quad y_2 = x^{a+1}.$

146. $\boldsymbol{x^2 y'''_{xxx} + (ax^{n+1} + bx)y''_{xx} + [a(b-2)x^n + c]y'_x + a(c-b+2)x^{n-1}y = 0.}$

Particular solutions: $\quad y_1 = x^{m_1}, \quad y_2 = x^{m_2},\quad$ where m_1 and m_2 are the roots of the quadratic equation: $m^2 + (b-3)m + c - b + 2 = 0$.

147. $\boldsymbol{x(ax+b)y'''_{xxx} + x(cx+d)y'_x - 2(cx+d)y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.2.103: $(ax+b)w''_{xx} + (cx+d)w = 0$.

148. $\boldsymbol{2x(x-1)y'''_{xxx} + 3(2x-1)y''_{xx} + (2ax+b)y'_x + ay = 0.}$

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 form a fundamental set of solutions of the equation

$$2x(x-1)w''_{xx} + (2x-1)w'_x + \Bigl(\frac{a}{2}x + \frac{b}{4} - \frac{1}{2}\Bigr)w = 0,$$

which is reduced, by means of the substitution $x = \cos^2\xi$, to the Mathieu equation 2.1.6.4: $2w''_{\xi\xi} = (a + b - 2 + a\cos 2\xi)w$.

149. $\boldsymbol{(a_2 x^2 + a_1 x + a_0)y'''_{xxx} + (b_1 x + b_0)y''_{xx} + (c_1 x + c_0)y'_x - mc_1 y = 0, \qquad c_1 \neq 0,}$ where m is a positive integer.

A solution of this eqaution is a polynomial of the degree m, which can presented as follows:

$$y = \sum_{k=0}^{m}\Bigl(-\frac{1}{c_1}\Bigr)^k \bigl\{x^m I x^{-m-1}[(ax^2 + a_1 x + a_0)D^3 + (b_1 x + b_0)D^2 + c_0 D]\bigr\}^k x^m,$$

where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu+1}$ with $\nu \neq -1$.

150. $\boldsymbol{x^3 y'''_{xxx} = a(a^2-1)y.}$

This is a special case of equation 3.1.2.161.

Solution:

$$y = x(C_1 x^{n_1} + C_2 x^{n_2} + C_3 x^a),$$

where n_1 and n_2 are the roots of the quadratic equation $n^2 + an + a^2 - 1 = 0$.

151. $\boldsymbol{x^3 y'''_{xxx} + (1-a^2)xy'_x + (bx^3 + a^2 - 1)y = 0.}$

For $a = \pm 1$, we have a constant coefficient equation of the form 3.1.2.1. For $b = 0$, we obtain the Euler equation 3.1.2.161.

If $b \neq 0$ and a is a positive integer greater than 1, then

$$y = x^{1-a} \sum_{k=1}^{3} C_k \exp(-\lambda_k x) P_k(x),$$

where λ_1, λ_2, λ_3 are the roots of the cubic equation $\lambda^3 = b$ and $P_k(x)$ are polynomials of the degree $\leq 3(a-1)$.

Denote the solution of the original equation for arbitrary (including complex) a by y_a. Then, the recurrence relation holds

$$y_{a+3} = by_a + (2a+3)x^{-1}y''_a - (a+1)(2a+3)(x^{-2}y'_a - x^{-3}y_a). \qquad (1)$$

Since $y_{\pm 1} = e^{-\lambda x}$, corresponding to three values of λ which satisfy the condition $\lambda^3 = b$, make up a fundamental set of solutions, then formula (1) makes it possible to find all y_n for any integer values of n not divisible by 3. In particular, $y_2 = (x^{-1} + \lambda)e^{-\lambda x}$ $(\lambda^3 = b)$.

152. $\boldsymbol{x^3 y'''_{xxx} + (4x^3 + ax)y'_x - ay = 0.}$

Solution:

$$y = C_1 x J_\nu^2(x) + C_2 x J_\nu(x) Y_\nu(x) + C_3 x Y_\nu^2(x),$$

where J_ν and Y_ν are Bessel functions; $4\nu^2 = 1 - a$.

153. $\boldsymbol{x^3 y'''_{xxx} + x[ax^2 + 3b(1-b)]y'_x + 2b(ax^2 + b^2 - 1)y = 0.}$

1°. Particular solutions with $a > 0$: $y_1 = x^b \sin\bigl(x\sqrt{a}\bigr)$, $y_2 = x^b \cos\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = x^b \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = x^b \exp\bigl(x\sqrt{-a}\bigr)$.

3°. Particular solutions with $a = 0$: $y_1 = x^b$, $y_2 = x^{b+1}$.

154. $\boldsymbol{x^3 y'''_{xxx} + x(ax^2 + bx + c)y'_x + (k-1)(ax^2 + bx + c + k^2 + k)y = 0.}$

The substitution $w = xy'_x + (k-1)y$ leads to a second order linear equation of the form 2.1.2.126:

$$x^2 w''_{xx} - (k+1)xw'_x + (ax^2 + bs + c + k^2 + k)w = 0.$$

155. $\boldsymbol{x^3 y'''_{xxx} + ax^n y'_x + (b-1)(ax^{n-1} + b^2 + b)y = 0.}$

The substitution $w = xy'_x + (b-1)y$ leads to a second order linear equation of the form 2.1.2.127:

$$x^2 w''_{xx} - (b+1)xw'_x + (ax^{n-1} + b^2 + b)w = 0.$$

156. $\boldsymbol{x^3 y'''_{xxx} + x(ax^n + b)y'_x - 2(ax^n + b)y = 0}$.

The substitution $w = xy'_x - 2y$ leads to a second order linear equation of the form 2.1.2.113: $x^2 w''_{xx} + (ax^n + b)w = 0$.

157. $\boldsymbol{x^3 y'''_{xxx} + x(ax^n + b - c)y'_x + (c-1)(ax^n + b + c^2)y = 0}$.

The substitution $w = xy'_x + (c-1)y$ leads to a second order linear equation of the form 2.1.2.127:

$$x^2 w''_{xx} - (c+1)xw'_x + (ax^n + b + c^2)w = 0.$$

158. $\boldsymbol{x^3 y'''_{xxx} + (ax^{2n} + 1 - n^2)xy'_x + [bx^{3n} + a(n-1)x^{2n} + n^2 - 1]y = 0}$.

The transformation $\xi = \dfrac{1}{n}x^n$, $z = x^{n-1}y$ leads to a constant coefficient equation: $z'''_{\xi\xi\xi} + az'_\xi + bz = 0$.

159. $\boldsymbol{x^3 y'''_{xxx} + 6x^2 y''_{xx} + (ax^3 - 12)y + 2b = 0}$.

Solution: $\quad y = \dfrac{d}{dx}\left(\dfrac{w}{x^2}\right)$, $\quad$ where $w = w(x)$ satisfies a constant coefficient equation of the form 3.1.2.2: $w'''_{xxx} + aw = b$.

160. $\boldsymbol{x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x + (a-2)by = 0}$.

This is a special case of equation 3.1.2.161.

Solution:

$$y = C_1 x^{2-a} + C_2 x^{n_1} + C_3 x^{n_2},$$

where n_1 and n_2 are the roots of the quadratic equation $n^2 - n + b = 0$.

161. $\boldsymbol{x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x + cy = 0}$.

The Euler equation.

The substitution $t = \ln|x|$ leads to a constant coefficient equation of the form 3.1.2.41: $y'''_{ttt} + (a-3)y''_{tt} + (2 - a + b)y'_t + cy = 0$.

162. $\boldsymbol{x^3 y'''_{xxx} + 3ax^2 y''_{xx} + 3a(a-1)xy'_x + [bx^3 + a(a-1)(a-2)]y = 0}$.

The substitution $w = x^a y$ leads to a constant coefficient equation of the form 3.1.2.1: $w'''_{xxx} + bw = 0$.

163. $\boldsymbol{x^3 y'''_{xxx} + 3ax^2 y''_{xx} + 3a(a-1)xy'_x + [bx^n + a(a-1)(a-2)]y = 0}$.

The substitution $w = x^a y$ leads to an equation of the form 3.1.2.7: $w'''_{xxx} + bx^{n-3}w = 0$.

164. $\boldsymbol{x^3 y'''_{xxx} + 3(1-a)x^2 y''_{xx} + x[4b^2c^2x^{2c} + 1 - 4\nu^2c^2 + 3a(a-1)]y'_x + [4b^2c^2(c-a)x^{2c} + a(4\nu^2c^2 - a^2)]y = 0}$.

Solution:

$$y = C_1 x^a J_\nu^2(u) + C_2 x^a J_\nu(u) Y_\nu(u) + C_3 x^a Y_\nu^2(u),$$

where $u = bx^c$; J_ν and Y_ν are Bessel functions.

165. $x^3y'''_{xxx}+(ax^2+b)y''_{xx}+2(2a-9)xy'_x+2(a-6)y=0.$

Integrating the equation twice, we arrive at a first order linear equation: $x^3y'_x+[(a-6)x^2+b]y=C_1+C_2x$.

166. $x^3y'''_{xxx}+x^2(ax+b)y''_{xx}+cxy'_x+c(ax+b-2)y=0.$

Particular solutions: $y_1=x^{n_1}$, $y_2=x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2-n+c=0$.

167. $x^3y'''_{xxx}+x^2(2ax+b)y''_{xx}+x(a^2x^2+2abx+c)y'_x+(a^2bx^2+bc-2c)y=0.$

Particular solutions: $y_1=e^{-ax}x^{n_1}$, $y_2=e^{-ax}x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2-n+c=0$.

168. $x^3y'''_{xxx}+ax^ny''_{xx}+bxy'_x+b(ax^{n-2}-2)y=0.$

Particular solutions: $y_1=x^{m_1}$, $y_2=x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2-m+b=0$.

169. $x^3y'''_{xxx}+x^2(ax^n+b)y''_{xx}+x(ax^n+b-1)y'_x+(ax^n+b-3)y=0.$

Particular solutions: $y_1=\cos(\ln x)$, $y_2=\sin(\ln x)$.

170. $x^3y'''_{xxx}+x^2(ax^n+b+c+1)y''_{xx}+x[\alpha x^{2n}+(ac+\beta)x^n+\gamma+bc]y'_x$
$+(c-1)(\alpha x^{2n}+\beta x^n+\gamma)y=0.$

The substitution $w=xy'_x+(c-1)y$ leads to a second order equation of the form 2.1.2.141: $x^2w''_{xx}+x(ax^n+b)w'_x+(\alpha x^{2n}+\beta x^n+\gamma)w=0$.

171. $x^2(ax+b)y'''_{xxx}+(cx-bm^2-bm)y'_x+(m-1)(c+am^2+am)y=0.$

The substitution $w=xy'_x+(m-1)y$ leads to a hypergeometric equation of the form 2.1.2.159: $x(ax+b)w''_{xx}-(m+1)(ax+b)w'_x+(c+am^2+am)w=0$.

172. $x(ax^2+bx+c)y'''_{xxx}+xy'_x-2y=0.$

The substitution $w=xy'_x-2y$ leads to a second order equation of the form 2.1.2.166: $(ax^2+bx+c)w''_{xx}+w=0$.

173. $(ax+b)x^3y'''_{xxx}+(cx+d)x^2y''_{xx}+s(ax+b)xy'_x+s[(c-2a)x+d-2b]y=0.$

Particular solutions: $y_1=x^{n_1}$, $y_2=x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2-n+s=0$.

174. $x^5y'''_{xxx}=a(xy'_x-2y).$

Solution:

$$y=\begin{cases} x^2\left[C_1+C_2\exp\left(\dfrac{\sqrt{a}}{x}\right)+C_3\exp\left(-\dfrac{\sqrt{a}}{x}\right)\right] & \text{if } a>0,\\ x^2\left[C_1+C_2\cos\left(\dfrac{\sqrt{-a}}{x}\right)+C_3\sin\left(\dfrac{\sqrt{-a}}{x}\right)\right] & \text{if } a<0.\end{cases}$$

175. $\boldsymbol{x^6 y'''_{xxx} = ay + bx^2}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a constant coefficient equation of the form 3.1.2.2: $w'''_{ttt} + aw + b = 0$.

176. $\boldsymbol{x^6 y'''_{xxx} + ax^2 y'_x + (b - 2ax)y = 0}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to a constant coefficient equation: $w'''_{ttt} + aw'_t - bw = 0$.

177. $\boldsymbol{x^6 y'''_{xxx} + 6x^5 y''_{xx} - ay + 2bx = 0}$.

The substitution $x = t^{-1}$ leads to an equation of the form 3.1.2.127: $t^2 y'''_{ttt} - 6y'_t + at^2 y - 2bt = 0$.

178. $\boldsymbol{(x-a)^3 (x-b)^3 y'''_{xxx} - cy = 0, \qquad a \neq b}$.

The transformation

$$t = \ln\left|\frac{x-a}{x-b}\right|, \qquad w = \frac{y}{(x-b)^2}$$

leads to a constant coefficient equation: $(a-b)^3(w'''_{ttt} - 3w''_{tt} + 2w'_t) - cw = 0$.

179. $\boldsymbol{(ax^2 + bx + c)^3 y'''_{xxx} = ky}$.

The transformation

$$\xi = \int \frac{dx}{ax^2 + bx + c}, \qquad w = \frac{y}{ax^2 + bx + c}$$

leads to a constant coefficient equation: $w'''_{\xi\xi\xi} + (4ac - b^2)w'_\xi = kw$.

180. $\boldsymbol{x^7 y'''_{xxx} = ay + bx^3}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 3.1.2.3: $w'''_{ttt} + atw + b = 0$.

181. $\boldsymbol{x^7 y'''_{xxx} + (ax + b)y = 0}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 3.1.2.4: $w'''_{ttt} - (bt + a)w = 0$.

182. $\boldsymbol{x^9 y'''_{xxx} + (a^3 - 3a^2 x^2)y = 0}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 3.1.2.6: $w'''_{ttt} + (3a^2 t - a^3 t^3)w = 0$.

183. $\boldsymbol{x^9 y'''_{xxx} = ay}$.

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the form 3.1.2.5: $w'''_{ttt} + at^3 w = 0$.

184. $\boldsymbol{x^{3/2} y'''_{xxx} = ay}$.

This is a special case of equation 5.1.2.25 with $n = 1$.

185. $x^{9/2}y'''_{xxx} = ay$.

This is a special case of equation 5.1.2.26 with $n = 1$.

186. $x^2(x^n + a)y'''_{xxx} + x(bx^{m+1} + 2nx^n + cx)y''_{xx}$
$+ [2bmx^{m+1} + n(n-1)x^n]y'_x + bm(m-1)x^m y = 0$.

The twofold integration yields a first order linear equation:

$$(x^n + a)y'_x + (bx^m + c)y = C_1 + C_2 x.$$

3.1.3. Equations Containing Exponential Functions

1. $y'''_{xxx} - ae^{\lambda x}(a^2 e^{2\lambda x} + 3a\lambda e^{\lambda x} + \lambda^2)y = 0$.

Particular solution: $y_0 = \exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)$.

The substitution $y = \exp\Big(\dfrac{a}{\lambda}e^{\lambda x}\Big)\displaystyle\int z(x)\,dx$ leads to a second order equation of the form 2.1.3.29: $z''_{xx} + 3ae^{\lambda x}z'_x + (3a^2e^{2\lambda x} + 3a\lambda e^{\lambda x})z = 0$.

2. $y'''_{xxx} + ae^{\lambda x}y'_x + a\lambda e^{\lambda x}y = be^{\mu}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + ae^{\lambda x}y = \frac{b}{\mu}e^{\mu} + C$$

(see 2.1.3.1 for the solution of the corresponding homogeneous equation).

3. $y'''_{xxx} + ae^{\lambda x}y'_x + b(ae^{\lambda x} + b^2)y = 0$.

The substitution $w = y'_x + by$ leads to a second order equation of the form 2.1.3.10: $w''_{xx} - bw'_x + (ae^{\lambda x} + b^2)w = 0$.

4. $y'''_{xxx} + ae^{\lambda x}y'_x + bx(ae^{\lambda x} + b^2x^2 - 3b)y = 0$.

Particular solution: $y_0 = \exp\Big(-\dfrac{bx^2}{2}\Big)$.

5. $y'''_{xxx} + ae^{\lambda x}y'_x + [a(\lambda - b)e^{\lambda x} - b^3]y = 0$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + by'_x + (ae^{\lambda x} + b^2)y = Ce^{bx}$$

(see 2.1.3.10 for the solution of the corresponding homogeneous equation).

6. $y'''_{xxx} + (ae^{\lambda x} - b^2)y'_x + abe^{\lambda x}y = 0$.

The substitution $w = y'_x + by$ leads to a second order equation of the form 2.1.3.10: $w''_{xx} - bw'_x + ae^{\lambda x}w = 0$.

7. $\boldsymbol{y'''_{xxx} + (ae^{\lambda x} - b^2)y'_x + a(\lambda - b)e^{\lambda x}y = 0.}$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + by'_x + ae^{\lambda x} = Ce^{bx}$$

(see 2.1.3.10 for the solution of the corresponding homogeneous equation).

8. $\boldsymbol{y'''_{xxx} + (ae^{\lambda x} + b)y'_x + c(ae^{\lambda x} + b + c^2)y = 0.}$

The substitution $w = y'_x + cy$ leads to a second order equation of the form 2.1.3.10: $w''_{xx} - cw'_x + (ae^{\lambda x} + b + c^2)w = 0$.

9. $\boldsymbol{y'''_{xxx} + (ax + b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0.}$

Particular solution: $y_0 = ax + b$.

10. $\boldsymbol{y'''_{xxx} + (ae^{2\lambda x} + be^{\lambda x})y'_x - c(ae^{2\lambda x} + be^{\lambda x} + c^2)y = 0.}$

The substitution $w = y'_x - cy$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + cw'_x + (ae^{2\lambda x} + be^{\lambda x} + c^2)w = 0$.

11. $\boldsymbol{y'''_{xxx} - 3ae^{\lambda x}(ae^{\lambda x} + \lambda)y'_x + ae^{\lambda x}(2a^2e^{2\lambda x} - \lambda^2)y = 0.}$

Particular solutions: $y_1 = \exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$, $y_2 = x\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

12. $\boldsymbol{y'''_{xxx} - (3a^2e^{2\lambda x} + 3a\lambda e^{\lambda x} + b)y'_x + ae^{\lambda x}(2a^2e^{2\lambda x} - 2b - \lambda^2)y = 0.}$

1°. Particular solutions with $b > 0$:

$$y_1 = \exp\Bigl(\frac{a}{\lambda}e^{\lambda x} - x\sqrt{b}\Bigr), \quad y_2 = \exp\Bigl(\frac{a}{\lambda}e^{\lambda x} + x\sqrt{b}\Bigr).$$

2°. Particular solutions with $b < 0$:

$$y_1 = \cos\bigl(x\sqrt{-b}\bigr)\exp\Bigl(\frac{a}{\lambda}e^{\lambda x}\Bigr), \quad y_2 = \sin\bigl(x\sqrt{-b}\bigr)\exp\Bigl(\frac{a}{\lambda}e^{\lambda x}\Bigr).$$

13. $\boldsymbol{y'''_{xxx} + ay''_{xx} + be^{\lambda x}y'_x + abe^{\lambda x}y = 0.}$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.3.1: $w''_{xx} + be^{\lambda x}w = 0$.

14. $\boldsymbol{y'''_{xxx} + ay''_{xx} + (be^{\lambda x} + c)y'_x + [b(a + \lambda)e^{\lambda x} + ac]y = 0.}$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (be^{\lambda x} + c)y = Ce^{-ax}$$

(see 2.1.3.2 for the solution of the corresponding homogeneous equation).

15. $\boldsymbol{y'''_{xxx} + ay''_{xx} + (be^{2\lambda x} + ce^{\lambda x})y'_x + a(be^{2\lambda x} + ce^{\lambda x})y = 0.}$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + (be^{2\lambda x} + ce^{\lambda x})w = 0$.

16. $y'''_{xxx} + ae^{\lambda x} y''_{xx} - b^2(ae^{\lambda x} + b)y = 0.$

The substitution $w = y'_x - by$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + (ae^{\lambda x} + b)w'_x + b(ae^{\lambda x} + b)w = 0$.

17. $y'''_{xxx} + ae^{\lambda x} y''_{xx} - by'_x - abe^{\lambda x} y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

18. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + abe^{\lambda x} y'_x + b^3 y = 0.$

The substitution $w = y'_x + by$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + (ae^{\lambda x} - b)w'_x + b^2 w = 0$.

19. $y'''_{xxx} + ae^{\lambda x} y''_{xx} - b(2ae^{\lambda x} + 3b)y'_x + b^2(ae^{\lambda x} + 2b)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

20. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + (be^{\mu x} - c^2)y'_x - c(ace^{\lambda x} + be^{\mu x})y = 0.$

Particular solution: $y_0 = e^{cx}$.

21. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + (abe^{\lambda x} - b^2 + c)y'_x + c(ae^{\lambda x} - b)y = 0.$

Particular solutions: $y_1 = e^{\beta_1 x}$, $y_2 = e^{\beta_2 x}$, where β_1 and β_2 are the roots of the quadratic equation $\beta^2 + b\beta + c = 0$.

22. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + [a(b + \lambda)e^{\lambda x} - b^2]y'_x + ab\lambda e^{\lambda x} y = 0.$

Particular solutions: $y_1 = e^{-bx}$, $y_2 = e^{-bx} \displaystyle\int \exp\Bigl(2bx - \frac{a}{\lambda} e^{\lambda x}\Bigr)\,dx$.

23. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + bx^n y'_x + bx^{n-1}(axe^{\lambda x} + n)y = 0.$

The substitution $w = y''_{xx} + bx^n y$ leads to a first order linear equation: $w'_x + ae^{\lambda x} w = 0$.

24. $y'''_{xxx} + axe^{\lambda x} y''_{xx} + (bx^2 - ae^{\lambda x})y'_x + bx(ax^2 e^{\lambda x} + 3)y = 0.$

1°. Particular solutions with $b > 0$:

$$y_1 = \cos\left(\frac{x^2\sqrt{b}}{2}\right), \quad y_2 = \sin\left(\frac{x^2\sqrt{b}}{2}\right).$$

2°. Particular solutions with $b < 0$:

$$y_1 = \exp\left(\frac{-x^2\sqrt{-b}}{2}\right), \quad y_2 = \exp\left(\frac{x^2\sqrt{-b}}{2}\right).$$

25. $y'''_{xxx} + ax^2 e^{\lambda x} y''_{xx} - 2axe^{\lambda x} y'_x + 2ae^{\lambda x} y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

26. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} - ab^2 e^{\lambda x} y = 0.$

The substitution $w = y'_x + by$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + ae^{\lambda x} w'_x - abe^{\lambda x} w = 0$.

27. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} - c^2(e^{\lambda x} + b + c)y = 0.$

The substitution $w = y'_x - cy$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + (ae^{\lambda x} + b + c)w'_x + c(ae^{\lambda x} + b + c)w = 0$.

28. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + c(ae^{\lambda x} + b)y'_x + c^3 y = 0.$

The substitution $w = y'_x + cy$ leads to a second order equation of the form 2.1.3.29: $w''_{xx} + (ae^{\lambda x} + b - c)w'_x + c^2 w = 0$.

29. $y'''_{xxx} + (be^{ax} + 2a)y''_{xx} - a(be^{ax} + a)y'_x - 2a^3 y = 0.$

Particular solutions: $\quad y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

30. $y'''_{xxx} = (e^{\lambda x} - a)y''_{xx} + (ae^{\lambda x} - b)y'_x + be^{\lambda x} y.$

Particular solutions: $\quad y_1 = e^{\beta_1 x}, \quad y_2 = e^{\beta_2 x}$, where β_1 and β_2 are the roots of the quadratic equation $\beta^2 + a\beta + b = 0$.

31. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{\lambda x} + d)y'_x - s[(as + c)e^{\lambda x} + bs + d + s^2]y = 0.$

The substitution $w = y'_x - sy$ leads to a second order equation of the form 2.1.3.29:

$$w''_{xx} + (ae^{\lambda x} + b + s)w'_x + [(as + c)e^{\lambda x} + bs + d + s^2]w = 0.$$

32. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{2\lambda x} + d)y'_x - s(ce^{2\lambda x} + ase^{\lambda x} + bs + d + s^2)y = 0.$

The substitution $w = y'_x - sy$ leads to a second order equation of the form 2.1.3.29:

$$w''_{xx} + (ae^{\lambda x} + b + s)w'_x + (ce^{2\lambda x} + ase^{\lambda x} + bs + d + s^2)w = 0.$$

33. $y'''_{xxx} + (ae^{\lambda x} + b)y''_{xx} + (ce^{\lambda x} + d)y'_x - ke^{\lambda x}[k(a + k)e^{2\lambda x} + (a\lambda + 3k\lambda + bk + c)e^{\lambda x} + \lambda^2 + b\lambda + d]y = 0.$

Particular solution: $\quad y_0 = \exp\left(\dfrac{k}{\lambda} e^{\lambda x}\right)$.

The substitution $y = \exp\left(\dfrac{k}{\lambda} e^{\lambda x}\right) \displaystyle\int z(x)\,dx$ lead to a second order linear equation of the form 2.1.3.29.

34. $y'''_{xxx} + (2ae^{\lambda x} + b)y''_{xx} + ae^{\lambda x}(ae^{\lambda x} + 2b + 3\lambda)y'_x + ae^{\lambda x}[a(b + 2\lambda)e^{\lambda x} + b\lambda + \lambda^2]y = 0.$

Particular solutions: $\quad y_1 = \exp\left(-\dfrac{a}{\lambda} e^{\lambda x}\right), \quad y_2 = x\exp\left(-\dfrac{a}{\lambda} e^{\lambda x}\right)$.

35. $y'''_{xxx} + (ae^{\lambda x} - b)y''_{xx} + ce^{\mu x} y'_x - b(abe^{\lambda x} + ce^{\mu x})y = 0.$

Particular solution: $\quad y_0 = e^{bx}$.

36. $y'''_{xxx} + (e^{\lambda x} + a)y''_{xx} + (ae^{\lambda x} + be^{\mu x})y'_x + abe^{\mu x}y = 0.$

Particular solution: $y_0 = e^{-ax}$.

37. $y'''_{xxx} + (ax + be^{\lambda x})y''_{xx} + a(bxe^{\lambda x} + 2)y'_x + abe^{\lambda x}y = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $\quad y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\,dx$.

38. $y'''_{xxx} + (abxe^{\lambda x} + be^{\lambda x} + a)y''_{xx} + ab^2xe^{\lambda x}y'_x - a^2be^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $\quad y_2 = e^{-ax}$.

39. $y'''_{xxx} + ax^n(be^{\lambda x} + 2\lambda)y''_{xx} - \lambda(abx^ne^{\lambda x} + \lambda)y'_x - 2a\lambda^3x^ny = 0.$

Particular solutions: $y_1 = e^{\lambda x}$, $\quad y_2 = e^{-\lambda x} + \dfrac{b}{\lambda}$.

40. $y'''_{xxx} + (ax^n - 2be^{\lambda x})y''_{xx} - be^{\lambda x}(2ax^n - be^{\lambda x} + 3\lambda)y'_x$
$\qquad + be^{\lambda x}[ax^n(be^{\lambda x} - \lambda) + 2b\lambda e^{\lambda x} - \lambda^2]y = 0.$

Particular solutions: $y_1 = \exp\Bigl(\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$, $\quad y_2 = x\exp\Bigl(\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

41. $y'''_{xxx} + (ae^{\lambda x} + be^{\mu x})y''_{xx} + cy'_x + c(ae^{\lambda x} + be^{\mu x})y = 0.$

The substitution $w = y''_{xx} + cy$ leads to a first order linear equation: $w'_x + (ae^{\lambda x} + be^{\mu x})w = 0$.

42. $y'''_{xxx} + (ae^{\lambda x} + be^{\mu x})y''_{xx} + [abe^{(\lambda+\mu)x} + b(c + \mu)e^{\mu x} - c^2]y'_x$
$\qquad + c[abe^{(\lambda+\mu)x} - ace^{\lambda x} + b\mu e^{\mu x}]y = 0.$

Particular solutions: $y_1 = e^{-cx}$, $\quad y_2 = e^{-cx}\displaystyle\int \exp\Bigl(2cx - \dfrac{b}{\mu}e^{\mu x}\Bigr)\,dx$.

43. $y'''_{xxx} + ae^{\lambda x}(be^{\mu x} + 2\mu)y''_{xx} - \mu(abe^{(\lambda+\mu)x} + \mu)y'_x - 2a\mu^3e^{\lambda x}y = 0.$

Particular solutions: $y_1 = e^{\mu x}$, $\quad y_2 = e^{-\mu x} + \dfrac{b}{\mu}$.

44. $xy'''_{xxx} + ay''_{xx} + x(be^{\lambda x} + c)y'_x + [b(\lambda x + a)e^{\lambda x} + ac]y = 0.$

The substitution $w = y''_{xx} + (be^{\lambda x} + c)y$ yields a first order linear equation: $xw'_x + aw = 0$.

45. $xy'''_{xxx} + axe^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.3.1: $w''_{xx} + ae^{\lambda x}w = 0$.

46. $xy'''_{xxx} = (e^{\lambda x} - ax)y''_{xx} + (ae^{\lambda x} - bx)y'_x + be^{\lambda x}y.$

Particular solutions: $y_1 = e^{\beta_1 x}$, $\quad y_2 = e^{\beta_2 x}$, $\quad$ where β_1 and β_2 are the roots of the quadratic equation $\beta^2 + a\beta + b = 0$.

47. $xy'''_{xxx} + (axe^{\lambda x} + 3)y''_{xx} + a(bx + 2)e^{\lambda x}y'_x + b(abxe^{\lambda x} + ae^{\lambda x} - b^2x)y = 0.$

Particular solutions:

$$y_1 = \frac{1}{x}\exp\left(-\frac{bx}{2}\right)\cos\left(\frac{bx\sqrt{3}}{2}\right), \quad y_2 = \frac{1}{x}\exp\left(-\frac{bx}{2}\right)\sin\left(\frac{bx\sqrt{3}}{2}\right).$$

48. $x^2y'''_{xxx} + x(axe^{\lambda x} + b)y''_{xx} + [a(b-2)xe^{\lambda x} + c]y'_x + a(c - b + 2)e^{\lambda x}y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 + (b-3)n + c - b + 2 = 0$.

49. $x^3y'''_{xxx} + bx^2e^{\lambda x}y''_{xx} + axy'_x + a(be^{\lambda x} - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 - n + a = 0$.

50. $x^3y'''_{xxx} + x^2(ae^{\lambda x} + b)y''_{xx} + x(abe^{\lambda x} + c - b)y'_x + c(ae^{\lambda x} - 2)y = 0.$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 + (b-1)n + c = 0$.

51. $(ae^x + b)y'''_{xxx} - ae^xy = 0.$

Particular solution: $y_0 = ae^x + b$.

52. $(bce^{ax} + a + c)y'''_{xxx} - (bc^3e^{ax} + a^3 + c^3)y'_x + ac(a^2 - c^2)y = 0.$

Particular solutions: $y_1 = e^{cx}$, $y_2 = e^{-ax} + b$.

53. $(ae^{\lambda x} + b)y'''_{xxx} + (ce^{\lambda x} + d)y''_{xx} + k(ae^{\lambda x} + b)y'_x + k(ce^{\lambda x} + d)y = 0.$

1°. Particular solutions with $k > 0$: $y_1 = \cos\left(x\sqrt{k}\right)$, $y_2 = \sin\left(x\sqrt{k}\right)$.

2°. Particular solutions with $k < 0$: $y_1 = \exp\left(-x\sqrt{-k}\right)$, $y_2 = \exp\left(x\sqrt{-k}\right)$.

54. $(ae^x + bx)y'''_{xxx} - ae^xy = 0.$

Particular solution: $y_0 = ae^x + bx$.

55. $(ae^x + bx^2)y'''_{xxx} - ae^xy = 0.$

Particular solution: $y_0 = ae^x + bx^2$.

56. $(axe^x + b)y'''_{xxx} + by = 0.$

Particular solution: $y_0 = ax + be^{-x}$.

57. $(ax^2e^x + b)y'''_{xxx} + by = 0.$

Particular solution: $y_0 = ax^2 + be^{-x}$.

3.1.4. Equations Containing Hyperbolic Functions

1. $\boldsymbol{y'''_{xxx} - a^3 \tanh(ax)y = 0}$.

Particular solution: $y_0 = \cosh(ax)$.

The substitution $y = \cosh(ax)\int z(x)\,dx$ leads to a second order equation of the form 2.1.4.44: $z''_{xx} + 3a\tanh(ax)z'_x + 3a^2 z = 0$.

2. $\boldsymbol{y'''_{xxx} - a^3 \coth(ax)y = 0}$.

Particular solution: $y_0 = \sinh(ax)$.

The substitution $y = \sinh(ax)\int z(x)\,dx$ leads to a second order equation of the form 2.1.4.45: $z''_{xx} + 3a\coth(ax)z'_x + 3a^2 z = 0$.

3. $\boldsymbol{y'''_{xxx} - 3a^2 y'_x + 2a^3 \tanh(ax)y = 0}$.

Particular solutions: $y_1 = \cosh(ax)$, $y_2 = x\cosh(ax)$.

4. $\boldsymbol{y'''_{xxx} - 3a^2 y'_x + 2a^3 \coth(ax)y = 0}$.

Particular solutions: $y_1 = \sinh(ax)$, $y_2 = x\sinh(ax)$.

5. $\boldsymbol{y'''_{xxx} + [a\cosh(2x) + b]y'_x + a\sinh(2x)y = 0}$.

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 form a fundamental set of solutions of the modified Mathieu equation 2.1.4.1: $4w''_{xx} + [a\cosh(2x) + b]w = 0$.

6. $\boldsymbol{y'''_{xxx} + ay''_{xx} + b\cosh(2x)y'_x + ab\cosh(2x)y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.4.1: $w''_{xx} + b\cosh(2x)w = 0$.

7. $\boldsymbol{y'''_{xxx} + ay''_{xx} + b\cosh^2 x\, y'_x + ab\cosh^2 x\, y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.4.2: $w''_{xx} + b\cosh^2 x\, w = 0$.

8. $\boldsymbol{y'''_{xxx} + ay''_{xx} + b\sinh^2 x\, y'_x + ab\sinh^2 x\, y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.4.5: $w''_{xx} + b\sinh^2 x\, w = 0$.

9. $\boldsymbol{y'''_{xxx} + ay''_{xx} + [b\tanh(\lambda x) + c]y'_x + a[b\tanh(\lambda x) + c]y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.4.6: $w''_{xx} + [b\tanh(\lambda x) + c]w = 0$.

10. $\boldsymbol{y'''_{xxx} + ay''_{xx} - \lambda[2a\tanh(\lambda x) + 3\lambda]y'_x + \lambda^2[2a\tanh^2(\lambda x) + 2\lambda\tanh(\lambda x) - a]y = 0}$.

Particular solutions: $y_1 = \cosh(\lambda x)$, $y_2 = x\cosh(\lambda x)$.

11. $y'''_{xxx} + ay''_{xx} + [b\coth(\lambda x) + c]y'_x + a[b\coth(\lambda x) + c]y = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.4.10: $w''_{xx} + [b\coth(\lambda x) + c]w = 0$.

12. $y'''_{xxx} + ay''_{xx} - \lambda[2a\coth(\lambda x) + 3\lambda]y'_x + \lambda^2[2a\coth^2(\lambda x) + 2\lambda\coth(\lambda x) - a]y = 0.$

Particular solutions: $y_1 = \sinh(\lambda x)$, $y_2 = x\sinh(\lambda x)$.

13. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + by'_x + ab\cosh^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

14. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cosh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first order linear equation: $w'_x + a\cosh^n(\lambda x)w = 0$.

15. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} + ab\cosh^n(\lambda x)y'_x + b^2[a\cosh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

16. $y'''_{xxx} + a\cosh^n(\lambda x)y''_{xx} - b[2a\cosh^n(\lambda x) + 3b]y'_x + b^2[a\cosh^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

17. $y'''_{xxx} + a\cosh^n x\, y''_{xx} + (ab\cosh^n x + c - b^2)y'_x + c(a\cosh^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

18. $y'''_{xxx} + ax\cosh^n x\, y''_{xx} + (bx^2 - a\cosh^n x)y'_x + bx(ax^2\cosh^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\,\bigr)$.

19. $y'''_{xxx} + ax^2\cosh^n(\lambda x)y''_{xx} - 2ax\cosh^n(\lambda x)y'_x + 2a\cosh^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

20. $y'''_{xxx} = (\cosh^n x - a)y''_{xx} + (a\cosh^n x - b)y'_x + b\cosh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

21. $y'''_{xxx} + (a\cosh^n x + bx)y''_{xx} + b(ax\cosh^n x + 2)y'_x + ab\cosh^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\,dx$.

22. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + by'_x + ab\sinh^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

23. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\sinh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first order linear equation: $w'_x + a\sinh^n(\lambda x)w = 0$.

24. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} + ab\sinh^n(\lambda x)y'_x + b^2[a\sinh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

25. $y'''_{xxx} + a\sinh^n(\lambda x)y''_{xx} - b[2a\sinh^n(\lambda x) + 3b]y'_x + b^2[a\sinh^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

26. $y'''_{xxx} + a\sinh^n x\, y''_{xx} + (ab\sinh^n x + c - b^2)y'_x + c(a\sinh^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

27. $y'''_{xxx} + ax\sinh^n x\, y''_{xx} + (bx^2 - a\sinh^n x)y'_x + bx(ax^2\sinh^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\left(\frac{1}{2}x^2\sqrt{b}\right)$, $y_2 = \sin\left(\frac{1}{2}x^2\sqrt{b}\right)$.

28. $y'''_{xxx} + ax^2\sinh^n(\lambda x)y''_{xx} - 2ax\sinh^n(\lambda x)y'_x + 2a\sinh^n(\lambda x)y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

29. $y'''_{xxx} = (\sinh^n x - a)y''_{xx} + (a\sinh^n x - b)y'_x + b\sinh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

30. $y'''_{xxx} + (a\sinh^n x + bx)y''_{xx} + b(ax\sinh^n x + 2)y'_x + ab\sinh^n x\, y = 0.$

Particular solutions: $y_1 = \exp\left(-\frac{1}{2}bx^2\right)$, $y_2 = \exp\left(-\frac{1}{2}bx^2\right)\displaystyle\int \exp\left(\tfrac{1}{2}bx^2\right)dx$.

31. $y'''_{xxx} - \tanh x\, y''_{xx} - ay'_x + a\tanh x\, y = 0.$

1°. Solution with $a > 0$: $y = C_1\exp(-x\sqrt{a}) + C_2\exp(x\sqrt{a}) + C_3\cosh x$.

2°. Solution with $a < 0$: $y = C_1\cos(x\sqrt{-a}) + C_2\sin(x\sqrt{-a}) + C_3\cosh x$.

32. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\tanh^n(\lambda x) + m]y = 0.$

The substitution $w = y''_{xx} + bx^m y$ leads to a first order linear equation: $w'_x + a\tanh^n(\lambda x)w = 0$.

33. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} + ab\tanh^n(\lambda x)y'_x + b^2[a\tanh^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

34. $y'''_{xxx} + a\tanh^n(\lambda x)y''_{xx} - b[2a\tanh^n(\lambda x) + 3b]y'_x + b^2[a\tanh^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

35. $y'''_{xxx}+a\tanh^n(\lambda x)y''_{xx}+[ab\tanh^n(\lambda x)+c-b^2]y'_x+c[a\tanh^n(\lambda x)-b]y=0.$

Particular solutions: $y_1=\exp(\beta_1 x)$, $y_2=\exp(\beta_2 x)$, where β_1 and β_2 are the roots of the quadratic equation $\beta^2+b\beta+c=0$.

36. $y'''_{xxx}+ax^n y''_{xx}-(2ax^n\tanh x+3)y'_x+[ax^n(2\tanh^2 x-1)+2\tanh x]y=0.$

Particular solutions: $y_1=\cosh x$, $y_2=x\cosh x$.

37. $y'''_{xxx}+a\tanh^n x\, y''_{xx}-(2a\tanh^{n+1}x+3)y'_x$
$$+(2a\tanh^{n+2}x-a\tanh^n x+2\tanh x)y=0.$$

Particular solutions: $y_1=\cosh x$, $y_2=x\cosh x$.

38. $y'''_{xxx}+ax\tanh^n x\, y''_{xx}+(bx^2-a\tanh^n x)y'_x+bx(ax^2\tanh^n x+3)y=0.$

Particular solutions: $y_1=\cos\left(\frac{1}{2}x^2\sqrt{b}\right)$, $y_2=\sin\left(\frac{1}{2}x^2\sqrt{b}\right)$.

39. $y'''_{xxx}+ax^2\tanh^n(\lambda x)y''_{xx}-2ax\tanh^n(\lambda x)y'_x+2a\tanh^n(\lambda x)y=0.$

Particular solutions: $y_1=x$, $y_2=x^2$.

40. $y'''_{xxx}=(\tanh^n x-a)y''_{xx}+(a\tanh^n x-b)y'_x+b\tanh^n x\, y.$

Particular solutions: $y_1=\exp(\lambda_1 x)$, $y_2=\exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2+a\lambda+b=0$.

41. $y'''_{xxx}+(a\tanh^n x+bx)y''_{xx}+b(ax\tanh^n x+2)y'_x+ab\tanh^n x\, y=0.$

Particular solutions: $y_1=\exp\left(-\frac{1}{2}bx^2\right)$, $y_2=\exp\left(-\frac{1}{2}bx^2\right)\int\exp\left(\frac{1}{2}bx^2\right)dx$.

42. $y'''_{xxx}+[ax^n(\tanh x-b)-b]y''_{xx}+[a(b^2-1)x^n-1]y'_x$
$$+b[ax^n(1-b\tanh x)+1]y=0.$$

Particular solutions: $y_1=e^{bx}$, $y_2=\cosh x$.

43. $y'''_{xxx}+[\lambda\tanh(\lambda x)(ax^n-1)-ax^{n-1}]y''_{xx}-a\lambda^2x^n y'_x+a\lambda^2x^{n-1}y=0.$

Particular solutions: $y_1=x$, $y_2=\cosh(\lambda x)$.

44. $y'''_{xxx}+(a\tanh^{n+1}x-ab\tanh^n x-b)y''_{xx}$
$$+[a(b^2-1)\tanh^n x-1]y'_x+b(-ab\tanh^{n+1}x+a\tanh^n x+1)y=0.$$

Particular solutions: $y_1=e^{bx}$, $y_2=\cosh x$.

45. $y'''_{xxx}-\coth x\, y''_{xx}-ay'_x+a\coth x\, y=0.$

1°. Solution with $a>0$: $y=C_1\exp(-x\sqrt{a})+C_2\exp(x\sqrt{a})+C_3\sinh x$.

2°. Solution with $a<0$: $y=C_1\cos(x\sqrt{-a})+C_2\sin(x\sqrt{-a})+C_3\sinh x$.

46. $y'''_{xxx}+(a\coth x-ab-b)y''_{xx}+(ab^2-a-1)y'_x+b(-ab\coth x+a+1)y=0.$

Particular solutions: $y_1=e^{bx}$, $y_2=\sinh x$.

47. $\boldsymbol{y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\coth^n(\lambda x) + m]y = 0.}$

The substitution $w = y''_{xx} + bx^m y$ leads to a first order linear equation: $w'_x + a\coth^n(\lambda x)w = 0$.

48. $\boldsymbol{y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} + ab\coth^n(\lambda x)y'_x + b^2[a\coth^n(\lambda x) - b]y = 0.}$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

49. $\boldsymbol{y'''_{xxx} + a\coth^n(\lambda x)y''_{xx} - b[2a\coth^n(\lambda x) + 3b]y'_x + b^2[a\coth^n(\lambda x) + 2b]y = 0.}$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

50. $\boldsymbol{y'''_{xxx} + ax\coth^n x\, y''_{xx} + (bx^2 - a\coth^n x)y'_x + bx(ax^2\coth^n x + 3)y = 0.}$

Particular solutions: $y_1 = \cos\left(\frac{1}{2}x^2\sqrt{b}\right)$, $y_2 = \sin\left(\frac{1}{2}x^2\sqrt{b}\right)$.

51. $\boldsymbol{y'''_{xxx} + ax^2\coth^n(\lambda x)y''_{xx} - 2ax\coth^n(\lambda x)y'_x + 2a\coth^n(\lambda x)y = 0.}$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

52. $\boldsymbol{y'''_{xxx} + ax^n y''_{xx} - (2ax^n\coth x + 3)y'_x + [ax^n(2\coth^2 x - 1) + 2\coth x]y = 0.}$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

53. $\boldsymbol{y'''_{xxx} + a\coth^n x\, y''_{xx} - (2a\coth^{n+1} x + 3)y'_x}$
$$\boldsymbol{+ (2a\coth^{n+2} x - a\coth^n x + 2\coth x)y = 0.}$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

54. $\boldsymbol{y'''_{xxx} + (a\coth^n x + bx)y''_{xx} + b(ax\coth^n x + 2)y'_x + ab\coth^n x\, y = 0.}$

Particular solutions: $y_1 = \exp\left(-\frac{1}{2}bx^2\right)$, $y_2 = \exp\left(-\frac{1}{2}bx^2\right)\displaystyle\int \exp\left(\tfrac{1}{2}bx^2\right)dx$.

55. $\boldsymbol{y'''_{xxx} + [\lambda\coth(\lambda x)(ax^n - 1) - ax^{n-1}]y''_{xx} - a\lambda^2 x^n y'_x + a\lambda^2 x^{n-1} y = 0.}$

Particular solutions: $y_1 = x$, $y_2 = \sinh(\lambda x)$.

56. $\boldsymbol{xy'''_{xxx} + x[a\cosh(2x) + b]y'_x - 2[a\cosh(2x) + b]y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.4.1: $w''_{xx} + [a\cosh(2x) + b]w = 0$.

57. $\boldsymbol{xy'''_{xxx} + x(a\cosh^2 x + b)y'_x - 2(a\cosh^2 x + b)y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.4.2: $w''_{xx} + (a\cosh^2 x + b)w = 0$.

58. $\boldsymbol{xy'''_{xxx} + x(a\sinh^2 x + b)y'_x - 2(a\sinh^2 x + b)y = 0.}$

The substitution $w = xy'_x - 2y$ leads to a second order equation of the form 2.1.4.5: $w''_{xx} + (a\sinh^2 x + b)w = 0$.

59. $xy'''_{xxx} = (\cosh^n x - ax)y''_{xx} + (a\cosh^n x - bx)y'_x + b\cosh^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

60. $x^2 y'''_{xxx} + (ax^2\cosh^n x + bx)y''_{xx} + [a(b-2)x\cosh^n x + c]y'_x + a(c-b+2)\cosh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

61. $x^2 y'''_{xxx} + (ax^2\sinh^n x + bx)y''_{xx} + [a(b-2)x\sinh^n x + c]y'_x + a(c-b+2)\sinh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

62. $x^2 y'''_{xxx} + (ax^2\tanh^n x + bx)y''_{xx} + [a(b-2)x\tanh^n x + c]y'_x + a(c-b+2)\tanh^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

63. $x^2 y'''_{xxx} + (ax^2\coth^n x + bx)y''_{xx} + [a(b-2)x\coth^n x + c]y'_x + a(c-b+2)\coth^n x\, y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

64. $x^3 y'''_{xxx} + x^2(a\cosh^n x + b)y''_{xx} + x(ab\cosh^n x + c - b)y'_x + c(a\cosh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

65. $x^3 y'''_{xxx} + x^2(a\sinh^n x + b)y''_{xx} + x(ab\sinh^n x + c - b)y'_x + c(a\sinh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

66. $x^3 y'''_{xxx} + x^2(a\tanh^n x + b)y''_{xx} + x(ab\tanh^n x + c - b)y'_x + c(a\tanh^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

67. $x^3 y'''_{xxx} + x^2(a\coth^n x + b)y''_{xx} + x(ab\coth^n x + c - b)y'_x + c(a\coth^n x - 2)y = 0.$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

68. $\cosh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cosh^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

69. $\cosh^n x\, y'''_{xxx} + a y''_{xx} + b\cosh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b>0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b<0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

70. $\cosh^n x\, y'''_{xxx} + a y''_{xx} - b(2a + 3b\cosh^n x)y'_x + b^2(a + 2b\cosh^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

71. $\cosh^n x\, y'''_{xxx} + y''_{xx} + [(b-a^2)\cosh^n x + a]y'_x + b(1 - a\cosh^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

72. $\cosh^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

73. $\cosh^n x\, y'''_{xxx} + (a\cosh^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

74. $\cosh^n x\, y'''_{xxx} + (ax\cosh^n x + 1)y''_{xx} + a(x + 2\cosh^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\frac{ax^2}{2}\Bigr)\,dx$.

75. $x\cosh^n x\, y'''_{xxx} + (3\cosh^n x + x)y''_{xx} + (ax\cosh^n x + 2)y'_x + a(\cosh^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr)$.

76. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cosh^n x\, y'_x + 2(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

77. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\cosh^n x\, y'_x + 6(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

78. $x^3\cosh^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cosh^n x)y'_x + a(a - 3\cosh^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|)$.

79. $x^3\cosh^n x\, y'''_{xxx} + x^2(\cosh^n x + a)y''_{xx} + x[a - (b+1)\cosh^n x]y'_x + b(2\cosh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

80. $\sinh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\sinh^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\Bigl(\dfrac{bx\sqrt{3}}{2}\Bigr)$, $y_2 = e^{-bx/2}\sin\Bigl(\dfrac{bx\sqrt{3}}{2}\Bigr)$.

81. $\sinh^n x\, y'''_{xxx} + ay''_{xx} + b\sinh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

82. $\sinh^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\sinh^n x)y'_x + b^2(a + 2b\sinh^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

83. $\sinh^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\sinh^n x + a]y'_x + b(1 - a\sinh^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

84. $\sinh^n(\lambda x)y'''_{xxx} + ax^2y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

85. $\sinh^n x\, y'''_{xxx} + (a\sinh^n x + ax + 1)y''_{xx} + a^2xy'_x - a^2y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

86. $\sinh^n x\, y'''_{xxx} + (ax\sinh^n x + 1)y''_{xx} + a(x + 2\sinh^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\,dx$.

87. $x\sinh^n x\, y'''_{xxx} + (3\sinh^n x + x)y''_{xx} + (ax\sinh^n x + 2)y'_x + a(\sinh^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr)$.

88. $x^3\sinh^n x\, y'''_{xxx} + ax^2y''_{xx} - 2x\sinh^n x\, y'_x + 2(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

89. $x^3\sinh^n x\, y'''_{xxx} + ax^2y''_{xx} - 6x\sinh^n x\, y'_x + 6(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

90. $x^3\sinh^n x\, y'''_{xxx} + ax^2y''_{xx} + x(a - \sinh^n x)y'_x + a(a - 3\sinh^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

91. $x^3\sinh^n x\, y'''_{xxx} + x^2(\sinh^n x + a)y''_{xx} +$
$x[a - (b+1)\sinh^n x]y'_x + b(2\sinh^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

92. $\tanh^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\tanh^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

93. $\tanh^n x\, y'''_{xxx} + a y''_{xx} + b\tanh^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

94. $\tanh^n x\, y'''_{xxx} + a y''_{xx} - b(2a + 3b\tanh^n x) y'_x + b^2(a + 2b\tanh^n x) y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = x e^{bx}$.

95. $\tanh^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\tanh^n x + a] y'_x + b(1 - a\tanh^n x) y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

96. $\tanh^n(\lambda x) y'''_{xxx} + a x^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

97. $\tanh^n x\, y'''_{xxx} + (a\tanh^n x + ax + 1) y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

98. $\tanh^n x\, y'''_{xxx} + (ax\tanh^n x + 1) y''_{xx} + a(x + 2\tanh^n x) y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr) \displaystyle\int \exp\Bigl(\frac{ax^2}{2}\Bigr)\, dx$.

99. $x\tanh^n x\, y'''_{xxx} + (3\tanh^n x + x) y''_{xx} + (ax\tanh^n x + 2) y'_x + a(\tanh^n x + x) y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr)$.

100. $x^3\tanh^n x\, y'''_{xxx} + a x^2 y''_{xx} - 2x\tanh^n x\, y'_x + 2(2\tanh^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

101. $x^3\tanh^n x\, y'''_{xxx} + a x^2 y''_{xx} - 6x\tanh^n x\, y'_x + 6(2\tanh^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

102. $x^3\tanh^n x\, y'''_{xxx} + a x^2 y''_{xx} + x(a - \tanh^n x) y'_x + a(a - 3\tanh^n x) y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|)$.

103. $x^3\tanh^n x\, y'''_{xxx} + x^2(\tanh^n x + a) y''_{xx} + x[a - (b+1)\tanh^n x] y'_x + b(2\tanh^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

104. $\coth^n x\, y'''_{xxx} + a y''_{xx} + aby'_x + b^2(a - b\coth^n x) y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\Bigl(\dfrac{bx\sqrt{3}}{2}\Bigr)$, $y_2 = e^{-bx/2}\sin\Bigl(\dfrac{bx\sqrt{3}}{2}\Bigr)$.

105. $\coth^n x\, y'''_{xxx} + a y''_{xx} + b \coth^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

106. $\coth^n x\, y'''_{xxx} + a y''_{xx} - b(2a + 3b \coth^n x) y'_x + b^2(a + 2b \coth^n x) y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = x e^{bx}$.

107. $\coth^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2) \coth^n x + a] y'_x + b(1 - a \coth^n x) y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

108. $\coth^n(\lambda x) y'''_{xxx} + a x^2 y''_{xx} - 2 a x y'_x + 2 a y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

109. $\coth^n x\, y'''_{xxx} + (a \coth^n x + a x + 1) y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

110. $\coth^n x\, y'''_{xxx} + (a x \coth^n x + 1) y''_{xx} + a(x + 2 \coth^n x) y'_x + a y = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr) \displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\, dx$.

111. $x \coth^n x\, y'''_{xxx} + (3 \coth^n x + x) y''_{xx} + (a x \coth^n x + 2) y'_x + a(\coth^n x + x) y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x} \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x} \sin\bigl(x\sqrt{a}\,\bigr)$.

112. $x^3 \coth^n x\, y'''_{xxx} + a x^2 y''_{xx} - 2x \coth^n x\, y'_x + 2(2 \coth^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

113. $x^3 \coth^n x\, y'''_{xxx} + a x^2 y''_{xx} - 6x \coth^n x\, y'_x + 6(2 \coth^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

114. $x^3 \coth^n x\, y'''_{xxx} + a x^2 y''_{xx} + x(a - \coth^n x) y'_x + a(a - 3 \coth^n x) y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

115. $x^3 \coth^n x\, y'''_{xxx} + x^2(\coth^n x + a) y''_{xx} + x[a - (b+1) \coth^n x] y'_x + b(2 \coth^n x - a) y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

3.1.5. Equations Containing Logarithmic Functions

1. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + by'_x + ab\ln^n(\lambda x)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

2. $y'''_{xxx} + a\ln^n x\, y''_{xx} + ab\ln^n x\, y'_x + b^2(a\ln^n x - b)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

3. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (bx + c)\, y'_x + (abx\ln^n x + ac\ln^n x + b)y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + (bx + c)y = C\exp\Bigl(-a\int \ln^n x\, dx\Bigr)$$

(see 2.1.2.2 for the solution of the corresponding homogeneous equation).

4. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} - b[2a\ln^n(\lambda x) + 3b]y'_x + b^2[a\ln^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

5. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (ab\ln^n x + c - b^2)y'_x + c(a\ln^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

6. $y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\ln^n(\lambda x) + m]y = 0.$

Assuming $w = y''_{xx} + bx^m y$, we obtain a first order linear equation: $w'_x + a\ln^n(\lambda x)w = 0$.

7. $y'''_{xxx} + (a\ln^n x + b)y''_{xx} + c\, y'_x + c(a\ln^n + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr), y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

8. $y'''_{xxx} = (\ln^n x - a)y''_{xx} + (a\ln^n x - b)y'_x + b\ln^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

9. $y'''_{xxx} + ax\ln^n x\, y''_{xx} + (bx^2 - a\ln^n x)y'_x + bx(ax^2\ln^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr)$.

10. $y'''_{xxx} + (a\ln^n x + bx)y''_{xx} + b(ax\ln^n x + 2)y'_x + ab\ln^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr)$, $y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\, dx$.

11. $\boldsymbol{y'''_{xxx} + (ax\ln^n x + b)y''_{xx} + a(bx-1)\ln^n x\, y'_x - ab\ln^n x\, y = 0.}$

The substitution $w = y'_x + by$ leads to a second order linear equation of the form 2.1.5.5: $w''_{xx} + ax\ln^n x\, w'_x - a\ln^n x\, w = 0$.

12. $\boldsymbol{y'''_{xxx} + (abx\ln^n x + a\ln^n x + b)y''_{xx} + ab^2x\ln^n x\, y'_x - ab^2\ln^n x\, y = 0.}$

Particular solutions: $y_1 = x, \quad y_2 = e^{-bx}$.

13. $\boldsymbol{y'''_{xxx} + ax^2\ln^n(\lambda x)y''_{xx} - 2ax\ln^n(\lambda x)y'_x + 2a\ln^n(\lambda x)y = 0.}$

Particular solutions: $y_1 = x, \quad y_2 = x^2$.

14. $\boldsymbol{xy'''_{xxx} + axy''_{xx} - b(bx\ln^2 x + 1)y'_x - ab(bx\ln^2 x + 1)y = 0.}$

The substitution $w = y'_x + ay$ leads to an equation of the form 2.1.5.7: $xw''_{xx} - (b^2x\ln^2 x + b)w = 0$.

15. $\boldsymbol{xy'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bxy'_x + ab\ln^n(\lambda x)y = 0.}$

1°. Particular solutions with $b>0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr), \quad y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b<0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr), \quad y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

16. $\boldsymbol{xy'''_{xxx} + ax\ln x\, y''_{xx} + (abx\ln x - b^2x + a)y'_x + aby = 0.}$

Particular solutions: $y_1 = e^{-bx}, \quad y_2 = e^{-bx}\displaystyle\int x^{-ax}e^{(a+2b)x}\,dx$.

17. $\boldsymbol{xy'''_{xxx} = (\ln^n x - ax)y''_{xx} + (a\ln^n x - bx)y'_x + b\ln^n x\, y.}$

Particular solutions: $y_1 = \exp(\lambda_1 x), \quad y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

18. $\boldsymbol{xy'''_{xxx} + (ax\ln^n x + 3)y''_{xx} + (2a\ln^n x + bx)y'_x + b(ax\ln^n x + 1)y = 0.}$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\,\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\,\bigr)$.

19. $\boldsymbol{xy'''_{xxx} + (ax\ln^n x + 3)y''_{xx} + (abx\ln^n x + 2a\ln^n x - b^2x)y'_x + b(a\ln^n x - b)y = 0.}$

Particular solutions: $y_1 = \dfrac{1}{x}, \quad y_2 = \dfrac{1}{x}e^{-bx}$.

20. $\boldsymbol{xy'''_{xxx} + [a(b - \ln x)x^n + 2]y''_{xx} + ax^{n-1}y'_x - ax^{n-2}y = 0.}$

Particular solutions: $y_1 = x, \quad y_2 = \ln x - b + 1$.

21. $\boldsymbol{x^2y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^2y'_x + ab\ln^n(\lambda x)y = 0.}$

1°. Particular solutions with $b>0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr), \quad y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b<0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr), \quad y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

22. $\boldsymbol{x^2 y'''_{xxx} + x^2(a\ln x + b)y''_{xx} + 2axy'_x - ay = 0}$.

Integrating the equation twice, we obtain a first order linear equation:

$$y'_x + (a\ln x + b)y = C_1 + C_2 x.$$

23. $\boldsymbol{x^2 y'''_{xxx} - 3ax[ax\ln^2(\lambda x) + 1]y'_x + [2a^2x^2\ln^3(\lambda x) + 1]y = 0}$.

Particular solutions: $y_1 = \exp\left[a\int \ln(\lambda x)\,dx\right]$, $y_2 = x\exp\left[a\int \ln(\lambda x)\,dx\right]$.

24. $\boldsymbol{x^2 y'''_{xxx} + x^2(a\ln x + bx)y''_{xx} + 2x(bx + a)y'_x - ay = 0}$.

Integrating the equation twice, we obtain a first order linear equation:

$$y'_x + (a\ln x + bx)y = C_1 + C_2 x.$$

25. $\boldsymbol{x^2 y'''_{xxx} + (ax^2\ln^n x + bx)y''_{xx} + [a(b-2)x\ln^n x + c]y'_x + a(c-b+2)\ln^n x\, y = 0}$.

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-3)m + c - b + 2 = 0$.

26. $\boldsymbol{x^3 y'''_{xxx} + x^2(a\ln x + b)y''_{xx} + 2axy'_x - ax = 0}$.

Integrating the equation twice, we obtain a first order linear equation:

$$xy'_x + (a\ln x + b - 2)y = C_1 + C_2 x.$$

27. $\boldsymbol{x^3 y'''_{xxx} + a\ln^n(\lambda x)y''_{xx} + bx^3 y'_x + ab\ln^n(\lambda x)y = 0}$.

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

28. $\boldsymbol{x^3 y'''_{xxx} + ax^2\ln^n x\, y''_{xx} - 2xy'_x + 2(2 - a\ln^n x)y = 0}$.

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

29. $\boldsymbol{x^3 y'''_{xxx} + ax^2\ln^n x\, y''_{xx} - 6xy'_x + 6(2 - a\ln^n x)y = 0}$.

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

30. $\boldsymbol{x^3 y'''_{xxx} + x^2(a\ln x + bx)y''_{xx} + 2x(bx + a)y'_x - ay = 0}$.

Integrating the equation twice, we obtain a first order linear equation:

$$xy'_x + (a\ln x + bx - 2)y = C_1 + C_2 x.$$

31. $\boldsymbol{x^3 y'''_{xxx} + x^2(a\ln^n x + b)y''_{xx} + x(ab\ln^n x + c - b)y'_x + c(a\ln^n x - 2)y = 0}$.

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 + (b-1)m + c = 0$.

32. $\boldsymbol{\ln^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\ln^n x)y = 0}$.

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

33. $\ln^n x\, y'''_{xxx} + +(ax \ln^n x + 1)y''_{xx} + a(x + 2\ln^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr) \displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\, dx.$

34. $\ln^n x\, y'''_{xxx} + (a\ln^n x + ax + 1)y''_{xx} + a^2 x\, y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

35. $\ln^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

36. $\ln^n x\, y'''_{xxx} + y''_{xx} + [(a - b^2)\ln^n x + b]y'_x + a(1 - b\ln^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + a = 0$.

37. $\ln^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\ln^n x)y'_x + b^2(a + 2b\ln^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

38. $\ln^n(\lambda x) y'''_{xxx} + ay''_{xx} + b\ln^n(\lambda x)y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

3.1.6. Equations Containing Trigonometric Functions

1. $y'''_{xxx} + a^3 \tan(ax)\, y = 0.$

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{\xi\xi} + \tan\xi\, y'_\xi - y = \frac{C}{\cos\xi}, \qquad \xi = ax$$

(see 2.1.6.29 for the solution of the corresponding homogeneous equation).

2. $y'''_{xxx} - a^3 \tan(ax)\, y = 0.$

Particular solution: $y_0 = \cos(ax)$.

The substitution $y = \cos(ax)\int z(x)\,dx$ leads to a second order linear equation of the form 2.1.6.29: $z''_{\xi\xi} - 3\tan\xi\, z'_\xi - 3z = 0$, where $\xi = ax$.

3. $y'''_{xxx} + a^3 \cot(ax)\, y = 0.$

The substitution $x = t + \dfrac{\pi}{2a}$ leads to a linear equation of the form 3.1.6.2: $y'''_{ttt} - a^3\tan(at)\, y = 0$.

4. $y'''_{xxx} - a^3 \cot(ax)\, y = 0.$

The substitution $x = t + \dfrac{\pi}{2a}$ leads to a linear equation of the form 3.1.6.1: $y'''_{ttt} + a^3\tan(at)\, y = 0$.

5. $\boldsymbol{y'''_{xxx} + 3a^2 y'_x + 2a^3 \tan(ax)\, y = 0}$.

Particular solutions: $y_1 = \cos(ax), \quad y_2 = x\cos(ax)$.

6. $\boldsymbol{y'''_{xxx} + a y'_x + \lambda(a - \lambda^2)\tan(\lambda x)\, y = 0}$.

Particular solution: $y_0 = \cos(\lambda x)$.

The substitution $y = \cos(\lambda x)\int z(x)\,dx$ leads to a second order equation of the form 2.2.6.29: $z''_{\xi\xi} - 3\tan\xi\, z'_\xi + (a\lambda^{-2} - 3)z = 0$, where $\xi = \lambda x$.

7. $\boldsymbol{y'''_{xxx} + 3a^2 y'_x - 2a^3 \cot(ax)\, y = 0}$.

Particular solutions: $y_1 = \sin(ax), \quad y_2 = x\sin(ax)$.

8. $\boldsymbol{y'''_{xxx} + a\cos(2x) y'_x - b[a\cos(2x) + b^2] y = 0}$.

The substitution $w = e^{bx/2}(y'_x - by)$ leads to the Mathieu equation 2.1.6.4: $w''_{xx} + [a\cos(2x) + \frac{3}{4}b^2]w = 0$.

9. $\boldsymbol{y'''_{xxx} + [a\cos(2x) + b] y'_x - a\sin(2x)\, y = 0}$.

Solution:

$$y = C_1 w_1^2 + C_2 w_1 w_2 + C_3 w_2^2,$$

where w_1 and w_2 is the fundamental set of solutions of the Mathieu equation 2.1.6.4: $4w''_{xx} + [a\cos(2x) + b]w = 0$.

10. $\boldsymbol{y'''_{xxx} + [a\cos(2x) + b] y'_x - 2a\sin(2x)\, y = 0}$.

By integrating, we obtain a nonhomogeneous Mathieu equation:

$$y''_{xx} + [a\cos(2x) + b]y = C.$$

11. $\boldsymbol{y'''_{xxx} + [a\cos(\lambda x) + b] y'_x - c[a\cos(\lambda x) + b + c^2] y = 0}$.

The transformation $\xi = \frac{1}{2}\lambda x$, $w = e^{cx/2}(y'_x - cy)$ leads to the Mathieu equation 2.1.6.4: $w''_{\xi\xi} + 4\lambda^{-2}[a\cos(2\xi) + b + \frac{3}{4}c^2]w = 0$.

12. $\boldsymbol{y'''_{xxx} + [a\cos(2x) - b^2] y'_x - a[b\cos(2x) + 2\sin(2x)] y = 0}$.

By integrating and substituting $w = ye^{bx/2}$, we obtain a nonhomogeneous Mathieu equation: $w''_{xx} + [a\cos(2x) - \frac{1}{4}b^2]w = Ce^{3bx/2}$.

13. $\boldsymbol{y'''_{xxx} + a\sin x\, y'_x - b(a\sin x + b^2) y = 0}$.

The substitution $w = e^{bx/2}(y'_x - by)$ leads to an equation of the form 2.1.6.3: $w''_{xx} + (a\sin x + \frac{3}{4}b^2)w = 0$.

14. $\boldsymbol{y'''_{xxx} + a\sin^2 x\, y'_x - b(a\sin^2 x + b^2) y = 0}$.

The substitution $w = e^{bx/2}(y'_x - by)$ leads to an equation of the form 2.1.6.5: $w''_{xx} + (a\sin^2 x + \frac{3}{4}b^2)w = 0$.

15. $\boldsymbol{y'''_{xxx} + [a\sin(\lambda x) + b]y'_x + a\lambda\cos(\lambda x)\, y = 0}$.

By integrating, we obtain a nonhomogeneous second-order linear equation:

$$y''_{xx} + [a\sin(\lambda x) + b]y = C$$

(see 2.1.6.3 for the solution of the corresponding homogeneous equation).

16. $\boldsymbol{y'''_{xxx} + [a\sin(\lambda x) - b^2]y'_x + a[\lambda\cos(\lambda x) - b\sin(\lambda x)]y = 0}$.

By integrating and substituting $w = ye^{bx/2}$, we obtain a nonhomogeneous second-order linear equation:

$$w''_{xx} + [a\sin(\lambda x) - \tfrac{1}{4}b^2]w = Ce^{3bx/2}$$

(see 2.1.6.3 for the solution of the corresponding homogeneous equation).

17. $\boldsymbol{y'''_{xxx} + [a\sin(\lambda x) + b]y'_x - c[a\sin(\lambda x) + b + c^2]y = 0}$.

The substitution $w = e^{cx/2}(y'_x - cy)$ leads to an equation of the form 2.1.6.3: $w''_{xx} + [a\sin(\lambda x) + b + \frac{3}{4}c^2]w = 0$.

18. $\boldsymbol{y'''_{xxx} - 3a[a\sin^2(bx) + b\cos(bx)]y'_x + a\sin(bx)[b^2 + 2a^2\sin^2(bx)]y = 0}$.

Particular solutions: $y_1 = \exp\Bigl[-\dfrac{a}{b}\cos(bx)\Bigr]$, $y_2 = x\exp\Bigl[-\dfrac{a}{b}\cos(bx)\Bigr]$.

19. $\boldsymbol{y'''_{xxx} + a\tan^2 x\, y'_x - b(a\tan^2 x + b^2)y = 0}$.

The substitution $w = e^{bx/2}(y'_x - by)$ leads to an equation of the form 2.1.6.10: $w''_{xx} + (a\tan^2 x + \frac{3}{4}b^2)w = 0$.

20. $\boldsymbol{y'''_{xxx} + [a\tan^2(\lambda x) + b]y'_x - c[a\tan^2(\lambda x) + b + c^2]y = 0}$.

The transformation $\xi = \lambda x$, $w = e^{cx/2}(y'_x - cy)$ leads to an equation of the form 2.1.6.10: $w''_{\xi\xi} + \lambda^{-2}(a\tan^2\xi + b + \frac{3}{4}c^2)w = 0$.

21. $\boldsymbol{y'''_{xxx} + a\cot^2 x\, y'_x - b(a\cot^2 x + b^2)y = 0}$.

The substitution $w = e^{bx/2}(y'_x - by)$ leads to an equation of the form 2.1.6.12: $w''_{xx} + (a\cot^2 x + \frac{3}{4}b^2)w = 0$.

22. $\boldsymbol{y'''_{xxx} + ay''_{xx} + (b\cos 2x + c)y'_x + a(b\cos 2x + c)y = 0}$.

The substitution $w = y'_x + ay$ reduces the original equation to the Mathieu equation 2.1.6.4: $w''_{xx} + (b\cos 2x + c)w = 0$.

23. $\boldsymbol{y'''_{xxx} + ay''_{xx} + [b\sin(\lambda x) + c]y'_x + a[b\sin(\lambda x) + c]y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order linear equation of the form 2.1.6.3: $w''_{xx} + [b\sin(\lambda x) + c]w = 0$.

24. $\boldsymbol{y'''_{xxx} + ay''_{xx} + b\sin^2(\lambda x)y'_x + ab\sin^2(\lambda x)\, y = 0}$.

The substitution $w = y'_x + ay$ leads to a second order linear equation of the form 2.1.6.5: $w''_{xx} + b\sin^2(\lambda x)\, w = 0$.

25. $y'''_{xxx} + ay''_{xx} + (b\tan^2 x + c)y'_x + a(b\tan^2 x + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second order linear equation of the form 2.1.6.10: $w''_{xx} + (b\tan^2 x + c)w = 0$.

26. $y'''_{xxx} + ay''_{xx} + \lambda[3\lambda + 2a\tan(\lambda x)]y'_x + \lambda^2\{a[1 + 2\tan^2(\lambda x)] + 2\lambda\tan(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cos(\lambda x)$, $y_2 = x\cos(\lambda x)$.

27. $y'''_{xxx} + ay''_{xx} + (b\cot^2 x + c)y'_x + a(b\cot^2 x + c)y = 0.$

The substitution $w = y'_x + ay$ leads to a second order linear equation of the form 2.1.6.12: $w''_{xx} + (b\cot^2 x + c)w = 0$.

28. $y'''_{xxx} + ay''_{xx} + \lambda[3\lambda - 2a\cot(\lambda x)]y'_x + \lambda^2\{a[1 + 2\cot^2(\lambda x)] - 2\lambda\cot(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sin(\lambda x)$, $y_2 = x\sin(\lambda x)$.

29. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + by'_x + ab\cos^n(\lambda x)\,y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

30. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + ab\cos^n(\lambda x)y'_x + b^2[a\cos^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

31. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} - b[2a\cos^n(\lambda x) + 3b]y'_x + b^2[a\cos^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

32. $y'''_{xxx} + a\cos^n x\, y''_{xx} + (ab\cos^n x + c - b^2)y'_x + c(a\cos^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

33. $y'''_{xxx} + a\cos^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cos^n(\lambda x) + m]y = 0.$

Assuming $w = y''_{xx} + bx^m y$ yields a first order linear equation: $w'_x + a\cos^n(\lambda x)\,w = 0$.

34. $y'''_{xxx} + (a\cos^n x + b)y''_{xx} + cy'_x + c(a\cos^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

35. $y'''_{xxx} = (\cos^n x - a)y''_{xx} + (a\cos^n x - b)y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

36. $y'''_{xxx} + (a\cos^n x + bx)y''_{xx} + b(ax\cos^n x + 2)y'_x + ab\cos^n x\, y = 0.$

Particular solutions: $y_1 = \exp(-\tfrac{1}{2}bx^2)$, $y_2 = \exp(-\tfrac{1}{2}bx^2)\displaystyle\int \exp(\tfrac{1}{2}bx^2)\,dx$.

37. $y'''_{xxx} + ax\cos^n x\, y''_{xx} + (bx^2 - a\cos^n x)y'_x + bx(ax^2\cos^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr), \quad y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr).$

38. $y'''_{xxx} + (abx\cos^n x + a\cos^n x + b)y''_{xx} + ab^2x\cos^n x\, y'_x - ab^2\cos^n x\, y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-bx}.$

39. $y'''_{xxx} + ax^2\cos^n(\lambda x)y''_{xx} - 2ax\cos^n(\lambda x)y'_x + 2a\cos^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2.$

40. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} - by'_x - ab\sin^n(\lambda x)\, y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\bigr), \quad y_2 = \exp\bigl(x\sqrt{b}\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr), \quad y_2 = \sin\bigl(x\sqrt{-b}\bigr).$

41. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} + ab\sin^n(\lambda x)y'_x + b^2[a\sin^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right), \quad y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right).$

42. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} - b[2a\sin^n(\lambda x) + 3b]y'_x + b^2[a\sin^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}, \quad y_2 = xe^{bx}.$

43. $y'''_{xxx} + a\sin^n x\, y''_{xx} + (ab\sin^n x + c - b^2)y'_x + c(a\sin^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x), \quad y_2 = \exp(\lambda_2 x),$ where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0.$

44. $y'''_{xxx} + a\sin^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\sin^n(\lambda x) + m]y = 0.$

Assuming $w = y''_{xx} + bx^m y$ yields a first order linear equation: $w'_x + a\sin^n(\lambda x)\, w = 0.$

45. $y'''_{xxx} + (a\sin^n x + b)y''_{xx} + cy'_x + c(a\sin^n + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr), \quad y_2 = \sin\bigl(x\sqrt{c}\bigr).$

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr), \quad y_2 = \exp\bigl(x\sqrt{-c}\bigr).$

46. $y'''_{xxx} = (\sin^n x - a)y''_{xx} + (a\sin^n x - b)y'_x + b\sin^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x), \quad y_2 = \exp(\lambda_2 x),$ where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0.$

47. $y'''_{xxx} + (a\sin^n x + bx)y''_{xx} + b(ax\sin^n x + 2)y'_x + ab\sin^n x\, y = 0.$

Particular solutions: $y_1 = \exp\bigl(-\frac{1}{2}bx^2\bigr), \quad y_2 = \exp\bigl(-\frac{1}{2}bx^2\bigr)\displaystyle\int \exp\bigl(\tfrac{1}{2}bx^2\bigr)\, dx.$

48. $y'''_{xxx} + ax\sin^n x\, y''_{xx} + (bx^2 - a\sin^n x)y'_x + bx(ax^2\sin^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\frac{1}{2}x^2\sqrt{b}\bigr), \quad y_2 = \sin\bigl(\frac{1}{2}x^2\sqrt{b}\bigr).$

49. $\boldsymbol{y'''_{xxx} + (abx\sin^n x + a\sin^n x + b)y''_{xx} + ab^2 x\sin^n x\, y'_x - ab^2\sin^n x\, y = 0}$.

Particular solutions: $\quad y_1 = x, \quad y_2 = e^{-bx}$.

50. $\boldsymbol{y'''_{xxx} + ax^2\sin^n(\lambda x)y''_{xx} - 2ax\sin^n(\lambda x)y'_x + 2a\sin^n(\lambda x)\, y = 0}$.

Particular solutions: $\quad y_1 = x, \quad y_2 = x^2$.

51. $\boldsymbol{y'''_{xxx} + \lambda\tan(\lambda x)y''_{xx} - ay'_x - a\lambda\tan(\lambda x)\, y = 0}$.

1°. Solution with $a > 0$: $\quad y = C_1\exp\bigl(-x\sqrt{a}\,\bigr) + C_2\exp\bigl(x\sqrt{a}\,\bigr) + C_3\cos(\lambda x)$.

2°. Solution with $a < 0$: $\quad y = C_1\cos\bigl(x\sqrt{-a}\,\bigr) + C_2\sin\bigl(x\sqrt{-a}\,\bigr) + C_3\cos(\lambda x)$.

52. $\boldsymbol{y'''_{xxx} + a\tan(\lambda x)y''_{xx} + by'_x + \lambda(a\lambda + b - \lambda^2)\tan(\lambda x)\, y = 0}$.

Particular solution: $\quad y_0 = \cos(\lambda x)$.

The transformation $x = \dfrac{\xi}{\lambda}$, $y = \cos(\lambda x)\displaystyle\int z\,dx$ leads to a second order equation of the form 2.1.6.55:

$$z''_{\xi\xi} + \Bigl(\frac{a}{\lambda} - 3\Bigr)\tan\xi\, z'_\xi + \Bigl(\frac{b^2}{\lambda^2} - 3 - \frac{2a}{\lambda}\tan^2\xi\Bigr)z = 0.$$

53. $\boldsymbol{y'''_{xxx} + a\tan^n(\lambda x)y''_{xx} + by'_x + ab\tan^n(\lambda x)\, y = 0}$.

1°. Particular solutions with $b > 0$: $\quad y_1 = \cos\bigl(x\sqrt{b}\,\bigr), \quad y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $\quad y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr), \quad y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

54. $\boldsymbol{y'''_{xxx} + a\tan^n(\lambda x)y''_{xx} + ab\tan^n(\lambda x)y'_x + b^2[a\tan^n(\lambda x) - b]y = 0}$.

Particular solutions: $\quad y_1 = e^{-bx/2}\cos\biggl(\dfrac{bx\sqrt{3}}{2}\biggr), \quad y_2 = e^{-bx/2}\sin\biggl(\dfrac{bx\sqrt{3}}{2}\biggr)$.

55. $\boldsymbol{y'''_{xxx} + a\tan^n(\lambda x)y''_{xx} - b[2a\tan^n(\lambda x) + 3b]y'_x + b^2[a\tan^n(\lambda x) + 2b]y = 0}$.

Particular solutions: $\quad y_1 = e^{bx}, \quad y_2 = xe^{bx}$.

56. $\boldsymbol{y'''_{xxx} + a\tan^n x\, y''_{xx} + (ab\tan^n x + c - b^2)y'_x + c(a\tan^n x - b)y = 0}$.

Particular solutions: $\quad y_1 = \exp(\lambda_1 x), \quad y_2 = \exp(\lambda_2 x), \quad$ where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

57. $\boldsymbol{y'''_{xxx} + a\tan^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\tan^n(\lambda x) + m]y = 0}$.

Assuming $w = y''_{xx} + bx^m y$ yields a first order linear equation: $w'_x + a\tan^n(\lambda x)\, w = 0$.

58. $\boldsymbol{y'''_{xxx} + (a\tan^n x + b)y''_{xx} + cy'_x + c(a\tan^n x + b)y = 0}$.

1°. Particular solutions with $c > 0$: $\quad y_1 = \cos\bigl(x\sqrt{c}\,\bigr), \quad y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $\quad y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr), \quad y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

59. $\boldsymbol{y'''_{xxx} = (\tan^n x - a)y''_{xx} + (a\tan^n x - b)y'_x + b\tan^n x\, y}$.

Particular solutions: $\quad y_1 = \exp(\lambda_1 x), \quad y_2 = \exp(\lambda_2 x), \quad$ where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

60. $y'''_{xxx} + (a\tan^n x + bx)y''_{xx} + b(ax\tan^n x + 2)y'_x + ab\tan^n x\, y = 0.$

Particular solutions: $y_1 = \exp\left(-\tfrac{1}{2}bx^2\right)$, $y_2 = \exp\left(-\tfrac{1}{2}bx^2\right)\int \exp\left(\tfrac{1}{2}bx^2\right)dx.$

61. $y'''_{xxx} + ax\tan^n x\, y''_{xx} + (bx^2 - a\tan^n x)y'_x + bx(ax^2\tan^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\left(\tfrac{1}{2}x^2\sqrt{b}\right)$, $y_2 = \sin\left(\tfrac{1}{2}x^2\sqrt{b}\right).$

62. $y'''_{xxx} + (abx\tan^n x + a\tan^n x + b)y''_{xx} + ab^2x\tan^n x\, y'_x - ab^2\tan^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}.$

63. $y'''_{xxx} + ax^2\tan^n(\lambda x)y''_{xx} - 2ax\tan^n(\lambda x)y'_x + 2a\tan^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2.$

64. $y'''_{xxx} - [b(a + \tan x)x^n + a]y''_{xx} + [b(a^2+1)x^n + 1]y'_x + a[b(a\tan x - 1)x^n - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x.$

65. $y'''_{xxx} - (ab\tan^n x + b\tan^{n+1} x + a)y''_{xx} + [b(a^2+1)\tan^n x + 1]y'_x$
$+ a(ab\tan^{n+1} x - b\tan^n x - 1)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x.$

66. $y'''_{xxx} + [\lambda\tan(\lambda x)(ax^n + 1) + ax^{n-1}]y''_{xx} - a\lambda^2x^n y'_x + a\lambda^2x^{n-1}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos(\lambda x).$

67. $y'''_{xxx} - \lambda\cot(\lambda x)y''_{xx} - ay'_x + a\lambda\cot(\lambda x)\, y = 0.$

1°. Solution with $a > 0$: $y = C_1\exp\left(-x\sqrt{a}\right) + C_2\exp\left(x\sqrt{a}\right) + C_3\sin(\lambda x).$

2°. Solution with $a < 0$: $y = C_1\cos\left(x\sqrt{-a}\right) + C_2\sin\left(x\sqrt{-a}\right) + C_3\sin(\lambda x).$

68. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} + ab\cot^n(\lambda x)y'_x + b^2[a\cot^n(\lambda x) - b]y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right).$

69. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} - b[2a\cot^n(\lambda x) + 3b]y'_x + b^2[a\cot^n(\lambda x) + 2b]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}.$

70. $y'''_{xxx} + a\cot^n x\, y''_{xx} + (ab\cot^n x + c - b^2)y'_x + c(a\cot^n x - b)y = 0.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + b\lambda + c = 0.$

71. $y'''_{xxx} + a\cot^n(\lambda x)y''_{xx} + bx^m y'_x + bx^{m-1}[ax\cot^n(\lambda x) + m]y = 0.$

Assuming $w = y''_{xx} + bx^m y$ yields a first order linear equation: $w'_x + a\cot^n(\lambda x)\, w = 0.$

72. $y'''_{xxx} + (a\cot^n x + b)y''_{xx} + cy'_x + c(a\cot^n x + b)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

73. $y'''_{xxx} = (\cot^n x - a)y''_{xx} + (a\cot^n x - b)y'_x + b\cot^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

74. $y'''_{xxx} + ax\cot^n x\, y''_{xx} + (bx^2 - a\cot^n x)y'_x + bx(ax^2\cot^n x + 3)y = 0.$

Particular solutions: $y_1 = \cos\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$, $y_2 = \sin\bigl(\tfrac{1}{2}x^2\sqrt{b}\bigr)$.

75. $y'''_{xxx} + (abx\cot^n x + a\cot^n x + b)y''_{xx} + ab^2 x\cot^n x\, y'_x - ab^2\cot^n x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-bx}$.

76. $y'''_{xxx} + ax^2\cot^n(\lambda x)y''_{xx} - 2ax\cot^n(\lambda x)y'_x + 2a\cot^n(\lambda x)\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

77. $xy'''_{xxx} + x(a\cos 2x + b)y'_x - 2(a\cos 2x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to the Mathieu equation 2.1.6.4:

$$w''_{xx} + (a\cos 2x + b)w = 0.$$

78. $xy'''_{xxx} + x(a\cos^2 x + b)y'_x - 2(a\cos^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to a second order linear equation of the form 2.1.6.6: $w''_{xx} + (a\cos^2 x + b)w = 0$.

79. $xy'''_{xxx} = (\cos^n x - ax)y''_{xx} + (a\cos^n x - bx)y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

80. $xy'''_{xxx} + (ax\cos^n x + 3)y''_{xx} + (2a\cos^n x + bx)y'_x + b(ax\cos^n x + 1)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\bigr)$.

81. $xy'''_{xxx} + x[a\sin(\lambda x) + b]y'_x - 2[a\sin(\lambda x) + b]y = 0.$

The substitution $w = xy'_x - 2y$ leads to an equation of the form 2.1.6.3: $w''_{xx} + [a\sin(\lambda x) + b]w = 0$.

82. $xy'''_{xxx} = (\sin^n x - ax)y''_{xx} + (a\sin^n x - bx)y'_x + b\sin^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

83. $xy'''_{xxx} + (ax\sin^n x + 3)y''_{xx} + (2a\sin^n x + bx)y'_x + b(ax\sin^n x + 1)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\bigr).$

84. $xy'''_{xxx} + x(a\tan^2 x + b)y'_x - 2(a\tan^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to an equation of the form 2.1.6.10: $w''_{xx} + (a\tan^2 x + b)w = 0$.

85. $xy'''_{xxx} = (\tan^n x - ax)y''_{xx} + (a\tan^n x - bx)y'_x + b\tan^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

86. $xy'''_{xxx} + (ax\tan^n x + 3)y''_{xx} + (2a\tan^n x + bx)y'_x + b(ax\tan^n x + 1)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\bigr).$

87. $axy'''_{xxx} + [1 - \lambda(a+1)x\cot(\lambda x)]y''_{xx} - \lambda^2 xy'_x + \lambda^2 y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sin(\lambda x).$

88. $xy'''_{xxx} + x(a\cot^2 x + b)y'_x - 2(a\cot^2 x + b)y = 0.$

The substitution $w = xy'_x - 2y$ leads to an equation of the form 2.1.6.12: $w''_{xx} + (a\cot^2 x + b)w = 0$.

89. $xy'''_{xxx} + (ax\cot^n x + 3)y''_{xx} + (2a\cot^n x + bx)y'_x + b(ax\cot^n x + 1)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{b}\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{b}\bigr).$

90. $x^2 y'''_{xxx} = (\cos^n x - ax^2)y''_{xx} + (a\cos^n x - bx^2)y'_x + b\cos^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

91. $x^2 y'''_{xxx} = (\sin^n x - ax^2)y''_{xx} + (a\sin^n x - bx^2)y'_x + b\sin^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

92. $x^2 y'''_{xxx} = (\tan^n x - ax^2)y''_{xx} + (a\tan^n x - bx^2)y'_x + b\tan^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

93. $x^2 y'''_{xxx} = (\cot^n x - ax^2)y''_{xx} + (a\cot^n x - bx^2)y'_x + b\cot^n x\, y.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

94. $\boldsymbol{x^3 y'''_{xxx} + ax^2 \cos^n(\lambda x) y''_{xx} + bxy'_x + b[a\cos^n(\lambda x) - 2]y = 0.}$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 - m + b = 0$.

95. $\boldsymbol{x^3 y'''_{xxx} + ax^2 \sin^n(\lambda x) y''_{xx} + bxy'_x + b[a\sin^n(\lambda x) - 2]y = 0.}$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 - m + b = 0$.

96. $\boldsymbol{x^3 y'''_{xxx} + ax^2 \tan^n(\lambda x) y''_{xx} + bxy'_x + b[a\tan^n(\lambda x) - 2]y = 0.}$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 - m + b = 0$.

97. $\boldsymbol{x^3 y'''_{xxx} + ax^2 \cot^n(\lambda x) y''_{xx} + bxy'_x + b[a\cot^n(\lambda x) - 2]y = 0.}$

Particular solutions: $y_1 = x^{m_1}$, $y_2 = x^{m_2}$, where m_1 and m_2 are the roots of the quadratic equation $m^2 - m + b = 0$.

98. $\boldsymbol{\cos^2 x\, y'''_{xxx} + a\cos^2 x\, y''_{xx} + by'_x + aby = 0.}$

The substitution $x = \xi + \frac{\pi}{2}$ leads to an equation of the form 3.1.6.112: $\sin^2\xi\, y'''_{\xi\xi\xi} + a\sin^2\xi\, y''_{\xi\xi} + by'_\xi + aby = 0$.

99. $\boldsymbol{\cos^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cos^n x)y = 0.}$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

100. $\boldsymbol{\cos^n x\, y'''_{xxx} + ay''_{xx} + b\cos^n x\, y'_x + aby = 0.}$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

101. $\boldsymbol{\cos^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\cos^n x)y'_x + b^2(a + 2b\cos^n x)y = 0.}$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

102. $\boldsymbol{\cos^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\cos^n x + a]y'_x + b(1 - a\cos^n x)y = 0.}$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

103. $\boldsymbol{\cos^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.}$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

104. $\boldsymbol{\cos^n x\, y'''_{xxx} + (a\cos^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.}$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

105. $\cos^n x\, y'''_{xxx} + (ax\cos^n x + 1)y''_{xx} + a(x + 2\cos^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\left(-\dfrac{ax^2}{2}\right)$, $y_2 = \exp\left(-\dfrac{ax^2}{2}\right)\displaystyle\int \exp\left(\dfrac{ax^2}{2}\right) dx.$

106. $x\cos^n x\, y'''_{xxx} + (3\cos^n x + x)y''_{xx} + (ax\cos^n x + 2)y'_x + a(\cos^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr).$

107. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cos^n x\, y'_x + 2(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2.$

108. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\cos^n x\, y'_x + 6(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3.$

109. $x^3\cos^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cos^n x)y'_x + a(a - 3\cos^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|).$

110. $x^3\cos^n x\, y'''_{xxx} + x^2(\cos^n x + a)y''_{xx} + x[a - (b+1)\cos^n x]y'_x + b(2\cos^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}.$

111. $\sin^2 x\, y'''_{xxx} + 3\sin x\cos x\, y''_{xx} + [\cos 2x + 4\nu(\nu+1)\sin^2 x]y'_x + 2\nu(\nu+1)\sin 2x\, y = 0.$

Solution:

$$y = C_1 y_1^2 + C_2 y_1 y_2 + C_3 y_2^2,$$

where y_1, y_2 form a fundamental set of solutions of the Legendre equation 2.1.2.148, with argument x of functions y_1 and y_2 substituted by $\cos x$.

112. $\sin^2 x\, y'''_{xxx} + a\sin^2 x\, y''_{xx} + by'_x + aby = 0.$

The substitution $w = y'_x + ay$ leads to a second order equation of the form 2.1.6.108: $\sin^2 x\, w''_{xx} + bw = 0.$

113. $\sin^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\sin^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right).$

114. $\sin^n x\, y'''_{xxx} + ay''_{xx} + b\sin^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr).$

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr).$

115. $\sin^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\sin^n x)y'_x + b^2(a + 2b\sin^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}.$

116. $\sin^n x\, y'''_{xxx} + y''_{xx} + [(b-a^2)\sin^n x + a]y'_x + b(1 - a\sin^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

117. $\sin^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

118. $\sin^n x\, y'''_{xxx} + (a\sin^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

119. $\sin^n x\, y'''_{xxx} + (ax\sin^n x + 1)y''_{xx} + a(x + 2\sin^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\frac{ax^2}{2}\Bigr)\,dx$.

120. $x\sin^n x\, y'''_{xxx} + (3\sin^n x + x)y''_{xx} + (ax\sin^n x + 2)y'_x + a(\sin^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr)$.

121. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\sin^n x\, y'_x + 2(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

122. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\sin^n x\, y'_x + 6(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

123. $x^3\sin^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \sin^n x)y'_x + a(a - 3\sin^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

124. $x^3\sin^n x\, y'''_{xxx} + x^2(\sin^n x + a)y''_{xx} + x[a - (b+1)\sin^n x]y'_x + b(2\sin^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

125. $\tan^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\tan^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\biggl(\dfrac{bx\sqrt{3}}{2}\biggr)$, $y_2 = e^{-bx/2}\sin\biggl(\dfrac{bx\sqrt{3}}{2}\biggr)$.

126. $\tan^n x\, y'''_{xxx} + ay''_{xx} + b\tan^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

127. $\tan^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\tan^n x)y'_x + b^2(a + 2b\tan^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

128. $\tan^n x\, y'''_{xxx} + y''_{xx} + [(b - a^2)\tan^n x + a]y'_x + b(1 - a\tan^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

129. $\tan^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

130. $\tan^n x\, y'''_{xxx} + (a\tan^n x + ax + 1)y''_{xx} + a^2 xy'_x - a^2 y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

131. $\tan^n x\, y'''_{xxx} + (ax\tan^n x + 1)y''_{xx} + a(x + 2\tan^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\frac{ax^2}{2}\Bigr)\,dx$.

132. $x\tan^n x\, y'''_{xxx} + (3\tan^n x + x)y''_{xx} + (ax\tan^n x + 2)y'_x + a(\tan^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr)$.

133. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\tan^n x\, y'_x + 2(2\tan^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

134. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\tan^n x\, y'_x + 6(2\tan^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

135. $x^3\tan^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \tan^n x)y'_x + a(a - 3\tan^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln|x|)$, $y_2 = \sin(\ln|x|)$.

136. $x^3\tan^n x\, y'''_{xxx} + x^2(\tan^n x + a)y''_{xx} +$
$$x[a - (b+1)\tan^n x]y'_x + b(2\tan^n x - a)y = 0.$$

Particular solutions: $y_1 = x^{-\sqrt{b}}$, $y_2 = x^{\sqrt{b}}$.

137. $\cot^n x\, y'''_{xxx} + ay''_{xx} + aby'_x + b^2(a - b\cot^n x)y = 0.$

Particular solutions: $y_1 = e^{-bx/2}\cos\left(\dfrac{bx\sqrt{3}}{2}\right)$, $y_2 = e^{-bx/2}\sin\left(\dfrac{bx\sqrt{3}}{2}\right)$.

138. $\cot^n x\, y'''_{xxx} + ay''_{xx} + b\cot^n x\, y'_x + aby = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

139. $\cot^n x\, y'''_{xxx} + ay''_{xx} - b(2a + 3b\cot^n x)y'_x + b^2(a + 2b\cot^n x)y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = xe^{bx}$.

140. $\cot^n x\, y'''_{xxx} + y''_{xx} + [(b-a^2)\cot^n x + a]y'_x + b(1 - a\cot^n x)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x},$ where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

141. $\cot^n(\lambda x) y'''_{xxx} + ax^2 y''_{xx} - 2axy'_x + 2ay = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2.$

142. $\cot^n x\, y'''_{xxx} + (a\cot^n x + ax + 1)y''_{xx} + a^2 x y'_x - a^2 y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-ax}.$

143. $\cot^n x\, y'''_{xxx} + (ax\cot^n x + 1)y''_{xx} + a(x + 2\cot^n x)y'_x + ay = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr), \quad y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\,dx.$

144. $x\cot^n x\, y'''_{xxx} + (3\cot^n x + x)y''_{xx} + (ax\cot^n x + 2)y'_x + a(\cot^n x + x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr).$

145. $x^3\cot^n x\, y'''_{xxx} + ax^2 y''_{xx} - 2x\cot^n x\, y'_x + 2(2\cot^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-1}, \quad y_2 = x^2.$

146. $x^3\cot^n x\, y'''_{xxx} + ax^2 y''_{xx} - 6x\cot^n x\, y'_x + 6(2\cot^n x - a)y = 0.$

Particular solutions: $y_1 = x^{-2}, \quad y_2 = x^3.$

147. $x^3\cot^n x\, y'''_{xxx} + ax^2 y''_{xx} + x(a - \cot^n x)y'_x + a(a - 3\cot^n x)y = 0.$

Particular solutions: $y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x).$

148. $x^3\cot^n x\, y'''_{xxx} + x^2(\cot^n x + a)y''_{xx} +$

$$x[a - (b+1)\cot^n x]y'_x + b(2\cot^n x - a)y = 0.$$

Particular solutions: $y_1 = x^{-\sqrt{b}}, \quad y_2 = x^{\sqrt{b}}.$

3.1.7. Equations Containing Inverse Trigonometric Functions

1. $y'''_{xxx} + ay''_{xx} + by'_x + cy = \arcsin^k x.$

This is a special case of equation 5.1.5.9.

2. $y'''_{xxx} + \arcsin^k x\, y''_{xx} + ay'_x + a\arcsin^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr), \quad y_1 = \sin\bigl(x\sqrt{a}\,\bigr).$

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr), \quad y_1 = \exp\bigl(x\sqrt{-a}\,\bigr).$

The substitution $w = y''_{xx} + ay$ leads to a first order linear equation: $w'_x + \arcsin^k x\, w = 0$.

3. $\boldsymbol{y'''_{xxx} + \arcsin^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \arcsin^k x + n)y = 0}$.

The substitution $w = y''_{xx} + ax^n y$ leads to a first order linear equation: $w'_x + \arcsin^k x\, w = 0$.

4. $\boldsymbol{y'''_{xxx} + \arcsin^k x\, y''_{xx} + a \arcsin^k x\, y'_x + a^2(\arcsin^k x - a)y = 0}$.

Particular solutions: $y_1 = e^{-ax/2} \cos\left(\dfrac{a\sqrt{3}}{2}x\right)$, $y_2 = e^{-ax/2} \sin\left(\dfrac{a\sqrt{3}}{2}x\right)$.

5. $\boldsymbol{y'''_{xxx} + \arcsin^k x\, y''_{xx} - a(2\arcsin^k x + 3a)y'_x + a^2(\arcsin^k x + 2a)y = 0}$.

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

6. $\boldsymbol{y'''_{xxx} + \arcsin^k x\, y''_{xx} + (a \arcsin^k x + b - a^2)y'_x + b(\arcsin^k x - a)y = 0}$.

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

7. $\boldsymbol{y'''_{xxx} = (\arcsin^k x - a)y''_{xx} + (a \arcsin^k x - b)y'_x + b \arcsin^k x\, y}$.

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first order linear equation: $w'_x = \arcsin^k x\, w$.

8. $\boldsymbol{y'''_{xxx} + x \arcsin^k x\, y''_{xx} + (ax^2 - \arcsin^k x)y'_x + ax(x^2 \arcsin^k x + 3)y = 0}$.

Particular solutions: $y_1 = \cos\left(\dfrac{x^2\sqrt{a}}{2}\right)$, $y_2 = \sin\left(\dfrac{x^2\sqrt{a}}{2}\right)$.

9. $\boldsymbol{y'''_{xxx} + (\arcsin^k x + ax)y''_{xx} + a(x \arcsin^k x + 2)y'_x + a \arcsin^k x\, y = 0}$.

Particular solutions: $y_1 = \exp\left(-\dfrac{ax^2}{2}\right)$, $y_2 = \exp\left(-\dfrac{ax^2}{2}\right) \displaystyle\int \exp\left(\dfrac{ax^2}{2}\right) dx$.

10. $\boldsymbol{y'''_{xxx} + x^2 \arcsin^k x\, y''_{xx} - 2x \arcsin^k x\, y'_x + 2 \arcsin^k x\, y = 0}$.

Solution:

$$y = C_1 x + C_2 x^2 + C_3\left(x^2 \int x^{-3}\psi\, dx - x \int x^{-2}\psi\, dx\right),$$

where $\psi = \exp(-\int x^2 \arcsin^k x\, dx)$.

11. $\boldsymbol{y'''_{xxx} + (ax \arcsin^k x + \arcsin^k x + a)y''_{xx} + a^2 x \arcsin^k x\, y'_x - a^2 \arcsin^k x\, y = 0}$.

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

12. $\boldsymbol{y'''_{xxx} + \arccos^k x\, y''_{xx} + ay'_x + a \arccos^k x\, y = 0}$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\left(x\sqrt{a}\right)$, $y_1 = \sin\left(x\sqrt{a}\right)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\left(-x\sqrt{-a}\right)$, $y_1 = \exp\left(x\sqrt{-a}\right)$.

The substitution $w = y''_{xx} + ay$ leads to a first order linear equation: $w'_x + \arccos^k x\, w = 0$.

13. $y'''_{xxx} + \arccos^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \arccos^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first order linear equation: $w'_x + \arccos^k x\, w = 0$.

14. $y'''_{xxx} + \arccos^k x\, y''_{xx} + a \arccos^k x\, y'_x + a^2(\arccos^k x - a)y = 0.$

Particular solutions: $\quad y_1 = e^{-ax/2} \cos\left(\dfrac{a\sqrt{3}}{2}x\right), \quad y_2 = e^{-ax/2} \sin\left(\dfrac{a\sqrt{3}}{2}x\right).$

15. $y'''_{xxx} + \arccos^k x\, y''_{xx} - a(2 \arccos^k x + 3a)y'_x + a^2(\arccos^k x + 2a)y = 0.$

Particular solutions: $\quad y_1 = e^{ax}, \quad y_2 = xe^{ax}.$

16. $y'''_{xxx} + \arccos^k x\, y''_{xx} + (a \arccos^k x + b - a^2)y'_x + b(\arccos^k x - a)y = 0.$

Particular solutions: $\quad y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

17. $y'''_{xxx} = (\arccos^k x - a)y''_{xx} + (a \arccos^k x - b)y'_x + b \arccos^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first order linear equation: $w'_x = \arccos^k x\, w$.

18. $y'''_{xxx} + x \arccos^k x\, y''_{xx} + (ax^2 - \arccos^k x)y'_x + ax(x^2 \arccos^k x + 3)y = 0.$

Particular solutions: $\quad y_1 = \cos\left(\dfrac{x^2\sqrt{a}}{2}\right), \quad y_2 = \sin\left(\dfrac{x^2\sqrt{a}}{2}\right).$

19. $y'''_{xxx} + (\arccos^k x + ax)y''_{xx} + a(x \arccos^k x + 2)y'_x + a \arccos^k x\, y = 0.$

Particular solutions: $\quad y_1 = \exp\left(-\dfrac{ax^2}{2}\right), \quad y_2 = \exp\left(-\dfrac{ax^2}{2}\right) \displaystyle\int \exp\left(\dfrac{ax^2}{2}\right) dx.$

20. $y'''_{xxx} + x^2 \arccos^k x\, y''_{xx} - 2x \arccos^k x\, y'_x + 2 \arccos^k x\, y = 0.$

Solution:

$$y = C_1 x + C_2 x^2 + C_3\left(x^2 \int x^{-3}\psi\, dx - x \int x^{-2}\psi\, dx\right),$$

where $\psi = \exp(-\int x^2 \arccos^k x\, dx)$.

21. $y'''_{xxx} + (ax \arccos^k x + \arccos^k x + a)y''_{xx} + a^2 x \arccos^k x\, y'_x - a^2 \arccos^k x\, y = 0.$

Particular solutions: $\quad y_1 = x, \quad y_2 = e^{-ax}.$

22. $y'''_{xxx} + \arctan^k x\, y''_{xx} + ay'_x + a \arctan^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $\quad y_1 = \cos\left(x\sqrt{a}\right), \quad y_1 = \sin\left(x\sqrt{a}\right).$

2°. Particular solutions with $a < 0$: $\quad y_1 = \exp\left(-x\sqrt{-a}\right), \quad y_1 = \exp\left(x\sqrt{-a}\right).$

The substitution $w = y''_{xx} + ay$ leads to a first order linear equation: $w'_x + \arctan^k x\, w = 0$.

23. $y'''_{xxx} + \arctan^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \arctan^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first order linear equation: $w'_x + \arctan^k x\, w = 0$.

24. $y'''_{xxx} + \arctan^k x\, y''_{xx} + a \arctan^k x\, y'_x + a^2(\arctan^k x - a)y = 0.$

Particular solutions: $y_1 = e^{-ax/2} \cos\Bigl(\dfrac{a\sqrt{3}}{2}x\Bigr)$, $y_2 = e^{-ax/2} \sin\Bigl(\dfrac{a\sqrt{3}}{2}x\Bigr)$.

25. $y'''_{xxx} + \arctan^k x\, y''_{xx} - a(2\arctan^k x + 3a)y'_x + a^2(\arctan^k x + 2a)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

26. $y'''_{xxx} + \arctan^k x\, y''_{xx} + (a \arctan^k x + b - a^2)y'_x + b(\arctan^k x - a)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

27. $y'''_{xxx} = (\arctan^k x - a)y''_{xx} + (a \arctan^k x - b)y'_x + b \arctan^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first order linear equation: $w'_x = \arctan^k x\, w$.

28. $y'''_{xxx} + x \arctan^k x\, y''_{xx} + (ax^2 - \arctan^k x)y'_x + ax(x^2 \arctan^k x + 3)y = 0.$

Particular solutions: $y_1 = \cos\Bigl(\dfrac{x^2\sqrt{a}}{2}\Bigr)$, $y_2 = \sin\Bigl(\dfrac{x^2\sqrt{a}}{2}\Bigr)$.

29. $y'''_{xxx} + (\arctan^k x + ax)y''_{xx} + a(x \arctan^k x + 2)y'_x + a \arctan^k x\, y = 0.$

Particular solutions: $y_1 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$, $y_2 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)\displaystyle\int \exp\Bigl(\dfrac{ax^2}{2}\Bigr)\,dx$.

30. $y'''_{xxx} + x^2 \arctan^k x\, y''_{xx} - 2x \arctan^k x\, y'_x + 2 \arctan^k x\, y = 0.$

Solution:

$$y = C_1 x + C_2 x^2 + C_3\Bigl(x^2 \int x^{-3}\psi\,dx - x \int x^{-2}\psi\,dx\Bigr),$$

where $\psi = \exp\bigl(-\int x^2 \arctan^k x\,dx\bigr)$.

31. $y'''_{xxx} + (ax \arctan^k x + \arctan^k x + a)y''_{xx} + a^2 x \arctan^k x\, y'_x - a^2 \arctan^k x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

32. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + ay'_x + a \operatorname{arccot}^k x\, y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr)$, $y_1 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr)$, $y_1 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a first order linear equation: $w'_x + \operatorname{arccot}^k x\, w = 0$.

33. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + ax^n y'_x + ax^{n-1}(x \operatorname{arccot}^k x + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first order linear equation: $w'_x + \operatorname{arccot}^k x\, w = 0$.

34. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + a \operatorname{arccot}^k x\, y'_x + a^2(\operatorname{arccot}^k x - a)y = 0.$

Particular solutions: $y_1 = e^{-ax/2} \cos\left(\frac{a\sqrt{3}}{2} x\right)$, $y_2 = e^{-ax/2} \sin\left(\frac{a\sqrt{3}}{2} x\right)$.

35. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} - a(2 \operatorname{arccot}^k x + 3a)y'_x + a^2(\operatorname{arccot}^k x + 2a)y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

36. $y'''_{xxx} + \operatorname{arccot}^k x\, y''_{xx} + (a \operatorname{arccot}^k x + b - a^2)y'_x + b(\operatorname{arccot}^k x - a)y = 0.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

37. $y'''_{xxx} = (\operatorname{arccot}^k x - a)y''_{xx} + (a \operatorname{arccot}^k x - b)y'_x + b \operatorname{arccot}^k x\, y.$

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first order linear equation: $w'_x = \operatorname{arccot}^k x\, w$.

38. $y'''_{xxx} + x \operatorname{arccot}^k x\, y''_{xx} + (ax^2 - \operatorname{arccot}^k x)y'_x + ax(x^2 \operatorname{arccot}^k x + 3)y = 0.$

Particular solutions: $y_1 = \cos\left(\frac{x^2\sqrt{a}}{2}\right)$, $y_2 = \sin\left(\frac{x^2\sqrt{a}}{2}\right)$.

39. $y'''_{xxx} + (\operatorname{arccot}^k x + ax)y''_{xx} + a(x \operatorname{arccot}^k x + 2)y'_x + a \operatorname{arccot}^k x\, y = 0.$

Particular solutions: $y_1 = \exp\left(-\frac{ax^2}{2}\right)$, $y_2 = \exp\left(-\frac{ax^2}{2}\right) \int \exp\left(\frac{ax^2}{2}\right) dx$.

40. $y'''_{xxx} + x^2 \operatorname{arccot}^k x\, y''_{xx} - 2x \operatorname{arccot}^k x\, y'_x + 2 \operatorname{arccot}^k x\, y = 0.$

Solution:

$$y = C_1 x + C_2 x^2 + C_3\left(x^2 \int x^{-3}\psi\, dx - x \int x^{-2}\psi\, dx\right),$$

where $\psi = \exp\left(-\int x^2 \operatorname{arccot}^k x\, dx\right)$.

41. $y'''_{xxx} + (ax \operatorname{arccot}^k x + \operatorname{arccot}^k x + a)y''_{xx} + a^2 x \operatorname{arccot}^k x\, y'_x - a^2 \operatorname{arccot}^k x\, y = 0.$

Particular solutions: $y_1 = x$, $y_2 = e^{-ax}$.

42. $xy'''_{xxx} + (ax^2 + b)y''_{xx} + 4axy'_x + 2ay = \arcsin^k x.$

Twice integrating yields a first order linear equation:

$$y'_x + (ax^2 + b - 2)y = C_1 + C_2 x + \int\left(\int \arcsin^k x\, dx\right) dx.$$

43. $xy'''_{xxx} + (x \arcsin^k x + 3)y''_{xx} + (2 \arcsin^k x + ax)y'_x + a(x \arcsin^k x + 1)y = 0.$

Particular solutions: $y_1 = \frac{1}{x} \cos\left(x\sqrt{a}\right)$, $y_2 = \frac{1}{x} \sin\left(x\sqrt{a}\right)$.

44. $xy'''_{xxx} + (x\arccos^k x + 3)y''_{xx} + (2\arccos^k x + ax)y'_x + a(x\arccos^k x + 1)y = 0.$

Particular solutions: $\quad y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr).$

45. $xy'''_{xxx} + (x\arctan^k x + 3)y''_{xx} + (2\arctan^k x + ax)y'_x + a(x\arctan^k x + 1)y = 0.$

Particular solutions: $\quad y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr).$

46. $xy'''_{xxx} + (x\,\mathrm{arccot}^k x + 3)y''_{xx} + (2\,\mathrm{arccot}^k x + ax)y'_x + a(x\,\mathrm{arccot}^k x + 1)y = 0.$

Particular solutions: $\quad y_1 = \dfrac{1}{x}\cos\bigl(x\sqrt{a}\,\bigr), \quad y_2 = \dfrac{1}{x}\sin\bigl(x\sqrt{a}\,\bigr).$

47. $x^3y'''_{xxx} + [(a+6)x^2 + b]y''_{xx} + 2(2a+3)xy'_x + 2ay = \arcsin^k x.$

Twice integrating yields a first order linear equation:

$$x^3y'_x + (ax^2 + b)y = C_1 + C_2x + \int\left(\int \arcsin^k x\,dx\right)dx.$$

48. $x^3y'''_{xxx} + x^2\arcsin^k xy''_{xx} - 2xy'_x + 2(2 - \arcsin^k x)y = 0.$

Particular solutions: $\quad y_1 = x^{-1}, \quad y_2 = x^2.$

49. $x^3y'''_{xxx} + x^2\arcsin^k x\,y''_{xx} - 6xy'_x + 6(2 - \arcsin^k x)y = 0.$

Particular solutions: $\quad y_1 = x^{-2}, \quad y_2 = x^3.$

50. $x^3y'''_{xxx} + x^2\arcsin^k x\,y''_{xx} + x(\arcsin^k x - 1)y'_x + (\arcsin^k x - 3)y = 0.$

Particular solutions: $\quad y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x).$

51. $x^3y'''_{xxx} + x^2(\arcsin^k x + 1)y''_{xx} + x(\arcsin^k x - a - 1)y'_x - (\arcsin^k x - 2)y = 0.$

Particular solutions: $\quad y_1 = x^{-\sqrt{a}}, \quad y_2 = x^{\sqrt{a}}.$

52. $x^3y'''_{xxx} + x^2(\arcsin^k x + a)y''_{xx} + x(a\arcsin^k x + b - a)y'_x + b(\arcsin^k x - 2)y = 0.$

Particular solutions: $\quad y_1 = x^{n_1}, \quad y_2 = x^{n_2}, \quad$ where n_1 and n_2 are the roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

53. $x^3y'''_{xxx} + x^2\arccos^k x\,y''_{xx} - 2xy'_x + 2(2 - \arccos^k x)y = 0.$

Particular solutions: $\quad y_1 = x^{-1}, \quad y_2 = x^2.$

54. $x^3y'''_{xxx} + x^2\arccos^k x\,y''_{xx} - 6xy'_x + 6(2 - \arccos^k x)y = 0.$

Particular solutions: $\quad y_1 = x^{-2}, \quad y_2 = x^3.$

55. $x^3y'''_{xxx} + x^2\arccos^k x\,y''_{xx} + x(\arccos^k x - 1)y'_x + (\arccos^k x - 3)y = 0.$

Particular solutions: $\quad y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x).$

56. $\boldsymbol{x^3 y'''_{xxx} + x^2(\operatorname{arccos}^k x + 1) y''_{xx} + x(\operatorname{arccos}^k x - a - 1) y'_x - (\operatorname{arccos}^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

57. $\boldsymbol{x^3 y'''_{xxx} + x^2(\operatorname{arccos}^k x + a) y''_{xx} + x(a \operatorname{arccos}^k x + b - a) y'_x + b(\operatorname{arccos}^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

58. $\boldsymbol{x^3 y'''_{xxx} + x^2 \arctan^k x\, y''_{xx} - 2x y'_x + 2(2 - \arctan^k x) y = 0.}$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

59. $\boldsymbol{x^3 y'''_{xxx} + x^2 \arctan^k x\, y''_{xx} - 6x y'_x + 6(2 - \arctan^k x) y = 0.}$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

60. $\boldsymbol{x^3 y'''_{xxx} + x^2 \arctan^k x\, y''_{xx} + x(\arctan^k x - 1) y'_x + (\arctan^k x - 3) y = 0.}$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

61. $\boldsymbol{x^3 y'''_{xxx} + x^2(\arctan^k x + 1) y''_{xx} + x(\arctan^k x - a - 1) y'_x - (\arctan^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

62. $\boldsymbol{x^3 y'''_{xxx} + x^2(\arctan^k x + a) y''_{xx} + x(a \arctan^k x + b - a) y'_x + b(\arctan^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

63. $\boldsymbol{x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x\, y''_{xx} - 2x y'_x + 2(2 - \operatorname{arccot}^k x) y = 0.}$

Particular solutions: $y_1 = x^{-1}$, $y_2 = x^2$.

64. $\boldsymbol{x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x\, y''_{xx} - 6x y'_x + 6(2 - \operatorname{arccot}^k x) y = 0.}$

Particular solutions: $y_1 = x^{-2}$, $y_2 = x^3$.

65. $\boldsymbol{x^3 y'''_{xxx} + x^2 \operatorname{arccot}^k x\, y''_{xx} + x(\operatorname{arccot}^k x - 1) y'_x + (\operatorname{arccot}^k x - 3) y = 0.}$

Particular solutions: $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$.

66. $\boldsymbol{x^3 y'''_{xxx} + x^2(\operatorname{arccot}^k x + 1) y''_{xx} + x(\operatorname{arccot}^k x - a - 1) y'_x - (\operatorname{arccot}^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{-\sqrt{a}}$, $y_2 = x^{\sqrt{a}}$.

67. $\boldsymbol{x^3 y'''_{xxx} + x^2(\operatorname{arccot}^k x + a) y''_{xx} + x(a \operatorname{arccot}^k x + b - a) y'_x + b(\operatorname{arccot}^k x - 2) y = 0.}$

Particular solutions: $y_1 = x^{n_1}$, $y_2 = x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2 + (a-1)n + b = 0$.

3.1.8. Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + (2ae^{\lambda x} \tan x + 3)y'_x + [ae^{\lambda x}(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x, \quad y_2 = x\cos x.$

2. $y'''_{xxx} + ae^{\lambda x} y''_{xx} + (3 - 2ae^{\lambda x} \cot x)y'_x + [ae^{\lambda x}(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x, \quad y_2 = x\sin x.$

3. $y'''_{xxx} + a\cosh^n x\, y''_{xx} + (2a\cosh^n x \tan x + 3)y'_x + [a\cosh^n x\,(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x, \quad y_2 = x\cos x.$

4. $y'''_{xxx} + a\cosh^n x\, y''_{xx} + (3 - 2a\cosh^n x \cot x)y'_x + [a\cosh^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x, \quad y_2 = x\sin x.$

5. $y'''_{xxx} + a\sinh^n x\, y''_{xx} + (2a\sinh^n x \tan x + 3)y'_x + [a\sinh^n x\,(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x, \quad y_2 = x\cos x.$

6. $y'''_{xxx} + a\sinh^n x\, y''_{xx} + (3 - 2a\sinh^n x \cot x)y'_x + [a\sinh^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x, \quad y_2 = x\sin x.$

7. $y'''_{xxx} + a\tanh^n x\, y''_{xx} + (2a\tanh^n x \tan x + 3)y'_x + [a\tanh^n x\,(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x, \quad y_2 = x\cos x.$

8. $y'''_{xxx} + a\tanh^n x\, y''_{xx} + (3 - 2a\tanh^n x \cot x)y'_x + [a\tanh^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x, \quad y_2 = x\sin x.$

9. $y'''_{xxx} + a\coth^n x\, y''_{xx} + (2a\coth^n x \tan x + 3)y'_x + [a\coth^n x\,(2\tan^2 x + 1) + 2\tan x]y = 0.$

Particular solutions: $y_1 = \cos x, \quad y_2 = x\cos x.$

10. $y'''_{xxx} + a\coth^n x\, y''_{xx} + (3 - 2a\coth^n x \cot x)y'_x + [a\coth^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x, \quad y_2 = x\sin x.$

11. $y'''_{xxx} + a\ln^n x\, y''_{xx} - (2a\ln^n x \tanh x + 3)y'_x + [a\ln^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$

Particular solutions: $y_1 = \cosh x, \quad y_2 = x\cosh x.$

12. $y'''_{xxx} + a\ln^n x\, y''_{xx} - (2a\ln^n x\coth x + 3)y'_x$
$$+ [a\ln^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

13. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (2a\ln^n x\tan x + 3)y'_x$
$$+ [a\ln^n x\,(2\tan^2 x + 1) + 2\tan x]y = 0.$$

Particular solutions: $y_1 = \cos x$, $y_2 = x\cos x$.

14. $y'''_{xxx} + a\ln^n x\, y''_{xx} + (3 - 2a\ln^n x\cot x)y'_x + [a\ln^n x\,(2\cot^2 x + 1) - 2\cot x]y = 0.$

Particular solutions: $y_1 = \sin x$, $y_2 = x\sin x$.

15. $y'''_{xxx} + a\cos^n x\, y''_{xx} - (2a\cos^n x\tanh x + 3)y'_x$
$$+ [a\cos^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

16. $y'''_{xxx} + a\cos^n x\, y''_{xx} - (2a\cos^n x\coth x + 3)y'_x$
$$+ [a\cos^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

17. $y'''_{xxx} + a\sin^n x\, y''_{xx} - (2a\sin^n x\tanh x + 3)y'_x$
$$+ [a\sin^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

18. $y'''_{xxx} + a\sin^n x\, y''_{xx} - (2a\sin^n x\coth x + 3)y'_x$
$$+ [a\sin^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

19. $y'''_{xxx} + a\tan^n x\, y''_{xx} - (2a\tan^n x\tanh x + 3)y'_x$
$$+ [a\tan^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

20. $y'''_{xxx} + a\tan^n x\, y''_{xx} - (2a\tan^n x\coth x + 3)y'_x$
$$+ [a\tan^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

21. $y'''_{xxx} + a\cot^n x\, y''_{xx} - (2a\cot^n x\tanh x + 3)y'_x$
$$+ [a\cot^n x\,(2\tanh^2 x - 1) + 2\tanh x]y = 0.$$

Particular solutions: $y_1 = \cosh x$, $y_2 = x\cosh x$.

22. $y'''_{xxx} + a\cot^n x\, y''_{xx} - (2a\cot^n x\coth x + 3)y'_x$
$$+ [a\cot^n x\,(2\coth^2 x - 1) + 2\coth x]y = 0.$$

Particular solutions: $y_1 = \sinh x$, $y_2 = x\sinh x$.

23. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\cosh^n x\, y''_{xx} - a(be^{ax}\cosh^n x + a)y'_x - 2a^3\cosh^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

24. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\sinh^n x\, y''_{xx} - a(be^{ax}\sinh^n x + a)y'_x - 2a^3\sinh^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

25. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\tanh^n x\, y''_{xx} - a(be^{ax}\tanh^n x + a)y'_x - 2a^3\tanh^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

26. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\coth^n x\, y''_{xx} - a(be^{ax}\coth^n x + a)y'_x - 2a^3\coth^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

27. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\ln^n x\, y''_{xx} - a(be^{ax}\ln^n x + a)y'_x - 2a^3\ln^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

28. $\boldsymbol{y'''_{xxx} + (a\ln^n x - 2be^x)y''_{xx} - be^x(2a\ln^n x - be^x + 3)y'_x}$
$\boldsymbol{+ be^x[a\ln^n x\,(be^x - 1) + 2be^x - 1]y = 0}$.

Particular solutions: $y_1 = \exp(be^x), \quad y_2 = x\exp(be^x)$.

29. $\boldsymbol{y'''_{xxx} + (a\cos^n x - 2be^x)y''_{xx} - be^x(2a\cos^n x - be^x + 3)y'_x}$
$\boldsymbol{+ be^x[a\cos^n x\,(be^x - 1) + 2be^x - 1]y = 0}$.

Particular solutions: $y_1 = \exp(be^x), \quad y_2 = x\exp(be^x)$.

30. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\cos^n x\, y''_{xx} - a(be^{ax}\cos^n x + a)y'_x - 2a^3\cos^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

31. $\boldsymbol{y'''_{xxx} + (a\sin^n x - 2be^x)y''_{xx} - be^x(2a\sin^n x - be^x + 3)y'_x}$
$\boldsymbol{+ be^x[a\sin^n x\,(be^x - 1) + 2be^x - 1]y = 0}$.

Particular solutions: $y_1 = \exp(be^x), \quad y_2 = x\exp(be^x)$.

32. $\boldsymbol{y'''_{xxx} + (be^{ax} + 2a)\sin^n x\, y''_{xx} - a(be^{ax}\sin^n x + a)y'_x - 2a^3\sin^n x\, y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = e^{-ax} + \dfrac{b}{a}$.

33. $\boldsymbol{y'''_{xxx} - [e^{\lambda x}(\tan x + a) + a]y''_{xx} + [(a^2+1)e^{\lambda x} + 1]y'_x + a[e^{\lambda x}(a\tan x - 1) - 1]y = 0}$.

Particular solutions: $y_1 = e^{ax}, \quad y_2 = \cos x$.

34. $\boldsymbol{y'''_{xxx} + [\tan x\,(axe^{\lambda x} + 1) + ae^{\lambda x}]y''_{xx} - axe^{\lambda x}y'_x + ae^{\lambda x}y = 0}$.

Particular solutions: $y_1 = x, \quad y_2 = \cos x$.

35. $$y'''_{xxx} + (a \tan^n x - 2be^x) y''_{xx} - be^x(2a \tan^n x - be^x + 3) y'_x + be^x[a \tan^n x\,(be^x - 1) + 2be^x - 1] y = 0.$$

Particular solutions: $y_1 = \exp(be^x)$, $y_2 = x \exp(be^x)$.

36. $$y'''_{xxx} + (be^{ax} + 2a) \tan^n x\, y''_{xx} - a(be^{ax} \tan^n x + a) y'_x - 2a^3 \tan^n x\, y = 0.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + \dfrac{b}{a}$.

37. $$y'''_{xxx} + [e^{\lambda x}(\cot x + a) + a] y''_{xx} + [(a^2+1)e^{\lambda x} + 1] y'_x + a[e^{\lambda x}(1 - a \cot x) + 1] y = 0.$$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

38. $$y'''_{xxx} + [ae^{\lambda x} - \cot x\,(axe^{\lambda x} + 1)] y''_{xx} - axe^{\lambda x} y'_x + ae^{\lambda x} y = 0.$$

Particular solutions: $y_1 = x$, $y_2 = \sin x$.

39. $$y'''_{xxx} + (a \cot^n x - 2be^x) y''_{xx} - be^x(2a \cot^n x - be^x + 3) y'_x + be^x[a \cot^n x\,(be^x - 1) + 2be^x - 1] y = 0.$$

Particular solutions: $y_1 = \exp(be^x)$, $y_2 = x \exp(be^x)$.

40. $$y'''_{xxx} + (be^{ax} + 2a) \cot^n x\, y''_{xx} - a(be^{ax} \cot^n x + a) y'_x - 2a^3 \cot^n x\, y = 0.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + \dfrac{b}{a}$.

41. $$y'''_{xxx} - [\cosh^n x\,(\tan x + a) + a] y''_{xx} + [(a^2 + 1) \cosh^n x + 1] y'_x + a[\cosh^n x\,(a \tan x - 1) - 1] y = 0.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

42. $$y'''_{xxx} + [\cosh^n x\,(\cot x + a) + a] y''_{xx} + [(a^2 + 1) \cosh^n x + 1] y'_x + a[\cosh^n x\,(1 - a \cot x) + 1] y = 0.$$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

43. $$y'''_{xxx} - [\sinh^n x\,(\tan x + a) + a] y''_{xx} + [(a^2 + 1) \sinh^n x + 1] y'_x + a[\sinh^n x\,(a \tan x - 1) - 1] y = 0.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

44. $$y'''_{xxx} + [\sinh^n x\,(\cot x + a) + a] y''_{xx} + [(a^2 + 1) \sinh^n x + 1] y'_x + a[\sinh^n x\,(1 - a \cot x) + 1] y = 0.$$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

45. $$y'''_{xxx} - [\tanh^n x\,(\tan x + a) + a] y''_{xx} + [(a^2 + 1) \tanh^n x + 1] y'_x + a[\tanh^n x\,(a \tan x - 1) - 1] y = 0.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

46. $y'''_{xxx} + [\tanh^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\tanh^n x + 1]y'_x + a[\tanh^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

47. $y'''_{xxx} - [\coth^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\coth^n x + 1]y'_x + a[\coth^n x\,(\tan x - 1) - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

48. $y'''_{xxx} + [\coth^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\coth^n x + 1]y'_x + a[\coth^n x\,(1 - a\cot x) + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

49. $y'''_{xxx} + [a\tan^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\tan^n x - 1]y'_x + b[a\tan^n x\,(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

50. $y'''_{xxx} + (a\tan^n x + b\tanh^m x)y''_{xx} + cy'_x + c(a\tan^n x + b\tanh^m x)y = 0.$

1°. Partical solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

51. $y'''_{xxx} + [a\tan^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\tan^n x - 1]y'_x + b[a\tan^n x\,(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

52. $y'''_{xxx} + (a\tan^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\tan^n x + b\coth^m x)y = 0.$

1°. Partical solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

53. $y'''_{xxx} + [a\cot^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\cot^n x - 1]y'_x + b[a\cot^n x\,(1 - b\tanh x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

54. $y'''_{xxx} + a\cot^n x\tanh^m x\,y''_{xx} - by'_x - ab\cot^n x\tanh^m x\,y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

55. $y'''_{xxx} + [a\cot^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\cot^n x - 1]y'_x + b[a\cot^n x\,(1 - b\coth x) + 1]y = 0.$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

56. $\boldsymbol{y'''_{xxx} + (a\cot^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\cot^n x + b\coth^m x)y = 0}$.

1°. Partical solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

57. $$\boldsymbol{y'''_{xxx} + [a\ln^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\ln^n x - 1]y'_x + b[a\ln^n x\,(1 - b\tanh x) + 1]y = 0}.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

58. $\boldsymbol{y'''_{xxx} + a\ln^n x\tanh^m x\,y''_{xx} - by'_x - ab\ln^n x\tanh^m x\,y = 0}$.

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

59. $$\boldsymbol{y'''_{xxx} + [a\ln^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\ln^n x - 1]y'_x + b[a\ln^n x\,(1 - b\coth x) + 1]y = 0}.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

60. $\boldsymbol{y'''_{xxx} + (a\ln^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\ln^n x + b\coth^m x)y = 0}$.

1°. Partical solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

61. $$\boldsymbol{y'''_{xxx} - [\ln^n x\,(\tan x + a) + a]y''_{xx} + [(a^2+1)\ln^n x + 1]y'_x + a[\ln^n x\,(\tan x - 1) - 1]y = 0}.$$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

62. $$\boldsymbol{y'''_{xxx} + [\ln^n x\,(\cot x + a) + a]y''_{xx} + [(a^2+1)\ln^n x + 1]y'_x + a[\ln^n x\,(1 - a\cot x) + 1]y = 0}.$$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

63. $$\boldsymbol{y'''_{xxx} + [a\cos^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\cos^n x - 1]y'_x + b[a\cos^n x\,(1 - b\tanh x) + 1]y = 0}.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

64. $\boldsymbol{y'''_{xxx} + a\cos^n x\tanh^m x\,y''_{xx} + by'_x + ab\cos^n x\tanh^m x\,y = 0}$.

1°. Partical solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Partical solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

65. $$\boldsymbol{y'''_{xxx} + [a\cos^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\cos^n x - 1]y'_x + b[a\cos^n x\,(1 - b\coth x) + 1]y = 0}.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

66. $y'''_{xxx} + (a\cos^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\cos^n x + b\coth^m x)y = 0.$

1°. Partical solutions with $c>0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c<0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

67. $$y'''_{xxx} + [a\sin^n x\,(\tanh x - b) - b]y''_{xx} + [a(b^2-1)\sin^n x - 1]y'_x + b[a\sin^n x\,(1 - b\tanh x) + 1]y = 0.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \cosh x$.

68. $y'''_{xxx} + a\sin^n x\tanh^m x\,y''_{xx} + by'_x + ab\sin^n x\tanh^m x\,y = 0.$

1°. Partical solutions with $b>0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Partical solutions with $b<0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

69. $$y'''_{xxx} + [a\sin^n x\,(\coth x - b) - b]y''_{xx} + [a(b^2-1)\sin^n x - 1]y'_x + b[a\sin^n x\,(1 - b\coth x) + 1]y = 0.$$

Particular solutions: $y_1 = e^{bx}$, $y_2 = \sinh x$.

70. $y'''_{xxx} + (a\sin^n x + b\coth^m x)y''_{xx} + cy'_x + c(a\sin^n x + b\coth^m x)y = 0.$

1°. Partical solutions with $c>0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Partical solutions with $c<0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

71. $xy'''_{xxx} + [ax^2e^{\lambda x}(b - \ln x) + 2]y''_{xx} + axe^{\lambda x}y'_x - ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - b + 1$.

72. $(e^{\lambda x} - 1)y'''_{xxx} - (ae^{\lambda x} + \tan x)y''_{xx} + (e^{\lambda x} + a^2)y'_x + a(a\tan x - e^{\lambda x})y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

73. $a\cosh^n x\,y'''_{xxx} + [\tan x\,(a\cosh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos x$.

74. $a\cosh^n x\,y'''_{xxx} + [1 - \cot x\,(a\cosh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sin x$.

75. $a\sinh^n x\,y'''_{xxx} + [\tan x\,(a\sinh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos x$.

76. $a\sinh^n x\,y'''_{xxx} + [1 - \cot x\,(a\sinh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sin x$.

77. $a\tanh^n x\,y'''_{xxx} + [\tan x\,(a\tanh^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos x$.

78. $a\tanh^n x\, y'''_{xxx} + [1 - \cot x\,(a\tanh^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sin x.$

79. $a\coth^n x\, y'''_{xxx} + [\tan x\,(a\coth^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cos x.$

80. $a\coth^n x\, y'''_{xxx} + [1 - \cot x\,(a\coth^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sin x.$

81. $a\ln^n x\, y'''_{xxx} + [\tanh x\,(x - a\ln^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cosh x.$

82. $a\ln^n x\, y'''_{xxx} + [\coth x\,(x - a\ln^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sinh x.$

83. $a\ln^n x\, y'''_{xxx} + [\tan x\,(a\ln^n x + x) + 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cos x.$

84. $a\ln^n x\, y'''_{xxx} + [1 - \cot x\,(a\ln^n x + x)]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sin x.$

85. $a\cos^n x\, y'''_{xxx} + [\tanh x\,(x - a\cos^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cosh x.$

86. $a\cos^n x\, y'''_{xxx} + [\coth x\,(x - a\cos^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sinh x.$

87. $ax\cos^n x\, y'''_{xxx} + (2a\cos^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \ln x - b + 1.$

88. $a\sin^n x\, y'''_{xxx} + [\tanh x\,(x - a\sin^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cosh x.$

89. $a\sin^n x\, y'''_{xxx} + [\coth x\,(x - a\sin^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \sinh x.$

90. $ax\sin^n x\, y'''_{xxx} + (2a\sin^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \ln x - b + 1.$

91. $a\tan^n x\, y'''_{xxx} + [\tanh x\,(x - a\tan^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = \cosh x.$

92. $a\tan^n x\, y'''_{xxx} + [\coth x\,(x - a\tan^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sinh x$.

93. $ax\tan^n x\, y'''_{xxx} + (2a\tan^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - b + 1$.

94. $a\cot^n x\, y'''_{xxx} + [\tanh x\,(x - a\cot^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cosh x$.

95. $a\cot^n x\, y'''_{xxx} + [\coth x\,(x - a\cot^n x) - 1]y''_{xx} - xy'_x + y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sinh x$.

96. $ax\cot^n x\, y'''_{xxx} + (2a\cot^n x - x^2\ln x + bx^2)y''_{xx} + xy'_x - y = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - b + 1$.

3.1.9. Equations Containing Arbitrary Functions

Notation: $f = f(x)$, $g = g(x)$, and $h = h(x)$ are arbitrary function of argument x; a, b, c, n, and λ are parameters.

1. $y'''_{xxx} + fy'_x - (af + a^3)y = 0.$

Particular solution: $y_0 = e^{ax}$.

The substitution $w = y'_x - ay$ leads to a second order linear equation: $w''_{xx} + aw'_x + (f + a^2)w = 0$.

2. $y'''_{xxx} + fy'_x + ax(f + a^2x^2 - 3a)y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{ax^2}{2}\right)$.

The substitution $y = \exp\left(-\dfrac{ax^2}{2}\right)\displaystyle\int z(x)\,dx$ leads to a second order linear equation: $z''_{xx} - 3axz'_x + (f + 3a^2x^2 - 3a)z = 0$.

3. $y'''_{xxx} + (f - a^2)y'_x + afy = 0.$

Particular solution: $y_0 = e^{-ax}$.

The substitution $w = y'_x + ay$ leads to a second order linear equation: $w''_{xx} - aw'_x + fw = 0$.

4. $y'''_{xxx} + xfy'_x - 2fy = 0.$

Particular solution: $y_0 = x^2$.

The substitution $w = xy'_x - 2y$ leads to a second order linear equation: $w''_{xx} + xfw = 0$.

5. $y'''_{xxx} + (ax + b)fy'_x - afy = 0.$

Particular solution: $y_0 = ax + b$.

6. $y'''_{xxx} + (f - a^2x^2)y'_x + ax(f - 3a)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^2}{2}\Bigr)$.

The substitution $y = \exp\Bigl(\dfrac{-ax^2}{2}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order linear equation: $z''_{xx} - 3axz'_x + (2a^2x^2 - 3a + f)z = 0$.

7. $y'''_{xxx} + (f - a^2x^{2n})y'_x - a[x^n f + 3anx^{2n-1} + n(n-1)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(\dfrac{a}{n+1}x^{n+1}\Bigr)$.

The substitution $y = \exp\Bigl(\dfrac{a}{n+1}x^{n+1}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order linear equation: $z''_{xx} + 3ax^n z'_x + (2a^2x^{2n} + 3anx^{n-1} + f)z = 0$.

8. $y'''_{xxx} + ay''_{xx} + by'_x + cy = f(x).$

This is a special case of equation 5.1.5.9.

9. $y'''_{xxx} + ay''_{xx} + fy'_x + afy = 0.$

The substitution $w = y'_x + ay$ leads to a second order linear equation: $w''_{xx} + fw = 0$.

10. $y'''_{xxx} + fy''_{xx} - a^2(f + a)y = 0.$

Particular solution: $y_0 = e^{ax}$.

The substitution $w = y'_x - ay$ leads to a second order linear equation: $w''_{xx} + (f + a)w'_x + a(f + a)w = 0$.

11. $y'''_{xxx} + fy''_{xx} + ay'_x + afy = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a first order linear equation: $w'_x + fw = 0$.

12. $y'''_{xxx} + fy''_{xx} + ax^n y'_x + ax^{n-1}(xf + n)y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a first order linear equation: $w'_x + fw = 0$.

13. $y'''_{xxx} + fy''_{xx} + afy'_x + a^3y = 0.$

The substitution $w = y'_x + ay$ leads to a second order linear equation: $w''_{xx} + (f - a)qw'_x + a^2w = 0$.

14. $y'''_{xxx} + fy''_{xx} + afy'_x + a^2(f - a)y = 0.$

Particular solutions: $y_1 = e^{-ax/2}\cos\biggl(\dfrac{a\sqrt{3}}{2}x\biggr)$, $y_2 = e^{-ax/2}\sin\biggl(\dfrac{a\sqrt{3}}{2}x\biggr)$.

15. $y'''_{xxx} + fy''_{xx} + gy'_x + h = 0.$

The substitution $w = y'_x$ leads to a second order linear equation: $w''_{xx} + fw'_x + gw + h = 0$.

16. $\boldsymbol{y'''_{xxx} + fy''_{xx} - a(2f + 3a)y'_x + a^2(f + 2a)y = 0}.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = xe^{ax}$.

17. $\boldsymbol{y'''_{xxx} + fy''_{xx} + xgy'_x - gy = 0}.$

The substitution $w = xy'_x - y$ leads to a second order linear equation: $xw''_{xx} + (xf - 1)w'_x + x^2 gw = 0$.

18. $\boldsymbol{y'''_{xxx} + fy''_{xx} + (g - a^2)y'_x - a(af + g)y = 0}.$

Particular solution: $y_0 = e^{ax}$.

The substitution $w = y'_x - ay$ leads to a second order linear equation: $w''_{xx} + (f + a)w'_x + (af + g)w = 0$.

19. $\boldsymbol{y'''_{xxx} + fy''_{xx} + (af + b - a^2)y'_x + b(f - a)y = 0}.$

Particular solutions: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$, where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

20. $\boldsymbol{y'''_{xxx} + (f - a)y''_{xx} - a^2 fy = 0}.$

Particular solution: $y_0 = e^{ax}$.

The substitution $w = y'_x - ay$ leads to a second order equation: $w''_{xx} + fw'_x + afw = 0$.

21. $\boldsymbol{y'''_{xxx} = (f - a)y''_{xx} + (af - b)y'_x + bfy}.$

Particular solutions: $y_1 = \exp(\lambda_1 x)$, $y_2 = \exp(\lambda_2 x)$, where λ_1 and λ_2 are roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$.

The substitution $w = y''_{xx} + ay'_x + by$ leads to a first order linear equation: $w'_x = fw$.

22. $\boldsymbol{y'''_{xxx} + (f - a)y''_{xx} + gy'_x - a(af + g)y = 0}.$

Particular solution: $y_0 = e^{ax}$.

23. $\boldsymbol{y'''_{xxx} + (f + a)y''_{xx} + (af + g)y'_x + agy = 0}.$

Particular solution: $y_0 = e^{-ax}$.

24. $\boldsymbol{y'''_{xxx} + xfy''_{xx} + (ax^2 - f)y'_x + ax(x^2 f + 3)y = 0}.$

Particular solutions: $y_1 = \cos\left(\dfrac{x^2\sqrt{a}}{2}\right)$, $y_2 = \sin\left(\dfrac{x^2\sqrt{a}}{2}\right)$.

25. $\boldsymbol{y'''_{xxx} + (ax + b)fy''_{xx} + xfy'_x - 2fy = 0}.$

Particular solution: $y_0 = x^2 + 2ax + b$.

26. $\boldsymbol{y'''_{xxx} + (f + ax)y''_{xx} + a(xf + 2)y'_x + afy = 0}.$

Particular solutions: $y_1 = \exp\left(-\dfrac{ax^2}{2}\right)$, $y_2 = \exp\left(-\dfrac{ax^2}{2}\right)\displaystyle\int \exp\left(\dfrac{ax^2}{2}\right) dx$.

27. $\boldsymbol{y'''_{xxx} + x^2 f y''_{xx} - 2xfy'_x + 2fy = 0}$.

Particular solutions: $y_1 = x, \quad y_2 = x^2$.

Solution:

$$y = C_1 x + C_2 x^2 + C_3\Big(x^2 \int x^{-3}\psi\,dx - x\int x^{-2}\psi\,dx\Big),$$

where $\psi = \exp(-\int x^2 f\,dx)$.

28. $\boldsymbol{y'''_{xxx} + (f + ax)y''_{xx} + (g + 2a)y'_x + a[xg + (1 - ax^2)f]y = 0}$.

Particular solution: $y_0 = \exp\Big(-\dfrac{ax^2}{2}\Big)$.

The substitution $w = y'_x + axy$ leads to a second order linear equation: $w''_{xx} + fw'_x + (g - axf)w = 0$.

29. $\boldsymbol{y'''_{xxx} + (axf + f + a)y''_{xx} + a^2 xfy'_x - a^2 fy = 0}$.

Particular solutions: $y_1 = x, \quad y_2 = e^{-ax}$.

30. $\boldsymbol{y'''_{xxx} + (ax^2 + bx + c)fy''_{xx} - 2afy = 0}$.

Particular solution: $y_0 = ax^2 + bx + c$.

31. $\boldsymbol{y'''_{xxx} + x(xf + g)y''_{xx} - gy'_x - 2fy = 0}$.

Particular solution: $y_0 = x^2$. The substitution $w = xy'_x - 2y$ leads to a second order linear equation: $w''_{xx} + x(xf + g)w'_x + xfw = 0$.

32. $\boldsymbol{y'''_{xxx} - x(ax + b)fy''_{xx} + (b - a^2)fy'_x + 2afy = 0}$.

Particular solution: $y_0 = x^2 + ax + \dfrac{1}{2}(a^2 - b)$.

33. $\boldsymbol{y'''_{xxx} - [(2x + a)f + (x^2 + ax + b)g]y''_{xx} + 2fy'_x + 2gy = 0}$.

Particular solution: $y_0 = x^2 + ax + b$.

34. $\boldsymbol{xy'''_{xxx} + 3y''_{xx} + x(ax^2 + 1)fy'_x - (ax^2 - 1)fy = 0}$.

Particular solution: $y_0 = ax + \dfrac{1}{x}$.

35. $\boldsymbol{xy'''_{xxx} + (ax^2 + b)y''_{xx} + 4axy'_x + 2ay = f}$.

Integrating the equation twice, we obtain a first order linear equation:

$$xy'_x + (ax^2 + b - 2)y = C_1 + C_2 x + \int\Big(\int f\,dx\Big)\,dx.$$

36. $\boldsymbol{xy'''_{xxx} + xfy'_x - [(ax + 1)f + a^3 x + 3a^2]y = 0}$.

Particular solution: $y_0 = xe^{ax}$.

37. $xy'''_{xxx} + x(f - 2a)y''_{xx} + x(g + a^2)y'_x - [a(ax + 2)f + (ax + 1)g]y = 0.$

Particular solution: $y_0 = xe^{ax}$.

38. $xy'''_{xxx} + (xf + 3)y''_{xx} + (2f + ax)y'_x + a(xf + 1)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}\cos(x\sqrt{a}), \quad y_2 = \dfrac{1}{x}\sin(x\sqrt{a})$.

39. $xy'''_{xxx} + (xf + 3)y''_{xx} + (ax + 2)fy'_x + a(axf + f - a^2x)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}e^{-ax/2}\cos\left(\dfrac{a\sqrt{3}}{2}x\right), \quad y_2 = \dfrac{1}{x}e^{-ax/2}\sin\left(\dfrac{a\sqrt{3}}{2}x\right)$.

40. $xy'''_{xxx} + (xf + 3)y''_{xx} + (axf + 2f - a^2x)y'_x + a(f - a)y = 0.$

Particular solutions: $y_1 = \dfrac{1}{x}, \quad y_2 = \dfrac{1}{x}e^{-ax}$.

41. $xy'''_{xxx} + (xf + 3)y''_{xx} + (2f + ax^{n+1})y'_x + ax^n(xf + n + 1)y = 0.$

The substitution $w = xy$ leads to an equation of the form 3.1.9.12: $w'''_{xxx} + fw''_{xx} + ax^n w'_x + ax^{n-1}(xf + n)w = 0$.

42. $xy'''_{xxx} + (x^2f + a + 2)y''_{xx} - a(a + 1)fy = 0.$

Particular solution: $y_0 = x^{-a}$.

The substitution $w = xy'_x + ay$ leads to a second order linear equation: $w''_{xx} + xfw'_x - (a + 1)fw = 0$.

43. $xy'''_{xxx} + [x^2(ax^2 + 1)f + 3]y''_{xx} - 2fy = 0.$

Particular solution: $y_0 = ax + \dfrac{1}{x}$.

44. $xy'''_{xxx} + [x(ax^2 - 1)f + x^2(ax^2 + 1)g + 3]y''_{xx} - 2fy'_x - 2gy = 0.$

Particular solution: $y_0 = ax + \dfrac{1}{x}$.

45. $(ax - 1)y'''_{xxx} + x[(ax - 2)f - a^2]y''_{xx} + [(2 - a^2x^2)f + a^2]y'_x + 2a(ax - 1)fy = 0.$

Particular solutions: $y_1 = x^2, \quad y_2 = e^{ax}$.

46. $x^2y'''_{xxx} + (xf - a^2 - a)y'_x + (a - 1)fy = 0.$

Particular solution: $y_0 = x^{1-a}$.

The substitution $w = xy'_x + (a - 1)y$ leads to a second order linear equation: $xw''_{xx} - (a + 1)w'_x + fw = 0$.

47. $x^2y'''_{xxx} + [x(ax + 1)f - 6]y'_x + fy = 0.$

Particular solution: $y_0 = a + \dfrac{1}{x}$.

48. $x^2 y'''_{xxx} + xfy''_{xx} + [x(ax+1)g + 2f - 6]y'_x + gy = 0.$

Particular solution: $y_0 = a + \dfrac{1}{x}$.

49. $x^2 y'''_{xxx} + x[x(ax+1)f + 3]y''_{xx} - 2fy = 0.$

Particular solution: $y_0 = a + \dfrac{1}{x}$.

50. $x^2 y'''_{xxx} + x(xf + a)y''_{xx} + [(a-2)xf + b]y'_x + (b - a + 2)fy = 0.$

By integrating, we obtain the nonhomogeneous Euler equation 2.1.8.15:

$$x^2 y''_{xx} + (a-2)xy'_x + (b-a+2)y = C\exp\Bigl(-\int f\,dx\Bigr).$$

51. $(ax+b)xy'''_{xxx} + (\alpha x + \beta)y''_{xx} + xy'_x + y = f.$

By integrating, we obtain a second order linear equation:

$$(ax+b)xy''_{xx} + [(\alpha - 2a)x + \beta - b]y'_x + (x + 2a - \alpha)y = \int f\,dx + C.$$

52. $x(x+1)y'''_{xxx} + x(f - x - 3)y'_x - (x+1)fy = 0.$

Particular solution: $y_0 = xe^x$.

53. $x^3 y'''_{xxx} + xfy'_x + (a-1)(f + a^2 + a)y = 0.$

Particular solution: $y_0 = x^{1-a}$.

The substitution $w = xy'_x + (a-1)y$ leads to a second order linear equation: $x^2 w''_{xx} - (a+1)xw'_x + (f + a^2 + a)w = 0$.

54. $x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x + cy = f(x).$

The nonhomogeneous Euler equation.

The substitution $t = \ln|x|$ leads to an equation of the form 3.1.9.8.

$$y'''_{ttt} + (a-3)y''_{tt} + (b-a+2)y'_t + cy = f(\pm e^t).$$

55. $x^3 y'''_{xxx} + (a+2)x^2 y''_{xx} + xfy'_x + afy = 0.$

Particular solution: $y_0 = x^{-a}$.

56. $x^3 y'''_{xxx} + [(a+6)x^2 + b]y''_{xx} + 2(2a+3)xy'_x + 2ay = f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$x^3 y'_x + (ax^2 + b)y = C_1 + C_2 x + \int\Bigl(\int f\,dx\Bigr)dx.$$

57. $x^3 y'''_{xxx} + x^2(bx^{2a+1} - 3a)y''_{xx} + 2a(a+1)(2a+1)y = f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$x^{-2a} y'_x + (ax^{-2a-1} + b)y = C_1 + C_2 x + \int\Bigl(\int f x^{-2a-3}\,dx\Bigr)dx.$$

58. $x^3y'''_{xxx}+x^2fy''_{xx}-2xy'_x+2(2-f)y=0.$

Particular solutions: $y_1=x^{-1}$, $y_2=x^2$.

59. $x^3y'''_{xxx}+x^2fy''_{xx}-6xy'_x+6(2-f)y=0.$

Particular solutions: $y_1=x^{-2}$, $y_2=x^3$.

60. $x^3y'''_{xxx}+x^2fy''_{xx}+x(f-1)y'_x+(f-3)y=0.$

Particular solutions: $y_1=\cos(\ln x)$, $y_2=\sin(\ln x)$.

61. $x^3y'''_{xxx}+x^2(f+1)y''_{xx}+x(f-a-1)y'_x-a(f-2)y=0.$

Particular solutions: $y_1=x^{-\sqrt{a}}$, $y_2=x^{\sqrt{a}}$.

62. $x^3y'''_{xxx}+x^2(f+a)y''_{xx}+x(af+b-a)y'_x+b(f-2)y=0.$

Particular solutions: $y_1=x^{n_1}$, $y_2=x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2+(a-1)n+b=0$.

63. $x^3y'''_{xxx}+x^2(f+a)y''_{xx}+x[g+(a-1)f]y'_x+(a-2)gy=0.$

Particular solution: $y_0=x^{2-a}$. The substitution $w=xy'_x+(a-2)y$ leads to a second order linear equation: $x^2w''_{xx}+xfw'_x+gw=0$.

64. $x^3y'''_{xxx}+x^2(f+2ax)y''_{xx}+x(2axf+a^2x^2+b)y'_x+(a^2x^2f+bf-2b)y=0.$

Particular solutions: $y_1=e^{-ax}x^{n_1}$, $y_2=e^{-ax}x^{n_2}$, where n_1 and n_2 are the roots of the quadratic equation $n^2-n+b=0$.

65. $x^6y'''_{xxx}+x^2fy'_x+(a^3+af-2xf)y=0.$

Particular solution: $y_0=x^2e^{a/x}$.

66. $y'''_{xxx}+ae^{\lambda x}y''_{xx}-3\lambda^2y'_x+2\lambda^3y=f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$e^{-\lambda x}y'_x+(a+2\lambda e^{-\lambda x})y=C_1+C_2x+\int\left(\int fe^{-\lambda x}\,dx\right)dx.$$

67. $y'''_{xxx}+(f-a^2e^{2\lambda x})y'_x-ae^{\lambda x}(f+3a\lambda e^{\lambda x}+\lambda^2)y=0.$

Particular solution: $y_0=\exp\left(\dfrac{a}{\lambda}e^{\lambda x}\right)$.

The substitution $y=\exp\left(\dfrac{a}{\lambda}e^{\lambda x}\right)\displaystyle\int z(x)\,dx$ leads to a second order linear equation: $z''_{xx}+3ae^{\lambda x}z'_x+(f+2a^2e^{2\lambda x}+3a\lambda e^{\lambda x})z=0$.

68. $y'''_{xxx}+[(1+be^{ax})f-a^2]y'_x+afy=0.$

Particular solution: $y_0=e^{-ax}+b$.

69. $y'''_{xxx} + (f + a)y''_{xx} + [af + (1 + be^{ax})g]y'_x + agy = 0.$

Particular solution: $y_0 = e^{-ax} + b$.

70. $y'''_{xxx} + (be^{ax} + 2a)fy''_{xx} - a(be^{ax}f + a)y'_x - 2a^3 fy = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = e^{-ax} + \dfrac{b}{a}$.

71. $y'''_{xxx} + (f - 2ae^{\lambda x})y''_{xx} - ae^{\lambda x}(2f - ae^{\lambda x} + 3\lambda)y'_x$
$+ ae^{\lambda x}[(ae^{\lambda x} - \lambda)f + 2a\lambda e^{\lambda x} - \lambda^2]y = 0.$

Particular solutions: $y_1 = \exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$, $y_2 = x\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

72. $y'''_{xxx} + (f - ae^{\lambda x})y''_{xx} + (g - 2a\lambda e^{\lambda x})y'_x - ae^{\lambda x}[(ae^{\lambda x} + \lambda)f + g + \lambda^2]y = 0.$

Particular solution: $y_0 = \exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

The substitution $y = \exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x}\Bigr)\displaystyle\int z(x)\,dx$ leads to a second order equation:

$$z''_{xx} + (f + 2ae^{\lambda x})z'_x + (2ae^{\lambda x}f + g + a^2e^{2\lambda x} + a\lambda e^{\lambda x})z = 0.$$

73. $y'''_{xxx} - [(a+c+be^{ax})f - a + c]y''_{xx} + [(c^2 - a^2 + bce^{ax})f - ac]y'_x + ac(a+c)fy = 0.$

Particular solutions: $y_1 = e^{cx}$, $y_2 = e^{-ax} + \dfrac{b}{c}$.

74. $e^{\lambda x}y'''_{xxx} + (2\lambda e^{\lambda x} + \beta e^{\mu x} + \gamma)y''_{xx} + (\lambda^2 e^{\lambda x} + 2\beta\mu e^{\mu x})y'_x + \beta\mu^2 e^{\mu x}y = f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$e^{\lambda x}y'_x + (\beta e^{\mu x} + \gamma)y = C_1 + C_2 x + \int\Bigl(\int f\,dx\Bigr)dx.$$

75. $y'''_{xxx} + fy''_{xx} + gy'_x - \lambda[\lambda f + \tanh(\lambda x)(g + \lambda^2)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x)$.

The substitution $y = \cosh(\lambda x)\int z(x)\,dx$ leads to a second order equation:

$$z''_{xx} + [f + 3\lambda\tanh(\lambda x)]z'_x + [g + 3\lambda^2 + 2\lambda f\tanh(\lambda x)]z = 0.$$

76. $y'''_{xxx} + fy''_{xx} - \lambda[2f\tanh(\lambda x) + 3\lambda]y'_x$
$+ \lambda^2\{[2\tanh^2(\lambda x) - 1]f + 2\lambda\tanh(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cosh(\lambda x)$, $y_2 = x\cosh(\lambda x)$.

77. $y'''_{xxx} + fy''_{xx} - \lambda[2f\coth(\lambda x) + 3\lambda]y'_x$
$+ \lambda^2\{[2\coth^2(\lambda x) - 1]f + 2\lambda\coth(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sinh(\lambda x)$, $y_2 = x\sinh(\lambda x)$.

78. $y'''_{xxx} + [(\tanh x - a)f - a]y''_{xx} + [(a^2 - 1)f - 1]y'_x + a[(1 - a\tanh x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cosh x$.

79. $y'''_{xxx} + [(\coth x - a)f - a]y''_{xx} + [(a^2-1)f - 1]y'_x + a[(1 - a\coth x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \sinh x$.

80. $y'''_{xxx} + [\lambda \tanh(\lambda x)(xf - 1) - f]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cosh(\lambda x)$.

81. $y'''_{xxx} + [\lambda \coth(\lambda x)(xf - 1) - f]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sinh(\lambda x)$.

82. $xy'''_{xxx} + [x^2(a - \ln x)f + 2]y''_{xx} + xfy'_x - fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \ln x - a + 1$.

83. $y'''_{xxx} + fy'_x + \tan x(f - 1)y = 0.$

Particular solution: $y_0 = \cos x$.

The substitution $y = \cos x \int z(x)\,dx$ leads to a second order linear equation: $z''_{xx} - 3\tan x\, z'_x + (f-3)z = 0$.

84. $y'''_{xxx} + fy'_x + \cot x(1 - f)y = 0.$

Particular solution: $y_0 = \sin x$.

85. $y'''_{xxx} + fy''_{xx} + gy'_x + \lambda[\lambda f + \tan(\lambda x)(g - \lambda^2)]y = 0.$

Particular solution: $y_0 = \cos(\lambda x)$.

The substitution $y = \cos(\lambda x) \int z(x)\,dx$ leads to a second order linear equation:

$$z''_{xx} + [f - 3\lambda \tan(\lambda x)]z'_x + [g - 3\lambda^2 - 2\lambda f \tan(\lambda x)]z = 0.$$

86. $y'''_{xxx} + fy''_{xx} + \lambda[2f\tan(\lambda x) + 3\lambda]y'_x + \lambda^2\{[1 + 2\tan^2(\lambda x)]f + 2\lambda\tan(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \cos(\lambda x)$, $y_2 = x\cos(\lambda x)$.

87. $y'''_{xxx} + fy''_{xx} + \lambda[3\lambda - 2f\cot(\lambda x)]y'_x + \lambda^2\{[1 + 2\cot^2(\lambda x)]f - 2\lambda\cot(\lambda x)\}y = 0.$

Particular solutions: $y_1 = \sin(\lambda x)$, $y_2 = x\sin(\lambda x)$.

88. $y'''_{xxx} - [(a + \tan x)f + a]y''_{xx} + [(a^2+1)f + 1]y'_x + a[(a\tan x - 1)f - 1]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos x$.

89. $y'''_{xxx} + [(\cot x + a)f + a]y''_{xx} + [(a^2+1)f + 1]y'_x + a[(1 - a\cot x)f + 1]y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = \sin x$.

90. $y'''_{xxx} + [f + \lambda\tan(\lambda x)(xf + 1)]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \cos(\lambda x)$.

91. $y'''_{xxx} + [f - \lambda\cot(\lambda x)(xf + 1)]y''_{xx} - \lambda^2 xfy'_x + \lambda^2 fy = 0.$

Particular solutions: $y_1 = x$, $y_2 = \sin(\lambda x)$.

92. $a\sin(\lambda x)y'''_{xxx} + by''_{xx} + 3a\lambda^2\sin(\lambda x)y'_x + 2a\lambda^3\cos(\lambda x)y = f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$a\sin(\lambda x)y'_x + [b - 2a\lambda\cos(\lambda x)]y = C_1 + C_2x + \int\Big(\int f\,dx\Big)\,dx.$$

93. $\sin(\lambda x)y'''_{xxx} + [a + (2\lambda+1)\cos(\lambda x)]y''_{xx} - (\lambda^2 + 2\lambda)\sin(\lambda x)y'_x$
$- \lambda^2\cos(\lambda x)y = f(x).$

Integrating the equation twice, we obtain a first order linear equation:

$$\sin(\lambda x)y'_x + [a + \cos(\lambda x)]y = C_1 + C_2x + \int\Big(\int f\,dx\Big)\,dx.$$

94. $(f-1)y'''_{xxx} - [af + \lambda\tan(\lambda x)]y''_{xx} + (\lambda^2 f + a^2)y'_x + a\lambda[a\tan(\lambda x) - \lambda f]y = 0.$

Particular solutions: $y_1 = e^{ax}$, $y_2 = \cos(\lambda x)$.

95. $y'''_{xxx} + fy'_x + f'_x y = g.$

Integration yeilds a second order linear equation: $y''_{xx} + fy = \int g\,dx + C$.

96. $y'''_{xxx} + 2fy'_x + f'_x y = 0.$

Solution: $y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2$, where w_1 and w_2 are linearly-independent solutions of the second order linear equation $2w''_{xx} + fw = 0$.

97. $y'''_{xxx} + f'_x y'_x + f(2f'_x - f^2)y = 0.$

Integration yeilds a second order linear equation: $y''_{xx} + fy'_x + f^2y = C\exp(\int f\,dx)$.

98. $y'''_{xxx} + (a-1)f^2y'_x - [f''_{xx} - (2a+1)ff'_x + af^3]y = 0.$

Integration yeilds a second order linear equation:

$$y''_{xx} + fy'_x + (af^2 - f'_x)y = C\exp\Big(\int f\,dx\Big).$$

99. $y'''_{xxx} + (f - a^2)y'_x + (f'_x - af)y = 0.$

The substitution $w = y''_{xx} + ay'_x + fy$ leads to a first order linear equation: $w'_x - aw = 0$.

100. $y'''_{xxx} + fy''_{xx} + gy'_x + (fg + g'_x)y = 0.$

Integration yeilds a second order linear equation: $y''_{xx} + gy = C\exp(-\int f\,dx)$.

101. $y'''_{xxx} + 3fy''_{xx} + (f'_x + 2f^2 + 2g)y'_x + (2fg + g'_x)y = 0.$

Solution:

$$y = C_1w_1^2 + C_2w_1w_2 + C_3w_2^2,$$

where w_1 and w_2 is the fundamental set of solutions of the second order linear equation $w''_{xx} + fw'_x + \frac{1}{2}gw = 0$.

102. $y'''_{xxx} + (f+g)y''_{xx} + (f'_x + fg + h)y'_x + (h'_x + gh)y = 0.$

Integration yeilds a second order equation: $y''_{xx} + fy'_x + hy = C\exp(-\int g\,dx)$.

103. $y'''_{xxx} + (f+g)y''_{xx} + (2g'_x + fg + h)y'_x + (g''_{xx} + fg'_x + gh)y = 0.$

The substitution $w = y'_x + gy$ leads to a second order equation: $w''_{xx} + fw'_x + hw = 0$.

104. $fy'''_{xxx} - f'''_{xxx}y = 0.$

Particular solution: $y_0 = f$.

The substitution $y = f\int z\,dx$ leads to a second order equation: $fz''_{xx} + 3f'_x z'_x + 3f''_{xx}z = 0$.

105. $fy'''_{xxx} + f'''_{xxx}y = g.$

Integration yeilds a second order equation: $fy''_{xx} - f'_x y'_x + f''_{xx}y = \int g\,dx + C$.

106. $y'''_{xxx} = f(x)y.$

The transformation $x = t^{-1}$, $y = wt^{-2}$ leads to an equation of the similar form: $w'''_{ttt} = -t^{-6}f\left(\dfrac{1}{t}\right)w$.

3.2. Equations of the Form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

3.2.1. Preliminary Comments. Classification Table

The value of the insignificant parameter A is in many cases definined in the form of a function of two (one) auxiliary coefficients a and b:

$$A = \varphi(a,\ b) \tag{1}$$

and the corresponding solutions are represented in the parametric form

$$x = f_1(\tau, C_1, C_2, C_3, a), \qquad y = f_2(\tau, C_1, C_2, C_3, b), \tag{2}$$

where τ is a parameter, C_1, C_2, and C_3 are arbitrary constants, f_1 and f_2 are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $b > 0$), the coefficient b should be expressed in terms of both A and a with the help of

$$b = \psi(A,\ a).$$

Substituting this formula into (2), we obtain a solution of the equation under consideration (where the concrete numerical value of the coefficient a may be chosen arbitrarily). The case $a < 0$ (or $b < 0$), which may lead to the branch of the solution or to a different domain of determining the variables x and y in (2), should be considered in a similar manner.

The following Table 3.1 represents all solvable equations whose solutions are outlined in Subsections 3.2.2–3.2.4. The two-parameter families (in the space of parameters α, β, γ, and δ), one-parameter famililies and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of δ, the growth of γ (for identical δ), the growth of β (for identical δ and γ), and the growth of α (for identical δ, γ, and β). The number of the equation sought is indicated in the last column in this table.

TABLE 3.1
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha} y^{\beta} (y'_x)^{\gamma} (y''_{xx})^{\delta}$

δ	γ	β	α	Equation
arbitrary	arbitrary	0	arbitrary	3.2.4.15*
arbitrary ($\delta \neq 2$)	arbitrary ($\gamma \neq -1$)	0	0	3.2.4.1
$\dfrac{\gamma+4\beta+5}{\gamma+2\beta+3}$	arbitrary ($\gamma \neq -1$)	arbitrary ($\beta \neq -1$)	0	3.2.4.174
$\dfrac{3\gamma+7}{2(\gamma+2)}$	arbitrary ($\gamma \neq -2$)	$-\frac{1}{2}$	0	3.2.4.10
$\dfrac{3\gamma+7}{2(\gamma+2)}$	arbitrary ($\gamma \neq -2$)	1	0	3.2.4.7
arbitrary ($\delta \neq 1,\ 2$)	-1	-1	0	3.2.4.175
arbitrary ($\delta \neq 2$)	-1	0	0	3.2.4.11
$\dfrac{3\beta+4}{2\beta+3}$	0	arbitrary ($\beta \neq -\frac{3}{2}$)	0	3.2.4.8
arbitrary ($\delta \neq \frac{3}{2}$)	0	$-\frac{1}{2}$	0	3.2.4.87
arbitrary ($\delta \neq 1$)	1	arbitrary ($\beta \neq -1$)	0	3.2.4.2
arbitrary ($\delta \neq 1$)	1	-1	0	3.2.4.13
arbitrary ($\delta \neq 2$)	1	1	0	3.2.4.4
$\dfrac{3\beta+4}{2\beta+3}$	3	arbitrary ($\beta \neq -\frac{3}{2}$)	0	3.2.4.9
-1	3	$-\frac{7}{5}$	0	3.2.4.168
-1	3	0	0	3.2.4.164
0	arbitrary ($\gamma \neq -1$)	0	0	3.2.4.3
0	arbitrary	$-\frac{1}{4}(\gamma+5)$	0	3.2.4.171
0	$-2\beta-5$	arbitrary ($\beta \neq -2$)	0	3.2.4.5
0	-13	1	0	3.2.4.153
0	-13	3	0	3.2.4.155
0	-7	0	0	3.2.4.141
0	-7	1	0	3.2.4.145

* given are formulae of reducing to the generalized Emden—Fowler equation

TABLE 3.1 *Continued*

Solvable equations of the form $y'''_{xxx} = Ax^{\alpha} y^{\beta} (y'_x)^{\gamma} (y''_{xx})^{\delta}$

δ	γ	β	α	Equation
0	-4	$-\frac{1}{2}$	0	3.2.4.127
0	-4	0	0	3.2.4.123
0	-3	-2	0	3.2.4.95
0	-3	-1	0	3.2.4.30
0	-3	0	0	3.2.4.26
0	-3	1	0	3.2.4.91
0	$-\frac{7}{3}$	$-\frac{10}{3}$	0	3.2.4.76
0	$-\frac{7}{3}$	$-\frac{7}{3}$	0	3.2.4.42
0	$-\frac{7}{3}$	$-\frac{4}{3}$	0	3.2.4.52
0	$-\frac{7}{3}$	$-\frac{5}{6}$	0	3.2.4.133
0	$-\frac{7}{3}$	$-\frac{1}{2}$	0	3.2.4.131
0	$-\frac{7}{3}$	0	0	3.2.4.48
0	$-\frac{7}{3}$	1	0	3.2.4.38
0	$-\frac{7}{3}$	2	0	3.2.4.70
0	$-\frac{9}{5}$	$-\frac{13}{5}$	0	3.2.4.64
0	$-\frac{9}{5}$	1	0	3.2.4.60
0	-1	-2	0	3.2.4.22
0	-1	0	0	3.2.4.18
0	0	$-\frac{7}{2}$	0	3.2.2.2
0	0	$-\frac{7}{2}$	3	3.2.3.3
0	0	$-\frac{5}{2}$	0	3.2.2.3
0	0	$-\frac{5}{2}$	1	3.2.3.4
0	0	-2	0	3.2.2.6
0	0	$-\frac{4}{3}$	$-\frac{4}{3}$	3.2.3.5
0	0	$-\frac{4}{3}$	0	3.2.2.4
0	0	$-\frac{5}{4}$	$-\frac{3}{2}$	3.2.3.7
0	0	$-\frac{5}{4}$	0	3.2.2.8
0	0	$-\frac{7}{6}$	$-\frac{5}{3}$	3.2.3.6
0	0	$-\frac{7}{6}$	0	3.2.2.5
0	0	$-\frac{1}{2}$	-3	3.2.3.8

TABLE 3.1 *Continued*

Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	3.2.3.9
0	0	$-\frac{1}{2}$	0	3.2.2.7
0	0	0	arbitrary	3.2.3.1
0	0	0	0	3.2.2.1
0	0	1	arbitrary	3.2.3.2
0	1	1	0	3.2.4.35
0	2	$-\frac{7}{2}$	0	3.2.4.165
0	2	0	0	3.2.4.161
0	3	arbitrary ($\beta \neq -2$)	0	3.2.4.85
0	3	-2	0	3.2.4.82
0	5	-5	0	3.2.4.105
0	5	$-\frac{20}{7}$	0	3.2.4.117
0	5	$-\frac{15}{7}$	0	3.2.4.111
0	5	0	0	3.2.4.101
$\frac{1}{2}$	0	$-\frac{5}{2}$	0	3.2.4.74
$\frac{1}{2}$	3	$-\frac{15}{8}$	0	3.2.4.114
$\frac{1}{2}$	3	$-\frac{20}{13}$	0	3.2.4.120
$\frac{1}{2}$	3	$-\frac{5}{4}$	0	3.2.4.108
$\frac{1}{2}$	3	0	0	3.2.4.104
$\frac{2}{3}$	0	$-\frac{7}{6}$	0	3.2.4.157
$\frac{4}{5}$	-4	$-\frac{1}{2}$	0	3.2.4.137
1	arbitrary ($\gamma \neq 1$)	-1	0	3.2.4.140
1	-3	$-\frac{1}{2}$	0	3.2.4.32
1	-3	1	0	3.2.4.24
1	-1	-1	0	3.2.4.177
1	1	arbitrary ($\beta \neq -1$)	0	3.2.4.14
1	1	-1	0	3.2.4.17
1	1	1	0	3.2.4.21
$\frac{8}{7}$	3	$-\frac{3}{4}$	0	3.2.4.159

TABLE 3.1 *Continued*
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
$\frac{8}{7}$	3	$-\frac{1}{2}$	0	3.2.4.151
$\frac{6}{5}$	0	$-\frac{2}{3}$	0	3.2.4.109
$\frac{5}{4}$	-4	$-\frac{1}{2}$	0	3.2.4.57
$\frac{5}{4}$	3	$-\frac{1}{2}$	0	3.2.4.148
$\frac{5}{4}$	3	0	0	3.2.4.144
$\frac{9}{7}$	$-\frac{9}{4}$	1	0	3.2.4.66
$\frac{9}{7}$	0	$-\frac{1}{3}$	0	3.2.4.169
$\frac{9}{7}$	0	1	0	3.2.4.62
$\frac{13}{10}$	0	$-\frac{5}{2}$	0	3.2.4.80
$\frac{27}{20}$	0	$-\frac{2}{3}$	0	3.2.4.121
$\frac{18}{13}$	0	$-\frac{7}{2}$	0	3.2.4.68
$\frac{7}{5}$	-7	1	0	3.2.4.54
$\frac{7}{5}$	$-\frac{5}{2}$	1	0	3.2.4.45
$\frac{7}{5}$	$-\frac{13}{7}$	1	0	3.2.4.78
$\frac{7}{5}$	$-\frac{1}{3}$	1	0	3.2.4.72
$\frac{7}{5}$	0	1	0	3.2.4.40
$\frac{7}{5}$	1	1	0	3.2.4.50
$\frac{7}{5}$	3	0	0	3.2.4.126
$\frac{7}{5}$	3	1	0	3.2.4.130
$\frac{7}{5}$	11	1	0	3.2.4.135
$\frac{10}{7}$	0	$-\frac{5}{2}$	0	3.2.4.43
$\frac{22}{15}$	0	$-\frac{2}{3}$	0	3.2.4.115
$\frac{3}{2}$	arbitrary	$\frac{1}{2}(\gamma-1)$	0	3.2.4.173
$\frac{3}{2}$	-3	$-\frac{1}{2}$	0	3.2.4.100
$\frac{3}{2}$	-3	1	0	3.2.4.97
$\frac{3}{2}$	0	-2	0	3.2.4.99
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	3.2.4.84
$\frac{3}{2}$	0	1	0	3.2.4.93
$\frac{3}{2}$	1	1	0	3.2.4.28
$\frac{3}{2}$	3	-2	0	3.2.4.98

TABLE 3.1 *Continued*
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
$\frac{3}{2}$	3	$-\frac{1}{2}$	0	3.2.4.94
$\frac{3}{2}$	3	0	0	3.2.4.29
$\frac{23}{15}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.116
$\frac{11}{7}$	-4	$-\frac{1}{2}$	0	3.2.4.47
$\frac{8}{5}$	1	1	0	3.2.4.125
$\frac{8}{5}$	3	-4	0	3.2.4.55
$\frac{8}{5}$	3	$-\frac{7}{4}$	0	3.2.4.46
$\frac{8}{5}$	3	$-\frac{10}{7}$	0	3.2.4.79
$\frac{8}{5}$	3	$-\frac{2}{3}$	0	3.2.4.73
$\frac{8}{5}$	3	$-\frac{1}{2}$	0	3.2.4.41
$\frac{8}{5}$	3	0	0	3.2.4.51
$\frac{8}{5}$	3	1	0	3.2.4.129
$\frac{8}{5}$	3	5	0	3.2.4.136
$\frac{21}{13}$	-6	$-\frac{1}{2}$	0	3.2.4.69
$\frac{33}{20}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.122
$\frac{17}{10}$	-4	$-\frac{1}{2}$	0	3.2.4.81
$\frac{12}{7}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.170
$\frac{12}{7}$	3	$-\frac{13}{8}$	0	3.2.4.67
$\frac{12}{7}$	3	$-\frac{1}{2}$	0	3.2.4.63
$\frac{7}{4}$	0	$-\frac{5}{2}$	0	3.2.4.56
$\frac{7}{4}$	0	1	0	3.2.4.147
$\frac{7}{4}$	1	1	0	3.2.4.143
$\frac{9}{5}$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	3.2.4.110
$\frac{13}{7}$	$-\frac{1}{2}$	1	0	3.2.4.160
$\frac{13}{7}$	0	1	0	3.2.4.152
2	arbitrary ($\gamma \neq -1$)	0	0	3.2.4.12
2	-1	arbitrary ($\beta \neq 0$)	0	3.2.4.139
2	-1	-1	0	3.2.4.176
2	-1	0	0	3.2.4.16

TABLE 3.1 *Continued*
Solvable equations of the form $y'''_{xxx} = Ax^\alpha y^\beta (y'_x)^\gamma (y''_{xx})^\delta$

δ	γ	β	α	Equation
2	0	-2	0	3.2.4.33
2	3	-2	0	3.2.4.25
2	3	0	0	3.2.4.19
$\frac{11}{5}$	0	$-\frac{5}{2}$	0	3.2.4.138
$\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{1}{2}$	0	3.2.4.158
$\frac{5}{2}$	-4	$-\frac{1}{2}$	0	3.2.4.75
$\frac{5}{2}$	$-\frac{11}{4}$	1	0	3.2.4.113
$\frac{5}{2}$	$-\frac{27}{13}$	1	0	3.2.4.119
$\frac{5}{2}$	$-\frac{3}{2}$	1	0	3.2.4.107
$\frac{5}{2}$	1	1	0	3.2.4.103
3	arbitrary ($\gamma \neq -3$)	1	0	3.2.4.86
3	$-2\beta - 5$	arbitrary ($\beta \neq -2$)	0	3.2.4.6
3	arbitrary	$-\gamma - 2$	0	3.2.4.172
3	-9	2	0	3.2.4.106
3	-6	$-\frac{1}{2}$	0	3.2.4.59
3	-6	$\frac{1}{2}$	0	3.2.4.166
3	$-\frac{17}{3}$	$-\frac{5}{3}$	0	3.2.4.77
3	$-\frac{33}{7}$	2	0	3.2.4.118
3	$-\frac{21}{5}$	$-\frac{7}{5}$	0	3.2.4.65
3	-4	$-\frac{1}{2}$	0	3.2.4.37
3	$-\frac{11}{3}$	$-\frac{5}{3}$	0	3.2.4.44
3	$-\frac{23}{7}$	2	0	3.2.4.112
3	-3	-2	0	3.2.4.96
3	-3	-1	0	3.2.4.23
3	-3	$-\frac{1}{2}$	0	3.2.4.90
3	-3	1	0	3.2.4.83
3	$-\frac{5}{3}$	$-\frac{5}{3}$	0	3.2.4.53
3	$-\frac{5}{3}$	$-\frac{1}{2}$	0	3.2.4.149
3	$-\frac{4}{3}$	$-\frac{1}{2}$	0	3.2.4.150

TABLE 3.1 *Continued*
Solvable equations of the form $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$

δ	γ	β	α	Equation
3	-1	-2	0	3.2.4.31
3	$-\frac{2}{3}$	$-\frac{5}{3}$	0	3.2.4.134
3	0	$-\frac{5}{2}$	0	3.2.4.128
3	0	$-\frac{5}{3}$	0	3.2.4.132
3	0	$-\frac{1}{2}$	0	3.2.4.89
3	0	1	-3	3.2.4.88
3	1	-4	0	3.2.4.142
3	1	$-\frac{5}{2}$	0	3.2.4.124
3	1	-2	0	3.2.4.27
3	1	$-\frac{5}{3}$	0	3.2.4.49
3	1	-1	0	3.2.4.20
3	1	$-\frac{1}{2}$	0	3.2.4.34
3	1	$\frac{1}{2}$	0	3.2.4.162
3	1	2	0	3.2.4.102
3	3	-7	0	3.2.4.154
3	3	-4	0	3.2.4.146
3	3	-2	0	3.2.4.92
3	3	$-\frac{5}{3}$	0	3.2.4.39
3	3	$-\frac{7}{5}$	0	3.2.4.61
3	3	$-\frac{1}{2}$	0	3.2.4.58
3	3	0	0	3.2.4.36
3	5	$-\frac{5}{3}$	0	3.2.4.71
3	7	-7	0	3.2.4.156
4	$-\frac{9}{5}$	1	0	3.2.4.167
4	1	1	0	3.2.4.163

3.2.2. Equations of the Form $y'''_{xxx} = Ay^{\beta}$

1. $y'''_{xxx} = A$.

Solution: $y = \frac{1}{6}Ax^3 + C_2x^2 + C_1x + C_0$.

2. $y'''_{xxx} = Ay^{-7/2}$.

Solution in the parametric form:

$$x = aC_1^3 \int \bigl[C_1 e^{2\sigma\tau} + C_2 e^{-\sigma\tau} \sin\bigl(\sqrt{3}\,\sigma\tau\bigr)\bigr]^{-3/2}\,d\tau + C_3,$$

$$y = bC_1^2 \bigl[C_1 e^{2\sigma\tau} + C_2 e^{-\sigma\tau} \sin\bigl(\sqrt{3}\,\sigma\tau\bigr)\bigr]^{-1},$$

where $A = -8a^{-3}b^{9/2}\sigma^3$.

3. $y'''_{xxx} = Ay^{-5/2}$.

Solution in the parametric form:

$$x = aC_1^7 \int (\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3, \qquad y = bC_1^6(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = -6a^{-3}b^{7/2}$.

4. $y'''_{xxx} = Ay^{-4/3}$.

Solution in the parametric form:

$$x = aC_1^7 \int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3, \qquad y = bC_1^9(2\tau I \mp R)^3,$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $I = \int \tau R^{-1}\,d\tau + C_2$, $A = \pm 18a^{-3}b^{7/3}$.

5. $y'''_{xxx} = Ay^{-7/6}$.

Solution in the parametric form:

$$x = aC_1^{13} \int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3, \qquad y = bC_1^{18}(2\tau I \mp R)^{-3},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $I = \int \tau R^{-1}\,d\tau + C_2$, $A = \mp 18a^{-3}b^{13/6}$.

► *In the solutions of equations 6–7, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions.

6. $y'''_{xxx} = Ay^{-2}$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{-1} Z^{-2}\,d\tau + C_3, \qquad y = bC_1 \tau^{-2/3} Z^{-2},$$

where $A = \pm\frac{4}{3}a^{-3}b^3$.

7. $y'''_{xxx} = Ay^{-1/2}$.

Solution in the parametric form:

$$x = aC_1 \int Z\,d\tau + C_3, \qquad y = bC_1^2 \tau^{2/3} Z^2,$$

where $A = \mp\frac{4}{3}a^{-3}b^{3/2}$.

8. $y'''_{xxx} = Ay^{-5/4}$.

This is a special case of equation 3.2.4.171 with $\gamma = 0$.

3.2.3. Equations of the Form $y'''_{xxx} = Ax^{\alpha}y^{\beta}$

See Subsection 3.2.2 for the case $\alpha = 0$.

1. $y'''_{xxx} = Ax^{\alpha}$.

Solution: $y = Af(x) + C_2x^2 + C_1x + C_0$, where

$$f(x) = \begin{cases} \dfrac{x^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} & \text{if } \alpha \neq -1,\ -2,\ -3; \\ \frac{1}{2}x^2 \ln|x| - \frac{3}{4}x^2 & \text{if } \alpha = -1; \\ -x\ln|x| + x & \text{if } \alpha = -2; \\ \frac{1}{2}\ln|x| & \text{if } \alpha = -3. \end{cases}$$

2. $y'''_{xxx} = Ax^{\alpha}y$.

See equation 3.1.2.7.

3. $y'''_{xxx} = Ax^3y^{-7/2}$.

Solution in the parametric form:

$$x = aC_1^3\left(\int f^{-3/2}\,d\tau + C_3\right)^{-1}, \qquad y = bC_1^4 f^{-1}\left(\int f^{-3/2}\,d\tau + C_3\right)^{-2},$$

where $f = C_1e^{2\sigma\tau} + C_2e^{-\sigma\tau}\sin\bigl(\sqrt{3}\,\sigma\tau\bigr)$, $A = 8a^{-6}b^{9/2}\sigma^3$.

4. $y'''_{xxx} = Axy^{-5/2}$.

Solution in the parametric form:

$$x = aC_1^7\left[\int(\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3\right]^{-1},$$

$$y = bC_1^8(\tau^3 - 3\tau + C_2)^{-1}\left[\int(\tau^3 - 3\tau + C_2)^{-3/2}\,d\tau + C_3\right]^{-2},$$

where $A = 6a^{-4}b^{7/2}$.

► *In the solutions of equations 5–6, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \qquad I = \int \tau R^{-1}\,d\tau + C_2.$$

5. $y'''_{xxx} = Ax^{-4/3}y^{-4/3}$.

Solution in the parametric form:

$$x = aC_1^7\left[\int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3\right]^{-1},$$

$$y = bC_1^5(2\tau I \mp R)^3\left[\int R^{-1}(2\tau I \mp R)^2\,d\tau + C_3\right]^{-2},$$

where $A = \mp 18a^{-5/3}b^{7/3}$.

6. $y'''_{xxx} = Ax^{-5/3}y^{-7/6}$.

Solution in the parametric form:

$$x = aC_1^{13}\left[\int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3\right]^{-1},$$

$$y = bC_1^{8}(2\tau I \mp R)^{-3}\left[\int R^{-1}(2\tau I \mp R)^{-5/2}\,d\tau + C_3\right]^{-2},$$

where $A = \mp 18a^{-4/3}b^{13/6}$.

7. $y'''_{xxx} = Ax^{-3/2}y^{-5/4}$.

Solution in the parametric form:

$$x = aC_1^{3}\left[\int \tau^{-1/2}z^{-1/2}f^{3/4}\,d\tau + C_3\right]^{-1}, \qquad y = bC_1^{2}f\left[\int \tau^{-1/2}z^{-1/2}f^{3/4}\,d\tau + C_3\right]^{-2},$$

where $z = C_2 + \frac{1}{4}\tau^2 + 4B\tau^{1/2}$, $f = \exp\left(\int z^{-1/2}\,d\tau\right)$, $A = \frac{1}{2}Ba^{-3/2}b^{9/4}$.

8. $y'''_{xxx} = Ax^{-3}y^{-1/2}$.

Solution in the parametric form:

$$x = C_1\left[\int Z\,d\tau + C_3\right]^{-1}, \qquad y = b\tau^{2/3}Z^2\left[\int Z\,d\tau + C_3\right]^{-2},$$

where

$$A = \pm\tfrac{4}{3}b^{3/2}, \qquad Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

($J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions).

9. $y'''_{xxx} = Ax^{-3/2}y^{-1/2}$.

Solution in the parametric form:

$$x = aC_3\exp\left(2\int P\,d\tau\right), \qquad x = bC_3P^2\exp\left(2\int P\,d\tau\right),$$

where $P = P(\tau, C_1, C_2)$ is the general solution of the second Painlevé transcendent:

$$P''_{\tau\tau} = \pm\tau P + 2P^3, \qquad A = \pm\tfrac{1}{4}a^{-3/2}b^{3/2}.$$

3.2.4. Equations with $|\gamma| + |\delta| \neq 0$

1. $y'''_{xxx} = A(y'_x)^{\gamma}(y''_{xx})^{\delta}, \qquad \gamma \neq -1, \quad \delta \neq 2.$

Solution in the parametric form:

$$x = aC_1^{\gamma+\delta-1}\int \tau^{-1/2}\left(1 \pm \tau^{\frac{\gamma+1}{2}}\right)^{\frac{1}{\delta-2}}\,d\tau + C_3,$$

$$y = bC_1^{\gamma+2\delta-3}\int \left(1 \pm \tau^{\frac{\gamma+1}{2}}\right)^{\frac{1}{\delta-2}}\,d\tau + C_2,$$

where $A = \pm\dfrac{\gamma+1}{2-\delta}2^{\delta-2}a^{\gamma+2\delta-3}b^{1-\gamma-\delta}$.

2. $\boldsymbol{y'''_{xxx} = Ay^\beta y'_x (y''_{xx})^\delta, \qquad \beta \neq -1, \quad \delta \neq 1.}$

Solution in the parametric form:

$$x = aC_1^{\beta+\delta} \int \left[\int \left(1 \pm \tau^{\beta+1}\right)^{\frac{1}{1-\delta}} d\tau + C_2 \right]^{-1/2} d\tau + C_3, \qquad y = bC_1^{2\delta-2}\tau,$$

where $A = \pm \dfrac{\beta+1}{1-\delta} 2^{\delta-1} a^{2\delta-2} b^{-\beta-\delta}$.

► *In the solutions of equations 3–10, the following notation is used:*

$$R = \sqrt{1 \pm \tau^{m+1}}, \quad E = \int \left(1 \pm \tau^{m+1}\right)^{-1/2} d\tau + C_2, \quad F = RE - \tau.$$

3. $\boldsymbol{y'''_{xxx} = A(y'_x)^\gamma, \qquad \gamma \neq -1.}$

Solution in the parametric form:

$$x = aC_1^m \int \tau^{-1/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{m-1} E,$$

where $m = \dfrac{\gamma - 1}{2}$, $A = \pm \dfrac{m+1}{4} a^{2m-2} b^{-2m}$.

4. $\boldsymbol{y'''_{xxx} = Ayy'_x (y''_{xx})^\delta, \qquad \delta \neq 2.}$

Solution in the parametric form:

$$x = aC_1^{3m+1} \int E^{-1/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{2m+2} R,$$

where $m = \dfrac{1}{\delta - 2}$, $A = -8ma^2 b^{-3} \left[\pm \dfrac{2a^2}{(m+1)b} \right]^{1/m}$.

5. $\boldsymbol{y'''_{xxx} = Ay^\beta (y'_x)^{-2\beta-5}, \qquad \beta \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{m-3} \int \tau^{-1/2} E^{-3/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{2m-2} E^{-1},$$

where $m = -\beta - 3$, $A = \pm\frac{1}{4}(-1)^{-2m}(m+1) a^{2m-2} b^{3-m}$.

6. $\boldsymbol{y'''_{xxx} = Ay^\beta (y'_x)^{-2\beta-5} (y''_{xx})^3, \qquad \beta \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{m+3} \int E^{-3/2} R^{-1} F\, d\tau + C_3, \qquad y = bC_1^{2m+2} \tau E^{-1},$$

where $m = \beta$, $A = \mp 2(m+1) a^{-2m-2} b^{m+3}$.

7. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{\gamma}(y''_{xx})^{\frac{3\gamma+7}{2\gamma+4}}}, \qquad \gamma \neq -2.$

Solution in the parametric form:

$$x = aC_1^{m^2+m+2} \int E^{-m/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^4 F,$$

where $m = -\dfrac{2(\gamma+2)}{\gamma+1}$, $A = -\dfrac{2mb^{-1}}{m+2}\left[\pm\dfrac{(m+1)b}{2a}\right]^{\frac{2}{m+2}}\left[\pm\dfrac{4a^2}{(m+1)(m+2)b}\right]^{\frac{1}{m}}$.

8. $\boldsymbol{y'''_{xxx} = Ay^{\beta}(y''_{xx})^{\frac{3\beta+4}{2\beta+3}}}, \qquad \beta \neq -3/2.$

Solution in the parametric form:

$$x = aC_1^{m^2+2m-1} \int \tau^m E^{-m-2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{(m+1)^2} \tau^{m+1} E^{-m-1},$$

where $m = -\dfrac{\beta}{\beta+1}$, $A = (m+3)a^{-1}b^{\frac{m}{m+1}}\left[\pm\dfrac{2a^2}{(m+1)^2 b}\right]^{\frac{1}{m+3}}$.

9. $\boldsymbol{y'''_{xxx} = Ay^{\beta}(y'_x)^3(y''_{xx})^{\frac{3\beta+4}{2\beta+3}}}, \qquad \beta \neq -3/2.$

Solution in the parametric form:

$$x = aC_1^{m^2+m-1} \int E^{m+1} F^{-1/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^{(m-1)(m+2)} E^{m+2},$$

where $m = -\dfrac{2\beta+3}{\beta+1}$, $A = \dfrac{m}{(m+2)^3} a^2 b^{-\frac{2m+1}{m+1}}\left[\pm\dfrac{(m+1)(m+2)b}{4a^2}\right]^{\frac{1}{m}}$.

10. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{\gamma}(y''_{xx})^{\frac{3\gamma+7}{2\gamma+4}}}, \qquad \gamma \neq -2.$

Solution in the parametric form:

$$x = aC_1^{m^2+2m-7} \int \tau^{\frac{m-1}{2}} R^{-1} E^{\frac{m+3}{2}} F\, d\tau + C_3, \qquad y = bC_1^{-8} F^2,$$

where $m = \dfrac{1-\gamma}{1+\gamma}$, $A = -\dfrac{2(m+3)}{m+1} b^{1/2}\left[\pm\dfrac{a}{(m+1)b}\right]^{\frac{2}{m+1}}\left[\pm\dfrac{(m+1)^2 b}{2a^2}\right]^{\frac{1}{m+3}}$.

11. $\boldsymbol{y'''_{xxx} = A(y'_x)^{-1}(y''_{xx})^{\delta}}, \qquad \delta \neq 2.$

Solution in the parametric form:

$$x = aC_1 \int \tau^{\frac{\delta}{\delta-2}} \exp(\mp\tau^2)\, d\tau + C_2, \qquad y = bC_1^2 \int \tau^{\frac{\delta}{\delta-2}} \exp(\mp 2\tau^2)\, d\tau + C_3,$$

where $A = \mp\dfrac{4b^2}{(2-\delta)a^4}\left(\mp\dfrac{a^2}{2b}\right)^{\delta}$.

12. $y'''_{xxx} = A(y'_x)^\gamma (y''_{xx})^2, \qquad \gamma \neq -1.$

Solution in the parametric form:

$$x = aC_1 \int \tau^{\frac{1-\gamma}{1+\gamma}} \exp(\mp\tau^2)\, d\tau + C_2, \qquad y = bC_1 \int \tau^{\frac{3-\gamma}{1+\gamma}} \exp(\mp\tau^2)\, d\tau + C_3,$$

where $A = \pm(\gamma+1)a^{\gamma+1}b^{-\gamma-1}$.

13. $y'''_{xxx} = Ay^{-1}y'_x (y''_{xx})^\delta, \qquad \delta \neq 1.$

Solution in the parametric form:

$$x = aC_1 \int \tau \exp(\mp\tau^2) \left[\int \tau^{\frac{3-\delta}{1-\delta}} \exp(\mp\tau^2)\, d\tau + C_2\right]^{-1/2} d\tau + C_3, \qquad y = bC_1^2 \exp(\mp\tau^2),$$

where $A = \dfrac{1}{1-\delta}(\mp 1)^{-\delta} a^{2\delta-2} b^{1-\delta}$.

14. $y'''_{xxx} = Ay^\beta y'_x y''_{xx}, \qquad \beta \neq -1.$

Solution in the parametric form:

$$x = C_1 \int \tau^{\frac{1-\beta}{1+\beta}} \left[\int \tau^{\frac{1-\beta}{1+\beta}} \exp(\mp\tau^2)\, d\tau + C_2\right]^{-1/2} d\tau + C_3, \qquad y = b\tau^{\frac{2}{1+\beta}},$$

where $A = \mp(\beta+1)b^{-1-\beta}$.

15. $y'''_{xxx} = Ax^\alpha (y'_x)^\gamma (y''_{xx})^\delta.$

Solution in the parametric form:

$$x = aC_1^{\gamma+\delta-1} X(\tau), \qquad y = bC_1^{\gamma+2\delta-\alpha-3} \int Y(\tau) \frac{dX(\tau)}{d\tau}\, d\tau + C_3,$$

where $X = X(\tau)$, $Y = Y(\tau)$ is the general solution of the generalized Emden—Fowler equation

$$Y''_{XX} = BX^\alpha Y^\gamma (Y'_X)^\delta, \qquad A = Ba^{\gamma+2\delta-\alpha-3} b^{1-\gamma-\delta}.$$

16. $y'''_{xxx} = A(y'_x)^{-1}(y''_{xx})^2.$

Solution:

$$y = \begin{cases} \dfrac{1-A}{(2-A)C_1}(C_1x + C_2)^{\frac{2-A}{1-A}} + C_3 & \text{if } A \neq 1,\ A \neq 2, \\ \dfrac{C_2}{C_1} \exp(C_1 x) + C_3 & \text{if } A = 1, \\ \dfrac{1}{C_1} \ln(C_1 x + C_2) + C_3 & \text{if } A = 2. \end{cases}$$

17. $y'''_{xxx} = Ay^{-1}y'_x y''_{xx}.$

Solution:

$$x = \begin{cases} \displaystyle\int (C_1 y^{A+1} + C_2)^{-1/2}\, dy + C_3 & \text{if } A \neq -1, \\ \displaystyle\int (C_1 \ln y + C_2)^{-1/2}\, dy + C_3 & \text{if } A = -1. \end{cases}$$

18. $y'''_{xxx} = A(y'_x)^{-1}$.

Solution in the parametric form:

$$x = aC_1 \int \exp(\mp\tfrac{1}{2}\tau^2)\, d\tau + C_2, \qquad y = bC_1^2 \int \exp(\mp\tau^2)\, d\tau + C_3,$$

where $A = \mp a^{-4}b^2$.

19. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^2$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{-1/2} \exp(\mp\tau^2)\, d\tau + C_2, \qquad y = bC_1 \int \exp(\mp\tau^2)\, d\tau + C_3,$$

where $A = \pm 4a^4b^{-4}$.

► *In the solutions of equations 20–25, the following notation is used:*

$$E = \int \exp(\mp\tau^2)\, d\tau + C_2, \qquad F = 2\tau E \pm \exp(\mp\tau^2).$$

20. $y'''_{xxx} = Ay^{-1}y'_x(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1 \int \tau \exp(\mp\tau^2) E^{-1/2}\, d\tau + C_3, \qquad y = bC_1^2 \exp(\mp\tau^2),$$

where $A = \pm\frac{1}{2}a^4b^{-2}$.

21. $y'''_{xxx} = Ayy'_xy''_{xx}$.

Solution in the parametric form:

$$x = C_1 \int E^{-1/2}\, d\tau + C_3, \qquad y = b\tau,$$

where $A = \mp 2b^{-2}$.

22. $y'''_{xxx} = Ay^{-2}(y'_x)^{-1}$.

Solution in the parametric form:

$$x = aC_1 \int E^{-3/2} \exp(\mp\tfrac{1}{2}\tau^2)\, d\tau + C_3, \qquad y = bC_1E^{-1},$$

where $A = \mp a^{-4}b^4$.

23. $y'''_{xxx} = Ay^{-1}(y'_x)^{-3}(y''_{xx})^3$.

Solution in the parametric form:

$$x = C_1 \int E^{-3/2} F \exp(\mp\tau^2)\, d\tau + C_3, \qquad y = bE^{-1}\exp(\mp\tau^2),$$

where $A = \mp 8b^2$.

24. $y'''_{xxx} = Ay(y'_x)^{-3}y''_{xx}$.

Solution in the parametric form:

$$x = aC_1 \int E^{1/2}\,d\tau + C_3, \qquad y = bC_1^2 F,$$

where $A = \mp 8a^{-4}b^2$.

25. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^2$.

Solution in the parametric form:

$$x = aC_1 \int F^{-1/2}\exp(\mp\tau^2)\,d\tau + C_3, \qquad y = bC_1^2 E,$$

where $A = \pm a^4 b^{-2}$.

► *In the solutions of equations 26–33, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\,\bigr) + C_2, \qquad R = \sqrt{\frac{\tau+1}{\tau}}, \qquad F = RE - \tau.$$

26. $y'''_{xxx} = A(y'_x)^{-3}$.

Solution in the parametric form:

$$x = 2aC_1^2\sqrt{\tau+1} + C_3, \qquad y = bC_1^3 E,$$

where $A = -\frac{1}{4}a^{-6}b^4$.

27. $y'''_{xxx} = Ay^{-2}y'_x(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1 \int E^{-1/2}\,d\tau + C_3, \qquad y = bC_1^4\tau,$$

where $A = 2a^4b^{-1}$.

28. $y'''_{xxx} = Ayy'_x(y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-2}R^{-1}E^{-1/2}\,d\tau + C_3, \qquad y = bC_1^2 R,$$

where $A = -8a(-b)^{-5/2}$.

29. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1^7 \int R^{-3/2}\,d\tau + C_3, \qquad y = bC_1^6 E,$$

where $A = 4a^3(-b)^{-7/2}$.

30. $\boldsymbol{y'''_{xxx} = Ay^{-1}(y'_x)^{-3}}$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-1/2} R^{-1} E^{-3/2}\, d\tau + C_3, \qquad y = bC_1^6 E^{-1},$$

where $A = -\frac{1}{4}a^{-6}b^5$.

31. $\boldsymbol{y'''_{xxx} = Ay^{-2}(y'_x)^{-1}(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^{-1} \int R^{-1} E^{-3/2} F\, d\tau + C_3, \qquad y = bC_1^2 \tau E^{-1},$$

where $A = 2a^2 b$.

32. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3} y''_{xx}}$.

Solution in the parametric form:

$$x = aC_1^7 \int \tau^{-3/2} R^{-1} E^{1/2} F\, d\tau + C_3, \qquad y = bC_1^8 F^2,$$

where $A = a^{-4}b^{7/2}$.

33. $\boldsymbol{y'''_{xxx} = Ay^{-2}(y''_{xx})^2}$.

Solution in the parametric form:

$$x = aC_1^{-1} \int \tau^{-2} R^{-1}\, d\tau + C_3, \qquad y = bC_1 \tau^{-1} E,$$

where $A = 2ab$.

34. $\boldsymbol{y'''_{xxx} = Ay^{-1/2} y'_x (y''_{xx})^3}$.

Solution in the parametric form:

$$x = \pm aC_1^5 \int \tau(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1^8 (\tau^2 - 1)^2,$$

where $A = \mp\frac{1}{144}a^4 b^{-5/2}$.

35. $\boldsymbol{y'''_{xxx} = Ayy'_x}$.

Solution in the parametric form:

$$x = aC_1^{-1} \int (\tau^3 - 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1^2 \tau,$$

where $A = 3a^{-2}b^{-1}$.

36. $\boldsymbol{y'''_{xxx} = A(y'_x)^3 (y''_{xx})^3.}$

Solution in the parametric form:

$$x = \pm\tfrac{2}{5} aC_1^5 \tau^{1/2}(\tau^2 - 5) + C_3, \qquad y = bC_1^6(\tau^3 - 3\tau + C_2),$$

where $A = -\frac{8}{243} a^6 b^{-5}$.

37. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4} (y''_{xx})^3.}$

Solution in the parametric form:

$$x = aC_1^5 \int (\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-3/2} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)\, d\tau + C_3,$$
$$y = bC_1^2 (\tau^2 - 1)^2 (\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \mp\frac{16}{9} a^{-1} b^{5/2}$.

38. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-7/3}.}$

Solution in the parametric form:

$$x = aC_1^7 \int (\tau^3 - 3\tau + C_2)^{1/4}\, d\tau + C_3, \qquad y = \pm bC_1^{16}(\tau^4 - 6\tau^2 + 4C_2\tau - 3),$$

where $A = \pm 72 a^{-5} b^2 \left(\dfrac{4b}{a}\right)^{1/3}$.

39. $\boldsymbol{y'''_{xxx} = Ay^{-5/3}(y'_x)^3 (y''_{xx})^3.}$

Solution in the parametric form:

$$x = \pm aC_1^5 \int (\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{1/2} [\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-1/2}\, d\tau + C_3,$$
$$y = bC_1^9 (\tau^3 - 3\tau + C_2)^{3/2},$$

where $A = \mp 8 \cdot 9^{-5} a^6 b^{-10/3}$.

40. $\boldsymbol{y'''_{xxx} = Ay(y''_{xx})^{7/5}.}$

Solution in the parametric form:

$$x = aC_1^{-7} \int (\tau^3 - 3\tau + C_2)^{-3/2}\, d\tau + C_3, \qquad y = \pm bC_1(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-1/2},$$

where $A = \pm\dfrac{15}{2} a^{-1} b^{-1} \left(\dfrac{a^2}{2b}\right)^{2/5}$.

41. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^3 (y''_{xx})^{8/5}.}$

Solution in the parametric form:

$$x = \pm aC_1^{31} \int [\pm(\tau^2 - 1)]^{-1/2} (\tau^3 - 3\tau + C_2)^{5/4} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)\, d\tau + C_3,$$
$$y = bC_1^{32} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^2,$$

where $A = -15 \cdot 2^{-10} a^2 b^{-5/2} \left(\dfrac{a^2}{2b}\right)^{3/5}$.

42. $y'''_{xxx} = Ay^{-7/3}(y'_x)^{-7/3}$.

Solution in the parametric form:

$$x = aC_1^{17} \int (\tau^3 - 3\tau + C_2)^{1/4} [\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-3/2}\, d\tau + C_3,$$

$$y = \pm bC_1^{16} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \pm 72a^{-5}b^{17/3}\left(\dfrac{a}{4}\right)^{-1/3}$.

43. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{10/7}$.

Solution in the parametric form:

$$x = \pm aC_1^{29} \int (\tau^3 - 3\tau + C_2)^{-3/2} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1/3}\, d\tau + C_3,$$

$$y = bC_1^2 (\tau^3 - 3\tau + C_2)^{-1} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{2/3},$$

where $A = -\dfrac{28}{3} a^{-1} b^{5/2} \left(\dfrac{2a^2}{3b}\right)^{3/7}$.

► *In the solutions of equations 44–47, the following notation is used:*

$$P_6(\tau) = \pm(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2).$$

44. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-11/3}(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^5 \int (\tau^3 - 3\tau + C_2)^{1/2} [\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{-3/2} P_6(\tau)\, d\tau + C_3,$$

$$y = \pm bC_1 (\tau^3 - 3\tau + C_2)^{3/2} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \mp \frac{9}{16} b^{10/3} (2a)^{-2/3}$.

45. $y'''_{xxx} = Ay(y'_x)^{-5/2}(y''_{xx})^{7/5}$.

Solution in the parametric form:

$$x = aC_1^{11} \int (\tau^3 - 3\tau + C_2)^{-3/2} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3}\, d\tau + C_3,$$

$$y = bC_1^{27} (\tau^3 - 3\tau + C_2)^{-1/2} P_6(\tau),$$

where $A = \dfrac{405}{8} a^{-3} b \left(\pm\dfrac{b}{2a}\right)^{1/2} \left(\dfrac{a^2}{12b}\right)^{2/5}$.

46. $y'''_{xxx} = Ay^{-7/4}(y'_x)^3(y''_{xx})^{8/5}$.

Solution in the parametric form:

$$x = aC_1^{37} \int (\tau^3 - 3\tau + C_2)^{5/4} [\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{1/3} [P_6(\tau)]^{-1/2}\, d\tau + C_3,$$

$$y = bC_1^{64} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3},$$

where $A = -45 \cdot 2^{-13} a^2 b^{-5/4} \left(\dfrac{a^2}{12b}\right)^{3/5}$.

47. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{11/7}}.$

Solution in the parametric form:

$$x = \pm a C_1^{55} \int (\tau^3 - 3\tau + C_2)^{-3/2} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{5/3} P_6(\tau)\, d\tau + C_3,$$
$$y = bC_1^{54}(\tau^3 - 3\tau + C_2)^{-1}[P_6(\tau)]^2,$$

where $A = \mp 28 \cdot 3^7 a^{-5} b^{9/2} \Bigl(\dfrac{2a^2}{3b}\Bigr)^{4/7}$.

48. $\boldsymbol{y'''_{xxx} = A(y'_x)^{-7/3}}.$

Solution in the parametric form:

$$x = aC_1^5 \int (\tau^2 \pm 1)^{1/4}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2),$$

where $A = \pm \dfrac{81}{2} a^{-5} b^3 \Bigl(\dfrac{3b}{a}\Bigr)^{1/3}$.

49. $\boldsymbol{y'''_{xxx} = Ay^{-5/3}y'_x(y''_{xx})^3}.$

Solution in the parametric form:

$$x = aC_1 \int \tau(\tau^2 \pm 1)^{1/2}(\tau^3 \pm 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1^3(\tau^2 \pm 1)^{3/2},$$

where $A = \mp \frac{4}{243} a^4 b^{-4/3}$.

50. $\boldsymbol{y'''_{xxx} = Ayy'_x(y''_{xx})^{7/5}}.$

Solution in the parametric form:

$$x = aC_1^3 \int (\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{-1/2}\, d\tau + C_3, \qquad y = bC_1\tau(\tau^2 \pm 1)^{-1/2},$$

where $A = \pm 5b^{-2}\Bigl(\dfrac{2a^2}{3b}\Bigr)^{2/5}$.

51. $\boldsymbol{y'''_{xxx} = A(y'_x)^3(y''_{xx})^{8/5}}.$

Solution in the parametric form:

$$x = aC_1^9 \int \tau^{-1/2}(\tau^2 \pm 1)^{5/4}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2),$$

where $A = \mp \dfrac{4}{27} a^2 b^{-3} \Bigl(\dfrac{2a^2}{3b}\Bigr)^{3/5}$.

52. $\boldsymbol{y'''_{xxx} = Ay^{-4/3}(y'_x)^{-7/3}}.$

Solution in the parametric form:

$$x = aC_1^7 \int (\tau^2 \pm 1)^{1/4}(\tau^3 \pm 3\tau + C_2)^{-3/2}\, d\tau + C_3, \qquad y = bC_1^8(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \pm \dfrac{81}{2} a^{-5} b^{13/3} \Bigl(\dfrac{3b}{a}\Bigr)^{1/3}$.

53. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-5/3}(y''_{xx})^3.$

Solution in the parametric form:

$$x = aC_1^{-1}\int(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{1/2}(\tau^3 \pm 3\tau + C_2)^{-3/2}\,d\tau + C_3,$$
$$y = bC_1(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \mp\frac{4}{27}a^2b^{2/3}\left(\frac{3b}{a}\right)^{2/3}$.

54. $y'''_{xxx} = Ay(y'_x)^{-7}(y''_{xx})^{7/5}.$

Solution in the parametric form:

$$x = aC_1^7\int(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{5/6}\,d\tau + C_3, \quad y = bC_1^9(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{-1/2},$$

where $A = \pm 5a^{-8}b^6\left(\frac{2a^2}{b}\right)^{2/5}$.

55. $y'''_{xxx} = Ay^{-4}(y'_x)^3(y''_{xx})^{8/5}.$

Solution in the parametric form:

$$x = aC_1^{-1}\int(\tau^2 \pm 1)^{5/4}(\pm\tau^2 + C_2\tau - 1)^{-1/2}(\tau^3 \pm 3\tau + C_2)^{-2/3}\,d\tau + C_3,$$
$$y = bC_1^8(\tau^3 \pm 3\tau + C_2)^{1/3},$$

where $A = \mp 5a^2b\left(\frac{2a^2}{b}\right)^{3/5}$.

56. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{7/4}.$

Solution in the parametric form:

$$x = aC_1^{-7}\int(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{-1/3}\,d\tau + C_3,$$
$$y = bC_1^2(\tau^2 \pm 1)^{-1}(\tau^3 \pm 3\tau + C_2)^{2/3},$$

where $A = 4a^{-1}b^{5/2}\left(\mp\frac{a^2}{2b}\right)^{3/4}$.

57. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{5/4}.$

Solution in the parametric form:

$$x = aC_1^{17}\int(\pm\tau^2 + C_2\tau - 1)(\tau^2 \pm 1)^{-3/2}(\tau^3 \pm 3\tau + C_2)^{2/3}\,d\tau + C_3,$$
$$y = bC_1^{18}(\pm\tau^2 + C_2\tau - 1)^2(\tau^2 \pm 1)^{-1},$$

where $A = -64a^{-5}b^{9/2}\left(\mp\frac{a^2}{2b}\right)^{1/4}$.

▶ *In the solutions of equations 58–69, the following notation is used:*

$$S_1 = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau), \qquad \omega = k\sqrt{3},$$
$$S_2 = 2kC_1 e^{2k\tau} + kC_2 e^{-k\tau}\bigl[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\bigr],$$
$$S_3 = 4k^2 C_1 e^{2k\tau} - 2k^2 C_2 e^{-k\tau}\bigl[\sqrt{3}\cos(\omega\tau) + \sin(\omega\tau)\bigr],$$
$$S_4 = S_2^2 - 2S_1S_3, \qquad S_5 = 5S_2S_4 + 32k^3S_1^3.$$

58. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^3\int S_1^{-1/2}S_2S_3\,d\tau + C_3, \qquad y = bC_1^4S_2^2,$$

where $A = -a^6b^{-9/2}k^3$.

59. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-6}(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^3\int S_1^{-3/2}S_2S_4\,d\tau + C_3, \qquad y = bC_1^2S_1^{-1}S_2^2,$$

where $A = 16a^{-3}b^{9/2}k^3$.

60. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-9/5}}$.

Solution in the parametric form:

$$x = aC_1^3\int S_1^{3/4}\,d\tau + C_3, \qquad y = bC_1^8S_4,$$

where $A = -160a^{-4}bk^6\Bigl(\dfrac{16bk^3}{a}\Bigr)^{4/5}$.

61. $\boldsymbol{y'''_{xxx} = Ay^{-7/5}(y'_x)^3(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^3\int S_1^{3/2}S_2S_4^{-1/2}\,d\tau + C_3, \qquad y = bC_1^5S_1^{5/2},$$

where $A = \frac{1}{4}\cdot 5^{-5}a^6b^{-18/5}k^{-6}$.

62. $\boldsymbol{y'''_{xxx} = Ay(y''_{xx})^{9/7}}$.

Solution in the parametric form:

$$x = aC_1^{-3}\int S_1^{-3/2}\,d\tau + C_3, \qquad y = bC_1S_1^{-1/2}S_2,$$

where $A = \dfrac{7}{2}a^{-1}b^{-1}\Bigl(\dfrac{a^2}{8bk^3}\Bigr)^{2/7}$.

63. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{12/7}}$.

Solution in the parametric form:

$$x = aC_1^{15}\int S_1^{9/4}S_2^{-1/2}S_4\,d\tau + C_3, \qquad y = bC_1^{16}S_4^2,$$

where $A = 7\cdot 16^{-4}a^2b^{-5/2}k^{-9}\Big(\dfrac{a^2}{8bk^3}\Big)^{5/7}$.

64. $\boldsymbol{y'''_{xxx} = Ay^{-13/5}(y'_x)^{-9/5}}$.

Solution in the parametric form:

$$x = aC_1^{9}\int S_1^{3/4}S_4^{-3/2}\,d\tau + C_3, \qquad y = bC_1^{8}S_4^{-1},$$

where $A = -160a^{-4}b^{23/5}k^6\Big(\dfrac{16bk^3}{a}\Big)^{4/5}$.

65. $\boldsymbol{y'''_{xxx} = Ay^{-7/5}(y'_x)^{-21/5}(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^{3}\int S_1^{3/2}S_4^{-3/2}S_5\,d\tau + C_3, \qquad y = bC_1S_1^{5/2}S_4^{-1},$$

where $A = \dfrac{5}{512}a^{-1}b^{17/5}k^{-6}\Big(\dfrac{b}{2a}\Big)^{1/5}$.

66. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-9/4}(y''_{xx})^{9/7}}$.

Solution in the parametric form:

$$x = aC_1^{9}\int S_1^{-3/2}S_4^{6/5}\,d\tau + C_3, \qquad y = bC_1^{25}S_1^{-1/2}S_5,$$

where $A = \dfrac{35}{8}a^{-3}b\Big(-\dfrac{5b}{2a}\Big)^{1/4}\Big(\dfrac{a^2}{32bk^3}\Big)^{2/7}$.

67. $\boldsymbol{y'''_{xxx} = Ay^{-13/8}(y'_x)^3(y''_{xx})^{12/7}}$.

Solution in the parametric form:

$$x = aC_1^{39}\int S_1^{9/4}S_4^{3/5}S_5^{-1/2}\,d\tau + C_3, \qquad y = bC_1^{64}S_4^{8/5},$$

where $A = 175\cdot 2^{-22}a^2b^{-11/8}k^{-9}\Big(\dfrac{a^2}{32bk^3}\Big)^{5/7}$.

68. $\boldsymbol{y'''_{xxx} = Ay^{-7/2}(y''_{xx})^{18/13}}$.

Solution in the parametric form:

$$x = aC_1^{27}\int S_1^{-3/2}S_4^{-3/5}\,d\tau + C_3, \qquad y = bC_1^{2}S_1^{-1}S_4^{2/5},$$

where $A = -\dfrac{208}{5}a^{-1}b^{7/2}k^3\Big(\dfrac{2a^2}{b}\Big)^{5/13}$.

69. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-6}(y''_{xx})^{21/13}$.

Solution in the parametric form:

$$x = aC_1^{51}\int S_1^{-3/2}S_4^{9/5}S_5\,d\tau + C_3, \qquad y = bC_1^{50}S_1^{-1}S_5^2,$$

where $A = 208\cdot 5^5 a^{-7}b^{13/2}k^3\left(\dfrac{2a^2}{b}\right)^{8/13}$.

► *In the solutions of equations 70–81, the following notation is used:*

$$\begin{aligned}
T_1 &= \cosh(\tau + C_2)\cos\tau, & \theta_1 &= \cosh\tau - \sin(\tau + C_2),\\
T_2 &= \tanh(\tau + C_2) + \tan\tau, & \theta_2 &= \sinh\tau + \cos(\tau + C_2),\\
T_3 &= \tanh(\tau + C_2) - \tan\tau, & \theta_3 &= \sinh\tau - \cos(\tau + C_2),\\
T_4 &= 3T_2T_3 - 4, & \theta_4 &= 3\theta_2\theta_3 - 2\theta_1^2.
\end{aligned}$$

70. $y'''_{xxx} = Ay^2(y'_x)^{-7/3}$.

1°. Solution in the parametric form:

$$x = aC_1\int T_1^{1/4}\,d\tau + C_3, \qquad y = bC_1^4T_1T_2,$$

where $A = -3a^{-5}b\left(\dfrac{2b}{a}\right)^{1/3}$.

2°. Solution in the parametric form:

$$x = aC_1\int \theta_1^{1/4}\,d\tau + C_3, \qquad y = bC_1^4\theta_2,$$

where $A = \dfrac{3}{8}a^{-5}b\left(\dfrac{b}{a}\right)^{1/3}$.

71. $y'''_{xxx} = Ay^{-5/3}(y'_x)^5(y''_{xx})^3$.

1°. Solution in the parametric form:

$$x = aC_1^2\int T_1T_2^{-1/2}T_3\,d\tau + C_3, \qquad y = bC_1^3T_1^{3/2},$$

where $A = 64\cdot 3^{-7}a^8b^{-16/3}$.

2°. Solution in the parametric form:

$$x = aC_1^2\int \theta_1^{1/2}\theta_2^{-1/2}\theta_3\,d\tau + C_3, \qquad y = bC_1^3\theta_1^{3/2},$$

where $A = -256\cdot 3^{-7}a^8b^{-16/3}$.

72. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-1/3}(y''_{xx})^{7/5}}$.

1°. Solution in the parametric form:

$$x = aC_1^{-2}\int T_1^{-1}T_2^{1/2}\,d\tau + C_3, \qquad y = bC_1T_1^{1/2}T_3,$$

where $A = -\dfrac{5}{2}a^{-1}b^{-1}\Big(\dfrac{b}{2a}\Big)^{1/3}\Big(\dfrac{2a^2}{3b}\Big)^{2/5}$.

2°. Solution in the parametric form:

$$x = aC_1^{-2}\int \theta_1^{-3/2}\theta_2^{1/2}\,d\tau + C_3, \qquad y = bC_1\theta_1^{-1/2}\theta_3,$$

where $A = \dfrac{5}{2}a^{-1}b^{-1}\Big(\dfrac{b}{2a}\Big)^{1/3}\Big(\dfrac{4a^2}{3b}\Big)^{2/5}$.

73. $\boldsymbol{y'''_{xxx} = Ay^{-2/3}(y'_x)^3(y''_{xx})^{8/5}}$.

1°. Solution in the parametric form:

$$x = aC_1^{11}\int T_1^{11/4}T_2^{2}T_3^{-1/2}\,d\tau + C_3, \qquad y = bC_1^{12}T_1^{3}T_2^{3},$$

where $A = \dfrac{5}{432}a^2b^{-7/3}\Big(\dfrac{2a^2}{3b}\Big)^{3/5}$.

2°. Solution in the parametric form:

$$x = aC_1^{11}\int \theta_1^{5/4}\theta_2^{2}\theta_3^{-1/2}\,d\tau + C_3, \qquad y = bC_1^{12}\theta_2^{3},$$

where $A = -\dfrac{5}{54}a^2b^{-7/3}\Big(\dfrac{4a^2}{3b}\Big)^{3/5}$.

74. $\boldsymbol{y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{1/2}}$.

1°. Solution in the parametric form:

$$x = aC_1^{3}\int T_1^{-3/2}\,d\tau + C_3, \qquad y = bC_1^{2}T_1^{-1},$$

where $A = 2a^{-2}b^{7/2}\Big(\dfrac{2}{b}\Big)^{1/2}$.

2°. Solution in the parametric form:

$$x = aC_1^{3}\int \theta_1^{-3/2}\,d\tau + C_3, \qquad y = bC_1^{2}\theta_1^{-1},$$

where $A = -a^{-2}b^{7/2}\Big(-\dfrac{2}{b}\Big)^{1/2}$.

75. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{5/2}.$

1°. Solution in the parametric form:

$$x = aC_1^3 \int T_1^{3/2} T_2^2 T_3 \, d\tau + C_3, \qquad y = bC_1^2 T_1 T_3^2,$$

where $A = -32a^{-2}b^{7/2}\left(\frac{b}{2}\right)^{-1/2}$.

2°. Solution in the parametric form:

$$x = aC_1^3 \int \theta_1^{-3/2} \theta_2^2 \theta_3 \, d\tau + C_3, \qquad y = bC_1^2 \theta_1^{-1} \theta_3^2,$$

where $A = 16a^{-2}b^{7/2}\left(-\frac{b}{2}\right)^{-1/2}$.

76. $y'''_{xxx} = Ay^{-10/3}(y'_x)^{-7/3}.$

1°. Solution in the parametric form:

$$x = aC_1^5 \int T_1^{-5/4} T_2^{-3/2} \, d\tau + C_3, \qquad y = bC_1^4 T_1^{-1} T_2^{-1},$$

where $A = -3a^{-5}b^{20/3}\left(\frac{2}{a}\right)^{1/3}$.

2°. Solution in the parametric form:

$$x = aC_1^5 \int \theta_1^{1/4} \theta_2^{-3/2} \, d\tau + C_3, \qquad y = bC_1^4 \theta_2^{-1},$$

where $A = \frac{3}{8}a^{-16/3}b^{20/3}$.

77. $y'''_{xxx} = Ay^{-5/3}(y'_x)^{-17/3}(y''_{xx})^3.$

1°. Solution in the parametric form:

$$x = aC_1^2 \int T_1 T_2^{-3/2} T_4 \, d\tau + C_3, \qquad y = bC_1 T_1^{1/2} T_2^{-1},$$

where $A = \dfrac{3}{16}a^{-2}b^{14/3}\left(\dfrac{b}{2a}\right)^{2/3}$.

2°. Solution in the parametric form:

$$x = aC_1^2 \int \theta_1^{1/2} \theta_2^{-3/2} \theta_4 \, d\tau + C_3, \qquad y = bC_1 \theta_1^{3/2} \theta_2^{-1},$$

where $A = -\dfrac{3}{4}a^{-2}b^{14/3}\left(\dfrac{b}{2a}\right)^{2/3}$.

78. $y'''_{xxx} = Ay(y'_x)^{-13/7}(y''_{xx})^{7/5}$.

1°. Solution in the parametric form:

$$x = aC_1^2 \int T_1^{1/3} T_2^{11/6}\, d\tau + C_3, \qquad y = bC_1^9 T_1^{3/2} T_4,$$

where $A = -\dfrac{5}{4} a^{-2} \Big(\dfrac{3b}{2a}\Big)^{6/7} \Big(\dfrac{2a^2}{7b}\Big)^{2/5}$.

2°. Solution in the parametric form:

$$x = aC_1^2 \int \theta_1^{-3/2} \theta_2^{11/6}\, d\tau + C_3, \qquad y = bC_1^9 \theta_1^{-1/2} \theta_4,$$

where $A = \dfrac{5}{4} a^{-2} \Big(\dfrac{3b}{2a}\Big)^{6/7} \Big(\dfrac{4a^2}{7b}\Big)^{2/5}$.

79. $y'''_{xxx} = Ay^{-10/7}(y'_x)^3(y''_{xx})^{8/5}$.

1°. Solution in the parametric form:

$$x = aC_1^{19} \int T_1^{19/12} T_2^{4/3} T_4^{-1/2}\, d\tau + C_3, \qquad y = bC_1^{28} T_1^{7/3} T_2^{7/3},$$

where $A = \dfrac{45}{16} \cdot 7^{-3} a^2 b^{-11/7} \Big(\dfrac{2a^2}{7b}\Big)^{3/5}$.

2°. Solution in the parametric form:

$$x = aC_1^{19} \int \theta_1^{5/4} \theta_2^{4/3} \theta_4^{-1/2}\, d\tau + C_3, \qquad y = bC_1^{28} \theta_2^{7/3},$$

where $A = -\dfrac{45}{2} \cdot 7^{-3} a^2 b^{-11/7} \Big(\dfrac{4a^2}{7b}\Big)^{3/5}$.

80. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{13/10}$.

1°. Solution in the parametric form:

$$x = aC_1^{-11} \int T_1^{-11/6} T_2^{-1/3}\, d\tau + C_3, \qquad y = bC_1^4 T_1^{-1/3} T_2^{2/3},$$

where $A = \dfrac{20}{3} a^{-1} b^{5/2} \Big(\dfrac{2a^2}{b}\Big)^{3/10}$.

2°. Solution in the parametric form:

$$x = aC_1^{-11} \int \theta_1^{-3/2} \theta_2^{-1/3}\, d\tau + C_3, \qquad y = bC_1^4 \theta_1^{-1} \theta_2^{2/3},$$

where $A = \dfrac{10}{3} a^{-1} b^{5/2} \Big(-\dfrac{2a^2}{b}\Big)^{3/10}$.

81. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{17/10}$.

1°. Solution in the parametric form:

$$x = aC_1^{19} \int T_1^{19/6} T_2^{8/3} T_4\, d\tau + C_3, \qquad y = bC_1^{18} T_1^3 T_4^2,$$

where $A = -540 a^{-5} b^{9/2} \Big(\dfrac{2a^2}{b}\Big)^{7/10}$.

2°. Solution in the parametric form:

$$x = aC_1^{19} \int \theta_1^{-3/2} \theta_2^{8/3} \theta_4\, d\tau + C_3, \qquad y = bC_1^{18} \theta_1^{-1} \theta_5^2,$$

where $A = -270 a^{-5} b^{9/2} \Big(-\dfrac{2a^2}{b}\Big)^{7/10}$.

▶ *In the solutions of equations 82–84, the following notation is used:*

$$L_1 = C_1\tau^k + C_2\tau^{-k}, \qquad N_1 = (1+k)C_1\tau^k + (1-k)C_2\tau^{-k},$$
$$L_2 = C_1\ln\tau + C_2, \qquad N_2 = C_1\ln\tau + C_1 + C_2,$$
$$L_3 = C_1\sin(k\ln\tau) + C_2\cos(k\ln\tau), \qquad N_3 = (C_1 - kC_2)\sin(k\ln\tau) + (C_2 + kC_1)\cos(k\ln\tau).$$

82. $y'''_{xxx} = Ay^{-2}(y'_x)^3.$

Solution in the parametric form:

$$x = \int \tau^{1/2} L_m^{-1/2}\,d\tau + C_3, \qquad y = \tau^2,$$

where $k = \sqrt{|1+8A|}$, $m = \begin{cases} 1 & \text{if } A > -1/8, \\ 2 & \text{if } A = -1/8, \\ 3 & \text{if } A < -1/8. \end{cases}$

83. $y'''_{xxx} = Ay(y'_x)^{-3}(y''_{xx})^3.$

Solution in the parametric form:

$$x = \int \tau^{-1} N_m\,d\tau + C_3, \qquad y = \tau L_m,$$

where $k = \sqrt{|A-1|}$, $m = \begin{cases} 1 & \text{if } A < 1, \\ 2 & \text{if } A = 1, \\ 3 & \text{if } A > 1. \end{cases}$

84. $y'''_{xxx} = Ay^{-1/2}(y''_{xx})^{3/2}.$

Solution in the parametric form:

$$x = \mp 4\int \tau^2 L_1\,d\tau + C_3, \qquad y = \tau^2 L_1^2,$$

where $k = \sqrt{1+8A^{-2}}$.

▶ *In the solutions of equations 85–100, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \textit{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \textit{for the lower sign,} \end{cases}$$

$$U_1 = \tau Z'_\tau + \nu Z, \qquad U_2 = U_1^2 \pm \tau^2 Z^2, \qquad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1U_2,$$

where J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

85. $y'''_{xxx} = Ay^\beta(y'_x)^3, \qquad \beta \neq -2.$

Solution in the parametric form:

$$x = C_1\int \tau^{\frac{3\nu-2}{2}} Z^{-1/2}\,d\tau + C_3, \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{\beta+2}$, $A = \mp\dfrac{1}{8\nu^2}b^{-\beta-2}$.

86. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{\gamma}(y''_{xx})^3, \qquad \gamma \neq -3.}$

Solution in the parametric form:

$$x = aC_1 \int \tau^{-1} U_1 \, d\tau + C_3, \qquad y = bC_1 \tau^{\nu} Z,$$

where $\nu = \dfrac{2}{\gamma + 3}$, $A = \pm \dfrac{1}{\nu^2} a^{\gamma+3} b^{-\gamma-3}$.

87. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y''_{xx})^{\delta}, \qquad \delta \neq 3/2.}$

Solution in the parametric form:

$$x = aC_1 \int \tau^{3\nu-1} Z \, d\tau + C_3, \qquad y = bC_1^2 \tau^{2\nu} Z^2,$$

where $\nu = \dfrac{1-\delta}{3-2\delta}$, $A = \mp \dfrac{4}{3-2\delta} a^{-3} b^{3/2} \left(\mp \dfrac{a^2}{2b}\right)^{\delta}$.

88. $\boldsymbol{y'''_{xxx} = Ax^{-3}y(y''_{xx})^3.}$

Solution in the parametric form:

$$x = aC_1 \int Z \, d\tau + C_3, \qquad y = bC_1^2 \tau^{-1/3} \left[\tau Z^2 - U_1 \int Z \, d\tau - C_3 U_1\right],$$

where $\nu = \frac{1}{3}$, $A = \frac{9}{4} a^6 b^{-3}$.

89. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y''_{xx})^3.}$

Solution in the parametric form:

$$x = aC_1 \int \tau Z \, d\tau + C_3, \qquad y = bC_1^2 \tau^{4/3} Z^2,$$

where $\nu = \frac{2}{3}$, $A = -\frac{1}{6} a^3 b^{-3/2}$.

90. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3}(y''_{xx})^3.}$

Solution in the parametric form:

$$x = C_1 \int \tau^{-2} Z^{-2} U_1 U_2 \, d\tau + C_3, \qquad y = b\tau^{-4/3} Z^{-2} U_1^2,$$

where $\nu = \frac{1}{3}$, $A = \pm \frac{4}{3} b^{3/2}$.

91. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-3}.}$

Solution in the parametric form:

$$x = aC_1 \int Z \, d\tau + C_3, \qquad y = bC_1^2 \tau^{-2/3} U_2,$$

where $\nu = \frac{1}{3}$, $A = -\frac{16}{81} a^{-6} b^3$.

92. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^3.$

Solution in the parametric form:

$$x = aC_1 \int ZU_1U_2^{-1/2}\,d\tau + C_3, \qquad y = bC_1^2\tau^{2/3}Z^2,$$

where $\nu = \frac{1}{3}$, $A = \frac{9}{32}a^6b^{-3}$.

93. $y'''_{xxx} = Ay(y''_{xx})^{3/2}.$

Solution in the parametric form:

$$x = C_1 \int \tau^{-1}Z^{-2}\,d\tau + C_3, \qquad y = b\tau^{-2/3}Z^{-1}U_1,$$

where $\nu = \frac{1}{3}$, $A = 4b^{-1}\left(\mp\dfrac{3}{2b}\right)^{1/2}$.

94. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{3/2}.$

Solution in the parametric form:

$$x = aC_1 \int Z^{5/2}U_1^{-1/2}U_2\,d\tau + C_3, \qquad y = bC_1\tau^{-4/3}U_2^2,$$

where $\nu = \frac{1}{3}$, $A = \mp\dfrac{27}{16}a^3b^{-5/2}\left(\mp\dfrac{3}{2b}\right)^{1/2}$.

95. $y'''_{xxx} = Ay^{-2}(y'_x)^{-3}.$

Solution in the parametric form:

$$x = aC_1 \int \tau ZU_2^{-3/2}\,d\tau + C_3, \qquad y = bC_1\tau^{2/3}U_2^{-1},$$

where $\nu = \frac{1}{3}$, $A = -\frac{16}{81}a^{-6}b^6$.

96. $y'''_{xxx} = Ay^{-2}(y'_x)^{-3}(y''_{xx})^3.$

Solution in the parametric form:

$$x = C_1 \int ZU_2^{-3/2}U_3\,d\tau + C_3, \qquad y = b\tau^{4/3}Z^2U_2^{-1},$$

where $\nu = \frac{1}{3}$, $A = 18b^{-3}$.

97. $y'''_{xxx} = Ay(y'_x)^{-3}(y''_{xx})^{3/2}.$

Solution in the parametric form:

$$x = aC_1 \int \tau^{-2}Z^{-2}U_2^{3/2}\,d\tau + C_3, \qquad y = bC_1^2\tau^{-4/3}Z^{-1}U_3,$$

where $\nu = \frac{1}{3}$, $A = -16a^{-3}b^2\left(\pm\dfrac{3}{2b}\right)^{1/2}$.

98. $y'''_{xxx} = Ay^{-2}(y'_x)^3(y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1 \int \tau Z^{5/2} U_3^{-1/2}\, d\tau + C_3, \qquad y = bC_1^2 \tau^{-2/3} U_2,$$

where $\nu = \frac{1}{3}$, $A = \pm\dfrac{27}{4} a^3 b^{-1} \Bigl(\pm\dfrac{3}{2b}\Bigr)^{1/2}$.

99. $y'''_{xxx} = Ay^{-2}(y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = C_1 \int \tau^{-1} Z^{-2}\, d\tau + C_3, \qquad y = b\tau^{-4/3} Z^{-2} U_2,$$

where $\nu = \frac{1}{3}$, $A = \pm\frac{4}{3} b^2 (2b)^{-1/2}$.

100. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-3}(y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{-3} Z^{-2} U_2^{3/2} U_3\, d\tau + C_3, \qquad y = bC_1 \tau^{-8/3} Z^{-2} U_3^2,$$

where $\nu = \frac{1}{3}$, $A = \mp\frac{256}{3} a^{-3} b^{7/2} (2b)^{-1/2}$.

► *In the solutions of equations 101–138, the following notation is used:*

$$f = \sqrt{\pm(4\wp^3 - 1)}, \qquad \tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

Function $\wp = \wp(\tau)$ is defined implicitly by the above elliptic integral of the first kind. For the upper sign, the function $\wp$ coincides with the classical elliptic Weirstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$. In the solution given below, we can assume $\wp$ as the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

101. $y'''_{xxx} = A(y'_x)^5$.

Solution in the parametric form:

$$x = aC_1^2 \int \wp^{-1/2}\, d\tau + C_3, \qquad y = bC_1 \tau,$$

where $A = \pm 3a^2 b^{-4}$.

102. $y'''_{xxx} = Ay^2 y'_x (y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-1/2} f\, d\tau + C_3, \qquad y = bC_1^4 \wp,$$

where $A = \mp 24 a^4 b^{-5}$.

103. $\boldsymbol{y'''_{xxx} = Ayy'_x(y''_{xx})^{5/2}}$.

Solution in the parametric form:

$$x = aC_1^7 \int \tau^{-1/2}\wp^2\, d\tau + C_3, \qquad y = bC_1^6 f,$$

where $\quad A = -\frac{1}{9}a^3 b^{-3}(\pm 3b)^{-1/2}$.

104. $\boldsymbol{y'''_{xxx} = A(y'_x)^3(y''_{xx})^{1/2}}$.

Solution in the parametric form:

$$x = aC_1^5 \int f^{-1/2}\, d\tau + C_3, \qquad y = bC_1^2 \tau,$$

where $\quad A = \pm 6ab^{-2}(\pm 3b)^{-1/2}$.

105. $\boldsymbol{y'''_{xxx} = Ay^{-5}(y'_x)^5}$.

Solution in the parametric form:

$$x = aC_1^{-1} \int \tau^{-3/2}\wp^{-1/2}\, d\tau + C_3, \qquad y = bC_1^2 \tau^{-1},$$

where $\quad A = \pm 3a^2 b$.

106. $\boldsymbol{y'''_{xxx} = Ay^2(y'_x)^{-9}(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-3/2}(\tau f - \wp)\, d\tau + C_3, \qquad y = bC_1^6 \tau^{-1}\wp,$$

where $\quad A = \mp 24a^{-6}b^5$.

107. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-3/2}(y''_{xx})^{5/2}}$.

Solution in the parametric form:

$$x = aC_1^2 \int \tau^{-1}\wp^2\, d\tau + C_3, \qquad y = bC_1(\tau f - \wp),$$

where $\quad A = \mp \dfrac{1}{2}ab\Big(\pm\dfrac{2}{a}\Big)^{1/2}$.

108. $\boldsymbol{y'''_{xxx} = Ay^{-5/4}(y'_x)^3(y''_{xx})^{1/2}}$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^3(\tau f - \wp)^{-1/2}\, d\tau + C_3, \qquad y = bC_1^4 \tau^4,$$

where $\quad A = \pm\frac{3}{16}ab^{-3/4}(\pm 3b)^{-1/2}$.

109. $y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{6/5}$.

Solution in the parametric form:

$$x = aC_1^7 \int \tau^{-4} \wp^2 \, d\tau + C_3, \qquad y = bC_1^9 \tau^{-3} \wp^3,$$

where $\quad A = \pm 5a^{-1}b^{-2/3}\Big(\dfrac{a^2}{18b}\Big)^{1/5}$.

110. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{9/5}$.

Solution in the parametric form:

$$x = aC_1^{-1} \int \tau^{5/2} \wp^{1/2}(\tau f - \wp) \, d\tau + C_3, \qquad y = bC_1^8 (\tau f - \wp)^2,$$

where $\quad A = \mp 5a^{-1}b^{1/2}\Big(\dfrac{12b}{a}\Big)^{1/3}\Big(\dfrac{a^2}{18b}\Big)^{4/5}$.

111. $y'''_{xxx} = Ay^{-15/7}(y'_x)^5$.

Solution in the parametric form:

$$x = aC_1^{13} \int \tau^{11/2} \bigl(\tau^2 \wp \mp 1\bigr)^{-1/2} \, d\tau + C_3, \qquad y = bC_1^{14} \tau^7,$$

where $\quad A = \pm 3 \cdot 7^{-4} a^2 b^{-13/7}$.

112. $y'''_{xxx} = Ay^2(y'_x)^{-23/7}(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^{-5} \int \tau^{-7/2} (\tau^3 f + 3\tau^2 \wp \mp 1) \, d\tau + C_3, \qquad y = bC_1^2 \tau (\tau^2 \wp \mp 1),$$

where $\quad A = \mp \frac{24}{49} a^{-2/7} b^{-5/7}$.

113. $y'''_{xxx} = Ay(y'_x)^{-11/4}(y''_{xx})^{5/2}$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{-3} \bigl(\tau^2 \wp \mp 1\bigr)^2 \, d\tau + C_3, \qquad y = bC_1^3 \tau^{-6} (\tau^3 f + 3\tau^2 \wp \mp 1),$$

where $\quad A = \pm \dfrac{3}{2}\Big(\pm \dfrac{6b}{a}\Big)^{3/4}\Big(\mp 6b\Big)^{-1/2}$.

114. $y'''_{xxx} = Ay^{-15/8}(y'_x)^3(y''_{xx})^{1/2}$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-6} \bigl(\tau^3 f + 3\tau^2 \wp \mp 1\bigr)^{-1/2} \, d\tau + C_3, \qquad y = bC_1^8 \tau^{-8},$$

where $\quad A = \pm \frac{3}{64} ab^{-1/8} (\mp 6b)^{-1/2}$.

115. $y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{22/15}$.

Solution in the parametric form:

$$x = aC_1^3 \int \tau^8 (\tau^2 \wp \mp 1)^2\, d\tau + C_3, \qquad y = bC_1 \tau^3 (\tau^2 \wp \mp 1)^3,$$

where $A = \mp 15 a^{-1/15} b^{2/3} (18b)^{-7/15}$.

116. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{23/15}$.

Solution in the parametric form:

$$x = aC_1^9 \int \tau^{-29/2} (\tau^3 f + 3\tau^2 \wp \mp 1)(\tau^2 \wp \mp 1)^{1/2}\, d\tau + C_3,$$
$$y = bC_1^8 \tau^{-12} (\tau^3 f + 3\tau^2 \wp \mp 1)^2,$$

where $A = \pm 15 a^{-1} b^{1/2} \left(\dfrac{12b}{a}\right)^{1/3} \left(\dfrac{a^2}{18b}\right)^{8/15}$.

117. $y'''_{xxx} = Ay^{-20/7}(y'_x)^5$.

Solution in the parametric form:

$$x = aC_1^4 \int \tau^{-5} (\tau^2 \wp \mp 1)^{-1/2}\, d\tau + C_3, \quad y = bC_1^7 \tau^{-7},$$

where $A = \pm 3 \cdot 7^{-4} a^2 b^{-8/7}$.

118. $y'''_{xxx} = Ay^2(y'_x)^{-33/7}(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^5 \int \tau^{-7/2} (\tau^3 f - 4\tau^2 \wp \pm 6)\, d\tau + C_3, \quad y = bC_1^{12} \tau^{-6} (\tau^2 \wp \mp 1),$$

where $A = \mp \frac{24}{49} a^{-12/7} b^{5/7}$.

119. $y'''_{xxx} = Ay(y'_x)^{-27/13}(y''_{xx})^{5/2}$.

Solution in the parametric form:

$$x = aC_1^{-11} \int \tau^{-13/2} (\tau^2 \wp \mp 1)^2\, d\tau + C_3, \quad y = bC_1^2 \tau (\tau^3 f - 4\tau^2 \wp \pm 6),$$

where $A = -\dfrac{24}{13} \left(\dfrac{6b}{a}\right)^{1/13} (\pm 39b)^{-1/2}$.

120. $y'''_{xxx} = Ay^{-20/13}(y'_x)^3(y''_{xx})^{1/2}$.

Solution in the parametric form:

$$x = aC_1^{25} \int \tau^{23/2} (\tau^3 f - 4\tau^2 \wp \pm 6)^{-1/2}\, d\tau + C_3, \quad y = bC_1^{26} \tau^{13},$$

where $A = \pm \frac{6}{169} a b^{-6/13} (\pm 39b)^{-1/2}$.

121. $y'''_{xxx} = Ay^{-2/3}(y''_{xx})^{27/20}$.

Solution in the parametric form:

$$x = aC_1^{19}\int \tau^{-20}(\tau^2\wp \mp 1)^2\,d\tau + C_3, \quad y = bC_1^{18}\tau^{-18}(\tau^2\wp \mp 1)^3,$$

where $A = 20a^{-1}b^{2/3}\Bigl(\pm\dfrac{a^2}{18b}\Bigr)^{7/20}$.

122. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-1/3}(y''_{xx})^{33/20}$.

Solution in the parametric form:

$$x = aC_1^{11}\int \tau^{10}(\tau^3 f - 4\tau^2\wp \pm 6)(\tau^2\wp \mp 1)^{1/2}\,d\tau + C_3, \quad y = bC_1^2\tau^2(\tau^3 f - 4\tau^2\wp \pm 6)^2,$$

where $A = \mp 20a^{-1}b^{1/2}\Bigl(\dfrac{12b}{a}\Bigr)^{1/3}\Bigl(\pm\dfrac{a^2}{18b}\Bigr)^{13/20}$.

123. $y'''_{xxx} = A(y'_x)^{-4}$.

Solution in the parametric form:

$$x = aC_1^5\int \wp^{-2}\,d\tau + C_3, \quad y = bC_1^7\wp^{-2}(f \pm 2\tau\wp^2),$$

where $A = -192a^{-7}b^5$.

124. $y'''_{xxx} = Ay^{-5/2}y'_x(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1\int f\wp^{-2}(f \pm 2\tau\wp^2)^{-1/2}\,d\tau + C_3, \quad y = bC_1^8\wp^{-2},$$

where $A = \pm\frac{3}{4}a^4b^{-1/2}$.

125. $y'''_{xxx} = Ayy'_x(y''_{xx})^{8/5}$.

Solution in the parametric form:

$$x = aC_1^{13}\int \wp^3(f \pm 2\tau\wp^2)^{-1/2}\,d\tau + C_3, \quad y = bC_1^6 f,$$

where $A = \mp\dfrac{5}{6}b^{-2}\Bigl(\dfrac{a^2}{6b}\Bigr)^{3/5}$.

126. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{7/5}$.

Solution in the parametric form:

$$x = aC_1^{17}\int \wp^{-3}f^{-1/2}\,d\tau + C_3, \quad y = bC_1^{14}\wp^{-2}(f \pm 2\tau\wp^2),$$

where $A = \mp\dfrac{5}{8}a^2b^{-3}\Bigl(\dfrac{a^2}{6b}\Bigr)^{2/5}$.

127. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}}$.

Solution in the parametric form:

$$x = aC_1^{11}\int \wp\bigl(f \pm 2\tau\wp^2\bigr)^{-3/2}\,d\tau + C_3, \quad y = bC_1^{14}\wp^2\bigl(f \pm 2\tau\wp^2\bigr)^{-1},$$

where $\quad A = 192a^{-7}b^{11/2}$.

128. $\boldsymbol{y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{3}}$.

Solution in the parametric form:

$$x = aC_1^{-1}\int \bigl(f \pm 2\tau\wp^2\bigr)^{-3/2}(\tau f + 2\wp)\,d\tau + C_3, \quad y = bC_1^{6}\bigl(f \pm 2\tau\wp^2\bigr)^{-1},$$

where $\quad A = -\frac{3}{16}a^3b^{1/2}$.

129. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{3}(y''_{xx})^{8/5}}$.

Solution in the parametric form:

$$x = aC_1^{23}\int \wp^{-1/2}\bigl(f \pm 2\tau\wp^2\bigr)^{5/4}\,d\tau + C_3, \quad y = bC_1^{16}(\tau f + 2\wp),$$

where $\quad A = \dfrac{10}{27}a^2b^{-4}\Bigl(\dfrac{2a^2}{3b}\Bigr)^{3/5}$.

130. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{3}(y''_{xx})^{7/5}}$.

Solution in the parametric form:

$$x = aC_1^{11}\int \bigl(f \pm 2\tau\wp^2\bigr)^{-3/2}(\tau f + 2\wp)^{-1/2}\,d\tau + C_3, \quad y = bC_1^{7}\wp\bigl(f \pm 2\tau\wp^2\bigr)^{-1/2},$$

where $\quad A = -10a^2b^{-4}\Bigl(\dfrac{2a^2}{3b}\Bigr)^{2/5}$.

131. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-7/3}}$.

Solution in the parametric form:

$$x = aC_1^{23}\int \bigl(f \pm 2\tau\wp^2\bigr)^{1/4}(\tau f + 2\wp)\,d\tau + C_3, \quad y = bC_1^{32}(\tau f + 2\wp)^{2},$$

where $\quad A = -648a^{-5}b^{7/2}\Bigl(\dfrac{6b}{a}\Bigr)^{1/3}$.

132. $\boldsymbol{y'''_{xxx} = Ay^{-5/3}(y''_{xx})^{3}}$.

Solution in the parametric form:

$$x = aC_1\int \wp\bigl(f \pm 2\tau\wp^2\bigr)^{1/2}\,d\tau + C_3, \quad y = bC_1^{9}(f \pm 2\tau\wp^2)^{3/2},$$

where $\quad A = \pm\frac{1}{324}a^3b^{-1/3}$.

133. $\boldsymbol{y'''_{xxx} = Ay^{-5/6}(y'_x)^{-7/3}}$.

Solution in the parametric form:

$$x = aC_1^{25}\int \bigl(f \pm 2\tau\wp^2\bigr)^{1/4}(\tau f + 2\wp)^{-2}\,d\tau + C_3, \quad y = bC_1^{32}(\tau f + 2\wp)^{-2},$$

where $\quad A = -648a^{-5}b^{23/6}\Bigl(\dfrac{6b}{a}\Bigr)^{1/3}.$

134. $\boldsymbol{y'''_{xxx} = Ay^{-5/3}(y'_x)^{-2/3}(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^{-1}\int (\tau^2\wp \mp 1)\bigl(f \pm 2\tau\wp^2\bigr)^{1/2}(\tau f + 2\wp)^{-2}\,d\tau + C_3,$$
$$y = bC_1^{7}\bigl(f \pm 2\tau\wp^2\bigr)^{3/2}(\tau f + 2\wp)^{-2},$$

where $\quad A = -\dfrac{1}{324}a^3b^{-1/3}\Bigl(\dfrac{6b}{a}\Bigr)^{2/3}.$

135. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{11}(y''_{xx})^{7/5}}$.

Solution in the parametric form:

$$x = aC_1^{31}\int \bigl(f \pm 2\tau\wp^2\bigr)^{-3/2}(\tau f + 2\wp)^{13/6}\,d\tau + C_3,$$
$$y = bC_1^{27}(\tau^2\wp \mp 1)\bigl(f \pm 2\tau\wp^2\bigr)^{-1/2},$$

where $\quad A = -20a^{10}b^{-12}\Bigl(\dfrac{2a^2}{b}\Bigr)^{2/5}.$

136. $\boldsymbol{y'''_{xxx} = Ay^5(y'_x)^3(y''_{xx})^{8/5}}$.

Solution in the parametric form:

$$x = aC_1^{43}\int (\tau^2\wp \mp 1)^{-1/2}\bigl(f \pm 2\tau\wp^2\bigr)^{5/4}(\tau f + 2\wp)^{-4/3}\,d\tau + C_3,$$
$$y = bC_1^{16}(\tau f + 2\wp)^{-1/3},$$

where $\quad A = 20a^2b^{-8}\Bigl(\dfrac{2a^2}{b}\Bigr)^{3/5}.$

137. $\boldsymbol{y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4}(y''_{xx})^{4/5}}$.

Solution in the parametric form:

$$x = aC_1^{47}\int (\tau^2\wp \mp 1)\bigl(f \pm 2\tau\wp^2\bigr)^{-3/2}(\tau f + 2\wp)^{4/3}\,d\tau + C_3,$$
$$y = bC_1^{54}(\tau^2\wp \mp 1)^2\bigl(f \pm 2\tau\wp^2\bigr)^{-1},$$

where $\quad A = -320a^{-7}b^{11/2}\Bigl(\dfrac{a^2}{4b}\Bigr)^{4/5}.$

138. $y'''_{xxx} = Ay^{-5/2}(y''_{xx})^{11/5}$.

Solution in the parametric form:

$$x = aC_1^{-13} \int \bigl(f \pm 2\tau\wp^2\bigr)^{-3/2} (\tau f + 2\wp)^{1/3}\, d\tau + C_3,$$
$$y = bC_1^{14} \bigl(f \pm 2\tau\wp^2\bigr)^{-1} (\tau f + 2\wp)^{4/3},$$

where $A = \dfrac{5}{4} ab^{3/2} \Bigl(\dfrac{a^2}{4b}\Bigr)^{1/5}$.

► *In the solutions of equations 139–140, the following notation is used:*

$$U = \int \frac{\tau^{k-1}\, d\tau}{z(\tau)}, \qquad z = \begin{cases} \dfrac{1}{k+1}\tau^{k+1} + \dfrac{1}{k}\tau^k + C_2 & \text{if } k \neq 0,\ -1; \\ \tau + \ln|\tau| + C_2 & \text{if } k = 0; \\ \ln|\tau| - \dfrac{1}{\tau} + C_2 & \text{if } k = -1. \end{cases}$$

139. $y'''_{xxx} = Ay^{\beta}(y'_x)^{-1}(y''_{xx})^2, \qquad \beta \neq 0$.

Solution in the parametric form:

$$x = C_1 \int \tau^{\frac{1-\beta}{\beta}} \exp\bigl(-\tfrac{1}{2}U\bigr)\, d\tau + C_3, \qquad y = b\tau^{1/\beta},$$

where $k = 1/\beta$, $A = -2b^{-\beta}$.

140. $y'''_{xxx} = Ay^{-1}(y'_x)^{\gamma} y''_{xx}, \qquad \gamma \neq 1$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{\frac{2-\gamma}{\gamma-1}} z^{-1} e^U\, d\tau + C_3, \qquad y = e^U,$$

where $k = \dfrac{2}{\gamma - 1}$, $A = a^{\gamma-1} b^{1-\gamma}$.

► *In the solutions of equations 141–170, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \qquad I_1 = 2\tau I \mp R, \qquad I_2 = \tau^{-1}(RI_1 - 1),$$
$$I_3 = 4\tau I_1^2 \mp I_2^2, \qquad I_4 = I_2 I_3 - 8I_1^2, \qquad I_5 = 2RI - \tau^2,$$

where $I = \int \tau R^{-1}\, d\tau + C_2$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

141. $y'''_{xxx} = A(y'_x)^{-7}$.

Solution in the parametric form:

$$x = aC_1^4 \int \tau^{-3/2} R^{-1}\, d\tau + C_3, \qquad y = bC_1^5 \tau^{-1} I_1,$$

where $A = \mp 3a^{-10}b^8$.

142. $\boldsymbol{y'''_{xxx} = Ay^{-4}y'_x(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1^{-1}\int \tau^{-3/2}I_1^{-1/2}\,d\tau + C_3, \qquad y = bC_1^4\tau^{-1},$$

where $\quad A = \pm 24a^4 b$.

143. $\boldsymbol{y'''_{xxx} = Ayy'_x(y''_{xx})^{7/4}}$.

Solution in the parametric form:

$$x = aC_1^{11}\int \tau^{5/2}I_1^{-1/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^6 R,$$

where $\quad A = \mp\dfrac{2}{3}b^{-2}\Big(\mp\dfrac{a^2}{3b}\Big)^{3/4}$.

144. $\boldsymbol{y'''_{xxx} = A(y'_x)^3(y''_{xx})^{5/4}}$.

Solution in the parametric form:

$$x = aC_1^{13}\int \tau^{-2}R^{-3/2}\,d\tau + C_3, \qquad y = bC_1^{10}\tau^{-1}I_1,$$

where $\quad A = -4a^2b^{-3}\Big(\mp\dfrac{a^2}{3b}\Big)^{1/4}$.

145. $\boldsymbol{y'''_{xxx} = Ay(y'_x)^{-7}}$.

Solution in the parametric form:

$$x = aC_1^7\int I_1^{-3/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^{10}\tau I_1^{-1},$$

where $\quad A = \mp 3a^{-10}b^7$.

146. $\boldsymbol{y'''_{xxx} = Ay^{-4}(y'_x)^3(y''_{xx})^3}$.

Solution in the parametric form:

$$x = aC_1\int \tau^{-1/2}I_1^{-3/2}I_2R^{-1}\,d\tau + C_3, \qquad y = bC_1^6 I_1^{-1},$$

where $\quad A = \pm 24a^6b^{-1}$.

147. $\boldsymbol{y'''_{xxx} = Ay(y''_{xx})^{7/4}}$.

Solution in the parametric form:

$$x = aC_1^7\int I_1^2R^{-1}\,d\tau + C_3, \qquad y = bC_1^2 I_2,$$

where $\quad A = -4a^{-1}b^{-1}\Big(\pm\dfrac{a^2}{6b}\Big)^{3/4}$.

148. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{5/4}$.

Solution in the parametric form:

$$x = aC_1^{11}\int \tau I_1^{-3}I_2^{-1/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^{10}\tau^2 I_1^{-2},$$

where $A = \dfrac{1}{2}a^2 b^{-5/2}\Bigl(\pm\dfrac{a^2}{6b}\Bigr)^{5/4}$.

149. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-5/3}(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^{-1}\int \tau I_1^{-1/2}I_2R^{-1}\,d\tau + C_3, \qquad y = bC_1^8 I_2^2,$$

where $A = \mp\dfrac{1}{27}a^2 b^{-1/2}\Bigl(\dfrac{12b}{a}\Bigr)^{2/3}$.

150. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4/3}(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1\int I_1^{-5/2}I_2I_3R^{-1}\,d\tau + C_3, \qquad y = bC_1^{10}I_1^{-3}I_2^2,$$

where $A = \mp\dfrac{16}{27}a^2 b^{-1/2}\Bigl(\pm\dfrac{3b}{a}\Bigr)^{1/3}$.

151. $y'''_{xxx} = Ay^{-1/2}(y'_x)^3(y''_{xx})^{8/7}$.

Solution in the parametric form:

$$x = aC_1^{37}\int I_1^{7/4}I_2^{-1/2}I_3R^{-1}\,d\tau + C_3, \qquad y = bC_1^{32}I_3^2,$$

where $A = \mp 7\cdot 2^{-10}a^2 b^{-5/2}\Bigl(\dfrac{a^2}{6b}\Bigr)^{1/7}$.

152. $y'''_{xxx} = Ay(y''_{xx})^{13/7}$.

Solution in the parametric form:

$$x = aC_1^{13}\int I_1^{-5/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^5 I_1^{-3/2}I_2,$$

where $A = \dfrac{7}{2}a^{-1}b^{-1}\Bigl(\dfrac{a^2}{6b}\Bigr)^{6/7}$.

153. $y'''_{xxx} = Ay(y'_x)^{-13}$.

Solution in the parametric form:

$$x = aC_1^{13}\int I_1^{3/4}R^{-1}\,d\tau + C_3, \qquad y = bC_1^{16}I_3,$$

where $A = \pm 3\cdot 2^{25}a^{-16}b^{13}$.

154. $y'''_{xxx} = Ay^{-7}(y'_x)^3(y''_{xx})^3.$

Solution in the parametric form:

$$x = aC_1^{-1}\int I_1^{-1/2}I_2I_3^{-1/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^3I_1^{1/2},$$

where $A = \mp 12a^6b^2$.

155. $y'''_{xxx} = Ay^3(y'_x)^{-13}.$

Solution in the parametric form:

$$x = aC_1^{11}\int I_1^{3/4}I_3^{-3/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^{16}I_3^{-1},$$

where $A = \pm 3\cdot 2^{25}a^{-16}b^{11}$.

156. $y'''_{xxx} = Ay^{-7}(y'_x)^7(y''_{xx})^3.$

Solution in the parametric form:

$$x = aC_1\int I_1^{-1/2}I_3^{-3/2}I_4R^{-1}\,d\tau + C_3, \qquad y = bC_1^5I_1^{1/2}I_3^{-1},$$

where $A = \mp 192a^{10}b^{-2}$.

157. $y'''_{xxx} = Ay^{-7/6}(y''_{xx})^{2/3}.$

Solution in the parametric form:

$$x = aC_1^9\int I_1^{-5/2}I_3^5R^{-1}\,d\tau + C_3, \qquad y = bC_1^{10}I_1^{-3}I_3^6,$$

where $A = \pm 54a^{-3}b^{13/6}\left(\dfrac{2a^2}{9b}\right)^{2/3}$.

158. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{-4/3}(y''_{xx})^{7/3}.$

Solution in the parametric form:

$$x = aC_1^3\int I_1^{-5/2}I_3^{-1}I_4R^{-1}\,d\tau + C_3, \qquad y = bC_1^2I_1^{-3}I_4^2,$$

where $A = 8b^{1/2}\left(\dfrac{2a}{3}\right)^{1/3}$.

159. $y'''_{xxx} = Ay^{-3/4}(y'_x)^3(y''_{xx})^{8/7}.$

Solution in the parametric form:

$$x = aC_1^{67}\int I_1^{7/4}I_3^{-5}I_4^{-1/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^{64}I_3^{-4},$$

where $A = \mp 7\cdot 2^{-13}a^2b^{-9/4}\left(\dfrac{a^2}{12b}\right)^{1/7}$.

160. $y'''_{xxx} = Ay(y'_x)^{-1/2}(y''_{xx})^{13/7}$.

Solution in the parametric form:

$$x = aC_1^{19}\int I_1^{-5/2} I_3^4 R^{-1}\,d\tau + C_3, \qquad y = bC_1^3 I_1^{-3/2} I_4,$$

where $\quad A = \dfrac{7}{2}a^{-1}b^{-1}\Bigl(\pm\dfrac{3b}{a}\Bigr)^{1/2}\Bigl(\dfrac{a^2}{12b}\Bigr)^{6/7}$.

161. $y'''_{xxx} = A(y'_x)^2$.

Solution in the parametric form:

$$x = aC_1^{-1}\int R^{-1}\,d\tau + C_3, \qquad y = bC_1\int \tau R^{-1}\,d\tau + C_2,$$

where $\quad A = \pm 6a^{-1}b^{-1}$.

162. $y'''_{xxx} = Ay^{1/2}y'_x(y''_{xx})^3$.

Solution in the parametric form:

$$x = aC_1^7\int \tau I^{-1/2}\,d\tau + C_3, \qquad y = bC_1^8\tau^2,$$

where $\quad A = \mp 24a^4b^{-7/2}$.

163. $y'''_{xxx} = Ayy'_x(y''_{xx})^4$.

Solution in the parametric form:

$$x = aC_1^5\int \tau^2 I^{-1/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^6 R,$$

where $\quad A = -\frac{1}{162}a^6b^{-5}$.

164. $y'''_{xxx} = A(y'_x)^3(y''_{xx})^{-1}$.

Solution in the parametric form:

$$x = aC_1^{-1}\int \tau R^{-3/2}\,d\tau + C_3, \qquad y = bC_1^2\int \tau R^{-1}\,d\tau + C_2,$$

where $\quad A = 9a^{-2}b^{-1}$.

165. $y'''_{xxx} = Ay^{-7/2}(y'_x)^2$.

Solution in the parametric form:

$$x = aC_1^5\int I^{-3/2}R^{-1}\,d\tau + C_3, \qquad y = bC_1^2 I^{-1},$$

where $\quad A = \mp 6a^{-1}b^{5/2}$.

166. $y'''_{xxx} = Ay^{1/2}(y'_x)^{-6}(y''_{xx})^3.$

Solution in the parametric form:

$$x = aC_1^7 \int \tau I^{-3/2} I_5 R^{-1}\,d\tau + C_3, \qquad y = bC_1^6 \tau^2 I^{-1},$$

where $A = \mp 48 a^{-3} b^{7/2}$.

167. $y'''_{xxx} = Ay(y'_x)^{-9/5}(y''_{xx})^4.$

Solution in the parametric form:

$$x = aC_1^{11} \int \tau^2 I^{-1/4} R^{-1}\,d\tau + C_3, \qquad y = bC_1^{16} I_5,$$

where $A = -\dfrac{2}{1125} a^4 b^{-3} \Bigl(\dfrac{12b}{a}\Bigr)^{4/5}$.

168. $y'''_{xxx} = Ay^{-7/5}(y'_x)^3(y''_{xx})^{-1}.$

Solution in the parametric form:

$$x = aC_1 \int \tau I^{3/2} I_5^{-1/2} R^{-1}\,d\tau + C_3, \qquad y = bC_1 I^{5/2},$$

where $A = \frac{36}{5} a^{-2} b^{2/5}$.

169. $y'''_{xxx} = Ay^{-1/3}(y''_{xx})^{9/7}.$

Solution in the parametric form:

$$x = aC_1 \int \tau^2 I^{-5/2} R^{-1}\,d\tau + C_3, \qquad y = bC_1^9 \tau^3 I^{-3/2},$$

where $A = \dfrac{7}{2} a^{-1} b^{1/3} \Bigl(\dfrac{a^2}{18b}\Bigr)^{2/7}$.

170. $y'''_{xxx} = Ay^{-1/2}(y'_x)^{1/3}(y''_{xx})^{12/7}.$

Solution in the parametric form:

$$x = aC_1^{23} \int \tau^{1/2} I^{7/4} I_5 R^{-1}\,d\tau + C_3, \qquad y = bC_1^{32} I_5^2,$$

where $A = -\dfrac{7}{4} a^{-1} b^{1/2} \Bigl(\dfrac{a}{3b}\Bigr)^{1/3} \Bigl(\dfrac{a^2}{18b}\Bigr)^{5/7}$.

► *In the solutions of equations 171–172, the following notation is used:*

$$f = \exp\Bigl(\int \frac{d\tau}{\sqrt{z}}\Bigr), \qquad z = \begin{cases} C_2 + \dfrac{1}{4}\tau^2 + \dfrac{2B}{k+1}\tau^{k+1} & \text{if } k \neq -1, \\ C_2 + \dfrac{1}{4}\tau^2 + 2B \ln|\tau| & \text{if } k = -1. \end{cases}$$

171. $y'''_{xxx} = Ay^{-\frac{\gamma+5}{4}}(y'_x)^\gamma$.

Solution in the parametric form:

$$x = aC_1^3 \int \tau^{-1/2} f^{3/4} z^{-1/2}\, d\tau + C_3, \qquad y = bC_1^4 f,$$

where $k = \dfrac{\gamma - 1}{2}$, $A = \dfrac{1}{2} B a^{2(k-1)} b^{\frac{3}{2}(1-k)}$.

172. $y'''_{xxx} = Ay^{-\gamma-2}(y'_x)^\gamma (y''_{xx})^3$.

Solution in the parametric form:

$$x = a \int (\tau z^{-1/2} + 2)\, d\tau + C_3, \qquad y = C_1 \tau f^{1/2},$$

where $k = -\gamma - 2$, $A = -2^{3-k} a^{1-k} B$.

173. $y'''_{xxx} = Ay^{\frac{\gamma-1}{2}}(y'_x)^\gamma (y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1^3 \int \frac{\left(V - \sqrt{V^2+4}\right)}{(\tau U)^{3/4}\sqrt{V^2+4}}\, d\tau + C_3, \qquad y = bC_1^2 \tau^{1/2} U^{1/2},$$

where

$$U = \exp\left(\int \frac{V\, d\tau}{\tau\sqrt{V^2+4}}\right), \quad V = \begin{cases} \tau^{-1/2}\left(C_2 + \dfrac{B}{\gamma+1}\tau^{\frac{\gamma+1}{2}}\right) & \text{if } \gamma \neq -1, \\ \tau^{-1/2}(C_2 + \frac{1}{2} B \ln|\tau|) & \text{if } \gamma = -1, \end{cases}$$

$$A = 2^{3/2} a b^{-\frac{\gamma}{2}-1} B\left(-\frac{2a}{b}\right)^{\gamma-1}.$$

174. $y'''_{xxx} = Ay^\beta (y'_x)^\gamma (y''_{xx})^{\frac{\gamma+4\beta+5}{\gamma+2\beta+3}}$, $\quad \beta,\ \gamma \neq -1$.

Solution in the parametric form:

$$x = aC_1^{\gamma+\beta+2} \int \tau^{-3/2} U^{-\frac{\gamma+4\beta+5}{2(\gamma+1)}} z^{-1}\, d\tau + C_3, \qquad y = bC_1^{\gamma+1} U,$$

where $A = a^{\gamma-1} b^{-\beta-\gamma} \left(\dfrac{2a^2}{b}\right)^{\frac{2(\beta+1)}{\gamma+2\beta+3}} B$, $U = \exp\left(\displaystyle\int \frac{d\tau}{\tau z}\right)$, $z = z(\tau)$ is the solution of the algebraic equation

$$(z + k - 1)(z + k)^{\frac{k}{1-k}} = \left(C_2 + \frac{2B}{\gamma + 2\beta + 3}\tau^{\frac{\gamma+1}{2}}\right)\tau^{\frac{1}{k-1}}, \qquad k = -\frac{2(\beta+1)}{\gamma+1}.$$

175. $y'''_{xxx} = Ay^{-1}(y'_x)^{-1}(y''_{xx})^\delta$, $\quad \delta \neq 1,\ 2$.

Solution in the parametric form:

$$x = aC_1^{\delta-3} \int \tau^{k-1} U^{\frac{2-k}{2k}} z^{-1}\, d\tau + C_3, \qquad y = bC_1^{2\delta-4} k(kz - \tau)^{-1} U^{1/k},$$

where $k = \dfrac{\delta - 1}{\delta - 2}$, $A = \dfrac{1-k}{2} a^{-4} b^3 \left(-\dfrac{2a^2}{b}\right)^\delta B$, $U = \exp\left(\displaystyle\int \frac{d\tau}{z}\right)$, $z = z(\tau)$ is the solution of the transcendental equation

$$\ln|kz - \tau| - \frac{\tau}{kz - \tau} = \frac{1}{k}\tau^k + C_2.$$

176. $y'''_{xxx} = Ay^{-1}(y'_x)^{-1}(y''_{xx})^2$.

Solution in the parametric form:

$$x = \pm C_1 \int e^{\tau} z^{-1/2} U^{-1/2}\, d\tau + C_3, \qquad y = \pm \tfrac{1}{2} e^{\tau},$$

where $\quad z = \mp A\tau + e^{\tau} + C_2, \quad U = \exp\biggl(\pm A \int \frac{d\tau}{z}\biggr).$

177. $y'''_{xxx} = Ay^{-1}(y'_x)^{-1}y''_{xx}$.

Solution in the parametric form:

$$x = C_1 \int e^{\tau/2} U\, d\tau + C_3, \qquad y = \pm C_1 z U,$$

where $\quad z = \pm A\tau + e^{\tau} + C_2, \quad U = \exp\biggl(\mp \int \frac{d\tau}{z}\biggr).$

3.2.5. Some Transformations

Let us consider some transformations of the equation $y'''_{xxx} = Ax^{\alpha}y^{\beta}(y'_x)^{\gamma}(y''_{xx})^{\delta}$.

1. In the special case $\gamma = \delta = 0$, the transformation

$$x = \frac{1}{t}, \qquad y = \frac{w}{t^2}$$

reduces the equation

$$y'''_{xxx} = Ax^{\alpha}y^{\beta}$$

to an equation of the similar form

$$w'''_{ttt} = -At^{-\alpha-2\beta-4}w^{\beta}.$$

2. In the special case $\alpha = \delta = 0$, the transformation

$$x = -\int \frac{d\tau}{[z(\tau)]^{3/2}}, \qquad y = \frac{1}{z(\tau)}$$

reduces the equation

$$y'''_{xxx} = Ay^{\beta}(y'_x)^{\gamma}$$

to an equation of the similar form

$$z'''_{\tau\tau\tau} = Az^{-\frac{2\beta+\gamma+5}{2}}(w'_{\tau})^{\gamma}.$$

3. In the special case $\beta = 0$, the substitution

$$u(x) = y'_x$$

reduces the equation

$$y'''_{xxx} = Ax^{\alpha}(y'_x)^{\gamma}(y''_{xx})^{\delta}$$

to the generalized Emden–Fowler equation

$$u''_{xx} = Ax^{\alpha} u^{\gamma} (u'_x)^{\delta}$$

(see Section 2.3 and Section 2.5).

4. In the special case $\alpha = 0$, the substitution

$$v(y) = (y'_x)^2$$

reduces the equation

$$y'''_{xxx} = Ay^{\beta} (y'_x)^{\gamma} (y''_{xx})^{\delta}$$

to the generalized Emden—Fowler equation

$$v''_{yy} = A \cdot 2^{1-\delta} y^{\beta} v^{\frac{\gamma-1}{2}} (v'_y)^{\delta}$$

(see Section 2.3 and Section 2.5).

3.3. Equations of the Form $y'''_{xxx} = f(y) g(y'_x) h(y''_{xx})$

3.3.1. Equations Containing Power Functions

1. $\quad y'''_{xxx} = (ay^2 + by + c)^{-5/4}$.

This is a special case of equation 3.5.2.17 with $f(w) = 1$.

2. $\quad y'''_{xxx} = (Ay^n + By^m) y'_x$.

This is a special case of equation 3.5.2.1 with $f(y) = Ay^n + By^m$.

3. $\quad y'''_{xxx} = (Ay^n + By^m) \big[a(y'_x)^3 + by'_x\big]$.

This is a special case of equation 3.5.2.4 with $f(y) = b(Ay^n + By^m)$, $g(y) = a(Ay^n + By^m)$.

4. $\quad y'''_{xxx} = y^{-2} \left[-\dfrac{(m+1)}{(m+3)^2} (y'_x)^3 + A(y'_x)^{2m+1} \right], \qquad m \neq -3,\ m \neq -1.$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.4:

$$w''_{yy} = y^{-2} \left[-\frac{2(m+1)}{(m+3)^2} w + 2Aw^m \right].$$

5. $\quad y'''_{xxx} = y^{-2} \big[\frac{15}{8} (y'_x)^3 + A(y'_x)^{-13}\big]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.35:

$$w''_{yy} = y^{-2} \left(\tfrac{15}{4} w + 2Aw^{-7}\right).$$

6. $\boldsymbol{y'''_{xxx} = y^{-2}\big[3(y'_x)^3 + A(y'_x)^{-7}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.31:

$$w''_{yy} = y^{-2}(6w + 2Aw^{-4}).$$

7. $\boldsymbol{y'''_{xxx} = y^{-2}\big[6(y'_x)^3 + A(y'_x)^{-4}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.64:

$$w''_{yy} = y^{-2}(12w + 2Aw^{-5/2}).$$

8. $\boldsymbol{y'''_{xxx} = y^{-2}\big[(y'_x)^3 + A(y'_x)^{-3}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.6:

$$w''_{yy} = y^{-2}(2w + 2Aw^{-2}).$$

9. $\boldsymbol{y'''_{xxx} = y^{-2}\big[-\tfrac{3}{32}(y'_x)^3 + A(y'_x)^{-7/3}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.26:

$$w''_{yy} = y^{-2}(-\tfrac{3}{16}w + 2Aw^{-5/3}).$$

10. $\boldsymbol{y'''_{xxx} = y^{-2}\big[-\tfrac{9}{200}(y'_x)^3 + A(y'_x)^{-7/3}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.10:

$$w''_{yy} = y^{-2}(-\tfrac{9}{100}w + 2Aw^{-5/3}).$$

11. $\boldsymbol{y'''_{xxx} = y^{-2}\big[\tfrac{3}{8}(y'_x)^3 + A(y'_x)^{-7/3}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.12:

$$w''_{yy} = y^{-2}(\tfrac{3}{4}w + 2Aw^{-5/3}).$$

12. $\boldsymbol{y'''_{xxx} = y^{-2}\big[\tfrac{63}{8}(y'_x)^3 + A(y'_x)^{-7/3}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.66:

$$w''_{yy} = y^{-2}(\tfrac{63}{4}w + 2Aw^{-5/3}).$$

13. $\boldsymbol{y'''_{xxx} = y^{-2}\big[-\tfrac{5}{72}(y'_x)^3 + A(y'_x)^{-9/5}\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.29:

$$w''_{yy} = y^{-2}(-\tfrac{5}{36}w + 2Aw^{-7/5}).$$

14. $\boldsymbol{y'''_{xxx} = y^{-2}\big[-\tfrac{1}{9}(y'_x)^3 + A\big]}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.14:

$$w''_{yy} = y^{-2}(-\tfrac{2}{9}w + 2Aw^{-1/2}).$$

15. $y'''_{xxx} = y^{-2}\left[-\frac{2}{25}(y'_x)^3 + A\right]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.8:

$$w''_{yy} = y^{-2}\left(-\tfrac{4}{25}w + 2Aw^{-1/2}\right).$$

16. $y'''_{xxx} = y^{-2}\left[10(y'_x)^3 + A\right]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.33:

$$w''_{yy} = y^{-2}(20w + 2Aw^{-1/2}).$$

17. $y'''_{xxx} = y^{-2}\left[-\frac{6}{49}(y'_x)^3 + A(y'_x)^2\right]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.37:

$$w''_{yy} = y^{-2}\left(-\tfrac{12}{49}w + 2Aw^{1/2}\right).$$

18. $y'''_{xxx} = y^{-2}\left[A(y'_x)^5 - \frac{3}{25}(y'_x)^3\right]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.60:

$$w''_{yy} = y^{-2}\left(2Aw^2 - \tfrac{6}{25}w\right).$$

19. $y'''_{xxx} = y^{-2}\left[A(y'_x)^5 + \frac{3}{25}(y'_x)^3\right]$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.62:

$$w''_{yy} = y^{-2}\left(2Aw^2 + \tfrac{6}{25}w\right).$$

20. $y'''_{xxx} = y^{-4/3}(Ay'_x + B)$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.40:

$$w''_{yy} = y^{-4/3}(2A + 2Bw^{-1/2}).$$

21. $y'''_{xxx} = (Ay^4 + By^3)(y'_x)^{-13}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.39:

$$w''_{yy} = (2Ay^4 + 2By^3)w^{-7}.$$

22. $y'''_{xxx} = (Ay^2 + B)(y'_x)^{-9}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.16:

$$w''_{yy} = (2Ay^2 + 2B)w^{-5}.$$

23. $y'''_{xxx} = (Ay^{-1} + By^{-2})(y'_x)^{-3}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.28:

$$w''_{yy} = (2Ay^{-1} + 2By^{-2})w^{-2}.$$

24. $\boldsymbol{y'''_{xxx} = (Ay^{-7/3} + By^{-10/3})(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.48:

$$w''_{yy} = (2Ay^{-7/3} + 2By^{-10/3})w^{-5/3}.$$

25. $\boldsymbol{y'''_{xxx} = (Ay^{-4/3} + By^{-10/3})(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.49:

$$w''_{yy} = (2Ay^{-4/3} + 2By^{-10/3})w^{-5/3}.$$

26. $\boldsymbol{y'''_{xxx} = (Ay^{-4/3} + By^{-7/3})(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.24:

$$w''_{yy} = (2Ay^{-4/3} + 2By^{-7/3})w^{-5/3}.$$

27. $\boldsymbol{y'''_{xxx} = (Ay^{-2/3} + By^{-4/3})(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.90:

$$w''_{yy} = (2Ay^{-2/3} + 2By^{-4/3})w^{-5/3}.$$

28. $\boldsymbol{y'''_{xxx} = (A + By^{-2/3})(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.89:

$$w''_{yy} = (2A + 2By^{-2/3})w^{-5/3}.$$

29. $\boldsymbol{y'''_{xxx} = (Ay^2 + B)(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.47:

$$w''_{yy} = (2Ay^2 + 2B)w^{-5/3}.$$

30. $\boldsymbol{y'''_{xxx} = (Ay^2 + By)(y'_x)^{-7/3}}$.

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.4.2.46:

$$w''_{yy} = (2Ay^2 + 2By)w^{-5/3}.$$

31. $\boldsymbol{y'''_{xxx} = (Ay^n + By^k)y'_x(y''_{xx})^m}$.

This is a special case of equation 3.5.4.15 with $f(y) = Ay^n + By^k$.

TABLE 3.2
Solvable equations of the form
$$y'''_{xxx} = Ae^y (y'_x)^\gamma (y''_{xx})^\delta$$

δ	γ	Equation
arbitrary ($\delta \neq 1$)	1	3.3.2.1
0	3	3.3.2.9
1	1	3.3.2.2
$\frac{3}{2}$	0	3.3.2.3
$\frac{3}{2}$	3	3.3.2.7
2	arbitrary ($\gamma \neq -1$)	3.3.2.11
2	-1	3.3.2.13

TABLE 3.3
Solvable equations of the form
$$y'''_{xxx} = Ay^\beta y'_x \exp[(y'_x)^2] (y''_{xx})^\delta$$

δ	β	Equation
arbitrary ($\delta \neq 2$)	0	3.3.2.4
1	arbitrary ($\beta \neq -1$)	3.3.2.12
1	-1	3.3.2.14
$\frac{3}{2}$	$-\frac{1}{2}$	3.3.2.5
$\frac{3}{2}$	1	3.3.2.8
2	0	3.3.2.6
3	1	3.3.2.10

TABLE 3.4
Other solvable equations of the type considered

Form of equation	Equation
$y'''_{xxx} = Ae^y y'_x \exp[(y'_x)^2] (y''_{xx})^\delta$	3.3.2.20
$y'''_{xxx} = A(y'_x)^\gamma \exp(y''_{xx}), \quad \gamma \neq -1$	3.3.2.15
$y'''_{xxx} = A(y'_x)^{-1} \exp(y''_{xx})$	3.3.2.16
$y'''_{xxx} = Ay^\beta y'_x \exp(y''_{xx}), \quad \beta \neq -1$	3.3.2.17
$y'''_{xxx} = Ay^{-1} y'_x \exp(y''_{xx})$	3.3.2.18
$y'''_{xxx} = Ae^y y'_x \exp[(y'_x)^2 + y''_{xx}]$	3.3.2.19

3.3.2. Equations Containing Exponential Functions

Tables 3.2–3.4 represent the equations whose solutions are given in this subsection.

► *In the solutions of equations 1–6, the following notation is used:*

$$E = \int \exp(\tau^2)\, d\tau + C_2, \qquad G = \int \tau^{-1}(1 \pm \tau)^k\, d\tau + C_2,$$
$$F = 2\tau E - \exp(\tau^2), \qquad H = \int \tau^{-1} \exp(\mp\tau)\, d\tau + C_2.$$

1. $\boldsymbol{y'''_{xxx} = Ae^y y'_x (y''_{xx})^\delta, \qquad \delta \neq 1.}$

Solution in the parametric form:

$$x = aC_1 \int \tau^{-1} G^{-1/2}\, d\tau + C_3, \qquad y = \ln(bC_1^{2\delta-2}\tau),$$

where $k = \dfrac{1}{1-\delta}$, $A = \pm\dfrac{1}{2(1-\delta)} a^{-2} b^{-1} (2a^2)^\delta$.

2. $y'''_{xxx} = Ae^{y} y'_x y''_{xx}$.

Solution in the parametric form:

$$x = C_1 \int \tau^{-1} H^{-1/2}\, d\tau + C_3, \qquad y = \ln\Bigl(\mp\frac{\tau}{A}\Bigr).$$

3. $y'''_{xxx} = Ae^{y} (y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = C_1 \int E^{-1}\, d\tau + C_3, \qquad y = \tau^2 + \ln\bigl(\sqrt{2}\, A^{-1} E^{-1}\bigr).$$

4. $y'''_{xxx} = Ay'_x \exp\bigl[(y'_x)^2\bigr] (y''_{xx})^{\delta}, \qquad \delta \neq 2$.

Solution in the parametric form:

$$x = aC_1 \int \tau^{-1}(1 \pm \tau)^{\frac{1}{\delta-2}} \bigl[\ln(bC_1^{\delta-2}\tau)\bigr]^{-1/2}\, d\tau + C_3, \qquad y = aC_1 G,$$

where $k = \dfrac{1}{\delta - 2}$, $A = \pm\dfrac{1}{2-\delta} a^{\delta-3} b^{-1}$.

5. $y'''_{xxx} = Ay^{-1/2} y'_x \exp\bigl[(y'_x)^2\bigr] (y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = 4C_1 \int EF\bigl[\tau^2 + \ln(aE^{-1})\bigr]^{-1/2}\, d\tau + C_3, \quad y = C_1 F^2, \qquad A = -\sqrt{2}\, a^{-1}.$$

6. $y'''_{xxx} = Ay'_x \exp\bigl[(y'_x)^2\bigr] (y''_{xx})^2$.

Solution in the parametric form:

$$x = C_1 \int \tau^{-1} \exp(\mp\tau) \Bigl[\ln\Bigl(\pm\frac{\tau}{A}\Bigr)\Bigr]^{-1/2}\, d\tau + C_3, \qquad y = C_1 H.$$

► *In the solutions of equations 7–8, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\bigl[C_2\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr)\bigr], \qquad R = \sqrt{\frac{\tau+1}{\tau}},$$

$$F = 1 - \sqrt{\frac{\tau+1}{\tau}} \ln\bigl[C_2\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr)\bigr].$$

7. $y'''_{xxx} = Ae^{y} (y'_x)^3 (y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = aC_1 \int R^{-1} E^{-1} F^{-1/2}\, d\tau + C_3, \qquad y = -\ln(bC_1^{-3} E),$$

where $A = 2^{3/2} a^3 b$.

8. $y'''_{xxx} = Ayy'_x \exp\bigl[(y'_x)^2\bigr] (y''_{xx})^{3/2}$.

Solution in the parametric form:

$$x = -\tfrac{1}{2} bC_1 \int \tau^{-2} R^{-1} E\bigl[\ln(aC_1^{-3/2} E^{-1})\bigr]^{-1/2}\, d\tau + C_3, \qquad y = bC_1 F,$$

where $A = -4a^{-1} b^{-3/2}$.

► *In the solutions of equations 9–10, the following notation is used:*

$$Z = \begin{cases} C_1 J_0(\tau) + C_2 Y_0(\tau) & \text{for the upper sign,} \\ C_1 I_0(\tau) + C_2 K_0(\tau) & \text{for the lower sign,} \end{cases}$$

where J_0 and Y_0 are Bessel functions, I_0 and K_0 are modified Bessel functions.

9. $y'''_{xxx} = Ae^y (y'_x)^3$.

Solution in the parametric form:

$$x = 2C_1 \int \tau^{-1} Z^{-1/2}\, d\tau + C_2, \qquad y = \ln\left(\mp \tfrac{1}{8} A^{-1} \tau^2\right).$$

10. $y'''_{xxx} = Ayy'_x \exp\bigl[(y'_x)^2\bigr] (y''_{xx})^3$.

Solution in the parametric form:

$$x = C_1 \int Z'_\tau \left[\ln\left(\pm \frac{\tau^2}{A}\right)\right]^{-1/2} d\tau + C_2, \qquad y = C_1 Z.$$

11. $y'''_{xxx} = Ae^y (y'_x)^\gamma (y''_{xx})^2, \qquad \gamma \neq -1$.

Solution in the parametric form:

$$x = a \int \tau^{-1/2} f^{-1} \exp\left(\frac{U}{\gamma + 1}\right) d\tau + C_3, \qquad y = U,$$

where $A = \frac{1}{2} a^{\gamma+1} k$, $U = \displaystyle\int \frac{d\tau}{f} + C_1$, $f = f(\tau)$ is the solution of the transcendental equation

$$\ln(\lambda f - \tau) - \frac{\tau}{\lambda f - \tau} = \frac{k}{\lambda} \tau^\lambda + C_2, \qquad \lambda = \frac{\gamma + 1}{2}.$$

12. $y'''_{xxx} = Ay^\beta y'_x \exp\bigl[(y'_x)^2\bigr] y''_{xx}, \qquad \beta \neq -1$.

Solution in the parametric form:

$$x = a \int f^{-1} \left(f - \frac{\tau}{\beta + 1}\right) U^{-1/2} \exp\left(-\frac{U}{\beta + 1}\right) d\tau + C_3, \qquad y = a\tau \exp\left(-\frac{U}{\beta + 1}\right),$$

where $A = a^{-\beta-1} k$, $U = \displaystyle\int \frac{d\tau}{f} + C_1$, $f = f(\tau)$ is the solution of the transcendental equation

$$\ln(\lambda f - \tau) - \frac{\tau}{\lambda f - \tau} = -\frac{k}{\lambda} \tau^\lambda + C_2, \qquad \lambda = \beta + 1.$$

13. $y'''_{xxx} = Ae^y (y'_x)^{-1} (y''_{xx})^2$.

Solution in the parametric form:

$$x = C_1 \int W^{-1/2}\, d\tau + C_3, \qquad y = \tau,$$

where $W = \exp\left(\displaystyle\int \frac{d\tau}{\tau - 2Ae^\tau + C_2}\right)$.

14. $y'''_{xxx} = Ay^{-1}y'_x \exp\bigl[(y'_x)^2\bigr]y''_{xx}$.

Solution in the parametric form:

$$x = C_1 \int \tau^{-1/2}(\tau + Ae^{\tau} + C_2)^{-1} W\, d\tau + C_3, \qquad y = C_1 W,$$

where $W = \exp\biggl(\displaystyle\int \frac{d\tau}{\tau + Ae^{\tau} + C_2}\biggr)$.

► *In the solutions of equations 15–19, the following notation is used:*

$$V = C_1 - \frac{1}{\lambda}(m+1)(\tau+1)e^{-\tau}, \qquad M = C_1 - \frac{1}{\lambda}(\tau+2)e^{-\tau/2},$$

$$W = \begin{cases} C_2 - \displaystyle\int \ln\Bigl(C_1 - \frac{\lambda}{n+1}\tau^{n+1}\Bigr)\, d\tau & \text{if } n \neq -1, \\ C_2 - \displaystyle\int \ln(C_1 - \lambda \ln|\tau|)\, d\tau & \text{if } n = -1, \end{cases}$$

$$N = \ln M - \frac{1}{2\lambda}\int e^{-\tau/2}M^{-1}\, d\tau - \frac{1}{2} + C_2.$$

15. $y'''_{xxx} = A(y'_x)^{\gamma} \exp(y''_{xx}), \qquad \gamma \neq -1.$

Solution in the parametric form:

$$x = \frac{1}{\lambda}\int e^{-\tau}V^{-\frac{2m+1}{2m+2}}\, d\tau + C_3, \quad y = \frac{2}{\lambda}\int e^{-\tau}V^{-\frac{m}{m+1}}\, d\tau + C_2,$$

where $m = \frac{1}{2}(\gamma - 1)$, $A = 2^{-\gamma}\lambda$.

16. $y'''_{xxx} = A(y'_x)^{-1} \exp(y''_{xx})$.

Solution in the parametric form:

$$x = \frac{1}{\lambda}\int \exp\Bigl(-\tau + \frac{1}{2}V\Bigr)\, d\tau + C_3, \qquad y = \frac{2}{\lambda}\int \exp(-\tau + V)\, d\tau + C_2,$$

where $m = 0$, $A = 2\lambda$.

17. $y'''_{xxx} = Ay^{\beta}y'_x \exp(y''_{xx}), \qquad \beta \neq -1.$

Solution in the parametric form:

$$x = \int W^{-1/2}\, d\tau + C_3, \qquad y = 2\tau,$$

where $n = \beta$, $A = 2^{-\beta-1}\lambda$.

18. $y'''_{xxx} = Ay^{-1}y'_x \exp(y''_{xx})$.

Solution in the parametric form:

$$x = \int W^{-1/2}\, d\tau + C_3, \qquad y = 2\tau,$$

where $n = -1$, $A = \lambda$.

19. $y'''_{xxx} = Ae^y y'_x \exp\bigl[(y'_x)^2 + y''_{xx}\bigr].$

Solution in the parametric form:

$$x = \frac{1}{2\lambda}\int e^{-\tau/2} M^{-1} N^{-1/2}\,d\tau + C_3, \qquad y = \frac{1}{2\lambda}\int e^{-\tau/2} M^{-1}\,d\tau + C_2,$$

where $A = \lambda$.

20. $y'''_{xxx} = Ae^y y'_x \exp[(y'_x)^2](y''_{xx})^\delta.$

Solution in the parametric form:

$$x = \int \tau^{-\delta} z^{-1}\Bigl(\ln\frac{z}{A} - U - C_1\Bigr)^{-1/2} d\tau + C_3, \qquad y = U,$$

where

$$U = \int \frac{d\tau}{z\tau^\delta}, \qquad z = \begin{cases} \dfrac{1}{2-\delta}\tau^{2-\delta} + \dfrac{1}{1-\delta}\tau^{1-\delta} + C_2 & \text{if } \delta \neq 2,\ \delta \neq 1, \\ \tau + \ln|\tau| + C_2 & \text{if } \delta = 1, \\ \ln|\tau| - \dfrac{1}{\tau} + C_2 & \text{if } \delta = 2. \end{cases}$$

3.3.3. Other Equations

1. $y'''_{xxx} = Ayy'_x\{\cosh[\omega(y'_x)^2]\}^{-2} y''_{xx}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.1: $u''_{yy} = Ay[\cosh(\omega u)]^{-2} u'_y$.

2. $y'''_{xxx} = Ayy'_x\{\sinh[\omega(y'_x)^2]\}^{-2} y''_{xx}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.2: $u''_{yy} = Ay[\sinh(\omega u)]^{-2} u'_y$.

3. $y'''_{xxx} = Ayy'_x \cosh[\omega(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.3: $u''_{yy} = \dfrac{A}{\sqrt{2}} y\cosh(\omega u)(u'_y)^{3/2}$.

4. $y'''_{xxx} = Ayy'_x \sinh[\omega(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.4: $u''_{yy} = \dfrac{A}{\sqrt{2}} y\sinh(\omega u)(u'_y)^{3/2}$.

5. $y'''_{xxx} = A\cosh(\omega y)\,(y'_x)^3 (y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.5: $u''_{yy} = \dfrac{A}{\sqrt{2}} \cosh(\omega y) u (u'_y)^{3/2}$.

6. $y'''_{xxx} = A\sinh(\omega y)\,(y'_x)^3 (y''_{xx})^{3/2}$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.6: $u''_{yy} = \dfrac{A}{\sqrt{2}} \sinh(\omega y) u (u'_y)^{3/2}$.

7. $y'''_{xxx} = A[\cosh(\omega y)]^{-2} (y'_x)^3 (y''_{xx})^2$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.7: $u''_{yy} = \frac{1}{2} A[\cosh(\omega y)]^{-2} u (u'_y)^2$.

8. $y'''_{xxx} = A[\sinh(\omega y)]^{-2} (y'_x)^3 (y''_{xx})^2$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.8: $u''_{yy} = \frac{1}{2} A[\sinh(\omega y)]^{-2} u (u'_y)^2$.

9. $y'''_{xxx} = A\cosh^n(\omega y)\, y'_x (y''_{xx})^m$.

This is a special case of equation 3.5.4.15 with $f(y) = A\cosh^n(\omega y)$.

10. $y'''_{xxx} = A\sinh^n(\omega y)\, y'_x (y''_{xx})^m$.

This is a special case of equation 3.5.4.15 with $f(y) = A\sinh^n(\omega y)$.

11. $y'''_{xxx} = A\tanh^n(\omega y)\, y'_x (y''_{xx})^m$.

This is a special case of equation 3.5.4.15 with $f(y) = A\tanh^n(\omega y)$.

12. $y'''_{xxx} = A\coth^n(\omega y)\, y'_x (y''_{xx})^m$.

This is a special case of equation 3.5.4.15 with $f(y) = A\coth^n(\omega y)$.

13. $y'''_{xxx} = A\ln^n(\omega y)\, y'_x (y''_{xx})^m$.

This is a special case of equation 3.5.4.15 with $f(y) = A\ln^n(\omega y)$.

14. $y''_{xx} = Ax[\cos(\omega y)]^{-2} y'_x$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.1: $u''_{yy} = Ay[\cos(\omega u)]^{-2} u'_y$.

15. $y'''_{xxx} = Ayy'_x \{\sin[\omega (y'_x)^2]\}^{-2} y''_{xx}$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.2: $u''_{yy} = Ay[\sin(\omega u)]^{-2} u'_y$.

16. $y'''_{xxx} = A[\cos(\omega y)]^{-2} (y'_x)^3 (y''_{xx})^2$.

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.3: $u''_{yy} = \frac{1}{2} A[\cos(\omega y)]^{-2} u (u'_y)^2$.

17. $y'''_{xxx} = A[\sin(\omega y)]^{-2}(y'_x)^3(y''_{xx})^2.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.4: $u''_{yy} = \frac{1}{2}A[\sin(\omega y)]^{-2}u(u'_y)^2$.

18. $y'''_{xxx} = Ayy'_x \cos[\omega(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.5: $u''_{yy} = \dfrac{A}{\sqrt{2}}y\cos(\omega u)(u'_y)^{3/2}$.

19. $y'''_{xxx} = Ayy'_x \sin[\omega(y'_x)^2]\,(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.6: $u''_{yy} = \dfrac{A}{\sqrt{2}}y\sin(\omega u)(u'_y)^{3/2}$.

20. $y'''_{xxx} = A\cos(\omega y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.5.7: $u''_{yy} = \dfrac{A}{\sqrt{2}}\cos(\omega y)u(u'_y)^{3/2}$.

21. $y'''_{xxx} = A\sin(\omega y)\,(y'_x)^3(y''_{xx})^{3/2}.$

The substitution $y'_x = \sqrt{u(y)}$ leads to a second order equation of the form 2.7.4.8: $u''_{yy} = \dfrac{A}{\sqrt{2}}\sin(\omega y)u(u'_y)^{3/2}$.

22. $y'''_{xxx} = A\cos^n(\omega y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A\cos^n(\omega y)$.

23. $y'''_{xxx} = A\sin^n(\omega y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A\sin^n(\omega y)$.

24. $y'''_{xxx} = A\tan^n(\omega y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A\tan^n(\omega y)$.

25. $y'''_{xxx} = A\cot^n(\omega y)\,y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A\cot^n(\omega y)$.

26. $y'''_{xxx} = A(\arcsin y)^n y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A(\arcsin y)^n$.

27. $y'''_{xxx} = A(\arctan y)^n y'_x(y''_{xx})^m.$

This is a special case of equation 3.5.4.15 with $f(y) = A(\arctan y)^n$.

3.4. Some Nonlinear Equations with Arbitrary Parameters

3.4.1. Equations Containing Power Functions

1. $\boldsymbol{y'''_{xxx} = ay^{-5/2} + by^{-7/2}}$.

Using the transformation given in 3.5.2.21, we reduce this equation to a nonhomogeneous constant-coefficient linear equation.

Solution in the parametric form ($b \neq 0$):

$$x = \int \frac{d\tau}{[\varphi(\tau)]^{3/2}} + C_3, \quad y = \frac{1}{\varphi(\tau)},$$

where

$$\varphi = -\frac{a}{b} + C_1 e^{-k\tau} + C_2 e^{k\tau/2} \sin\frac{k\tau\sqrt{3}}{2}, \quad k = b^{1/3}.$$

2. $\boldsymbol{y'''_{xxx} = axy^{-5/2} + bx^3 y^{-7/2}}$.

The transformation $x = 1/t$, $y = w/t^2$ leads to an equation of the form 3.4.1.1: $w'''_{ttt} = -aw^{-5/2} - bw^{-7/2}$.

3. $\boldsymbol{y'''_{xxx} = (y + ax^2 + bx + c)^n}$.

The substitution $z = y + ax^2 + bx + c$ leads to the equation $z'''_{xxx} = z^n$ whose solvable cases are outlined in Section 3.2.

4. $\boldsymbol{y'''_{xxx} = (ax + b)^n (cx + d)^{-n-2m-4} y^m}$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^2}$ leads to a simpler equation (see Section 3.2):

$$w'''_{\xi\xi\xi} = \Delta^{-3}\xi^n w^m, \qquad \text{where} \quad \Delta = ad - bc.$$

5. $\boldsymbol{y'''_{xxx} = x(ay^2 + bx^2 y + cx^4)^{-5/4}}$.

This is a special case of equation 3.5.2.18 with $f(\xi) \equiv 1$.

6. $\boldsymbol{y'''_{xxx} = bx^{2n-1}(x - a)^{-3} y^{-n}}$.

This is a special case of equation 3.5.1.9 with $f(\xi) = b\xi^{-n}$.

7. $\boldsymbol{y'''_{xxx} = ax^{-n-2} y^n y'_x - ax^{-n-3} y^{n+1}}$.

This is a special case of equation 3.5.2.2 with $f(\xi) = a\xi^n$.

8. $\boldsymbol{y'''_{xxx} = ax^{-2n-4} y^n y'_x - 2ax^{-2n-5} y^{n+1}}$.

This is a special case of equation 3.5.2.3 with $f(\xi) = a\xi^n$.

9. $\boldsymbol{y'''_{xxx} = \lambda y^{-3} y'_x + ay^{-5/2} + by^{-7/2}}$.

The transformation $x = \int [\varphi(\tau)]^{-3/2}\, d\tau$, $y = [\varphi(\tau)]^{-1}$ leads to a constant coefficient equation: $\varphi'''_{\tau\tau\tau} - \lambda\varphi'_\tau + b\varphi + a = 0$.

10. $y'''_{xxx} = -x^{-2}y'_x + x^{-3}y + ax^{1/2}y^{-5/2}.$

This is a special case of equation 3.5.2.20 with $f(\xi) = a$.

11. $y'''_{xxx} = -x^{-2}y'_x + x^{-3}y + ax^{-3/4}y^{-5/4}.$

This is a special case of equation 3.5.2.19 with $f(\xi) = a$.

12. $y'''_{xxx} = ay^n y'_x + by^m (y'_x)^3.$

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = by^m$.

13. $y'''_{xxx} = by^n (y'_x)^3 + a(y'_x)^{-5}.$

This is a special case of equation 3.5.2.9 with $f(y) = by^n$.

14. $y'''_{xxx} = (ax^2 + bx + c)^{-\frac{m+5}{4}} (y'_x)^m.$

This is a special case of equation 3.5.2.17 with $f(\xi) = \xi^m$.

15. $y'''_{xxx} = ax(xy'_x - y)^n.$

This is a special case of equation 3.5.2.13 with $f(\xi) = a\xi^n$.

16. $y'''_{xxx} = ax^{-n-2}(xy'_x - y)^n.$

This is a special case of equation 3.5.2.14 with $f(\xi) = a\xi^n$.

17. $y'''_{xxx} = ax^{-n-4}(xy'_x - 2y)^n.$

This is a special case of equation 3.5.2.16 with $f(\xi) = a\xi^n$.

18. $y'''_{xxx} = ax^{2n-7}y^{-n}(xy'_x - 2y)^3.$

This is a special case of equation 3.5.2.6 with $f(\xi) = a\xi^{-n}$.

19. $y'''_{xxx} = ax^{n-5}y^{-n}(xy'_x - y)^3.$

This is a special case of equation 3.5.2.5 with $f(\xi) = a\xi^{-n}$.

20. $y'''_{xxx} = ax^n y^m (xy'_x - 2y)^l.$

The transformation $t = x^{-1}$, $z = yx^{-2}$ leads to the equation

$$z'''_{ttt} = -a(-1)^l t^{-n-2m-l-4} z^m (z'_t)^l$$

which is discussed in Section 3.2.

21. $xy'''_{xxx} + 3y''_{xx} = ax^n y^n.$

The substitution $w(x) = xy$ leads to the equation $w'''_{xxx} = aw^n$ which is discussed in Section 3.2.

22. $xy'''_{xxx} = -\frac{3}{2}y''_{xx} + ax^{-n-2}y^{2n}(2xy'_x - y).$

This is a special case of equation 3.5.3.13 with $f(\xi) = a\xi^{2n}$.

23. $xy'''_{xxx} = -\frac{3}{2}y''_{xx} + ax^{-n-3}y^{2n}(2xy'_x - y)^3.$

This is a special case of equation 3.5.3.16 with $f(\xi) = a\xi^{2n}$.

24. $xy'''_{xxx} + y''_{xx} = ax^{-n-3}(xy'_x - y)^n.$

This is a special case of equation 3.5.3.15 with $f(\xi) = a\xi^n$.

25. $xy'''_{xxx} + (1-a)y''_{xx} = bx^{2a}(xy'_x - y)^n.$

This is a special case of equation 3.5.3.6 with $f(\xi) = b\xi^n$.

26. $x^2y'''_{xxx} + 6xy''_{xx} + 6y'_x = ax^{2n}y^n.$

The substitution $w(x) = x^2y$ leads to the equation $w'''_{xxx} = aw^n$ which is discussed in Section 3.2.

27. $yy'''_{xxx} + \frac{1}{2}y'_xy''_{xx} = ax + b.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a nonhomogeneous constant-coefficient linear equation of the fourth order of the form 4.1.2.2: $2x''''_{tttt} = ax + b$.

28. $yy'''_{xxx} + \frac{1}{2}y'_xy''_{xx} = Ax^{-5/3}.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to the equation of the form 4.2.1.1: $2x''''_{tttt} = Ax^{-5/3}$.

29. $yy'''_{xxx} + \frac{1}{2}y'_xy''_{xx} = k\sqrt{y}\,y''_{xx} + my'_x + a\sqrt{y} + bx + c.$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a constant coefficient linear equation of the fourth order: $2x''''_{tttt} = \pm 2kx'''_{ttt} + 2mx''_{tt} \pm ax'_t + bx + c$, where "+" corresponds to $x'_t > 0$, and "−" corresponds to $x'_t < 0$.

30. $yy'''_{xxx} + 3y'_xy''_{xx} + ax^nyy'_x = bx^m.$

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = bx^m$.

31. $yy'''_{xxx} + 3y'_xy''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = bx^n.$

This is a special case of equation 3.5.3.20 with $f(x) = bx^n$.

32. $yy'''_{xxx} + (3y'_x + 2ay)y''_{xx} + 2a(y'_x)^2 + a^2yy'_x = bx^n.$

This is a special case of equation 3.5.3.33 with $f(x) = e^{ax}$, $g(x) = bx^ne^{ax}$.

33. $yy'''_{xxx} + (3y'_x + ax^ny)y''_{xx} + ax^n(y'_x)^2 = 0.$

This is a special case of equation 3.5.3.21 with $f(x) = ax^n$.

34. $(y + a)y'''_{xxx} + by'_xy''_{xx} + cy^ny'_x = 0.$

This is a special case of equation 3.5.3.25 with $f(y) = cy^n$.

35. $(y + ax + b)y'''_{xxx} + 3(y'_x + a)y''_{xx} = cx^n.$

This is a special case of equation 3.5.3.27 with $f(x) = cx^n$.

36. $x(yy'''_{xxx} + 3y'_x y''_{xx}) + a\big[yy''_{xx} + (y'_x)^2\big] = bx^n.$

This is a special case of equation 3.5.3.28 with $f(x) = bx^n$.

37. $x^2 yy'''_{xxx} + x(3xy'_x + 2ay)y''_{xx} + 2ax(y'_x)^2 + a(a-1)yy'_x = bx^n.$

This is a special case of equation 3.5.3.33 with $f(x) = x^a$, $g(x) = bx^{n+a-2}$.

38. $y^2 y'''_{xxx} - 3yy'_x y''_{xx} + 2(y'_x)^3 = ax^n y^3.$

This is a special case of equation 3.5.3.29 with $f(x) = ax^n$.

39. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = ax^k y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x) = ax^k$, $n = m + 1$.

40. $2y'_x y'''_{xxx} - (y''_{xx})^2 = \lambda(y'_x)^2 + ay^2 + by + c.$

Differentiating with respect to x and dividing by y'_x, we arrive at a constant coefficient linear equation: $2y''''_{xxxx} = 2\lambda y''_{xx} + 2ay + b$.

41. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^{4-2n}(y'_x)^n.$

This is a special case of equation 3.5.4.6 with $f(\xi) = a\xi^n$.

42. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^{2n-8}y^{4-2n}(y'_x)^n.$

This is a special case of equation 3.5.4.8 with $f(\xi) = a\xi^n$.

43. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^{n-4}y^{4-2n}(y'_x)^n.$

This is a special case of equation 3.5.4.7 with $f(\xi) = a\xi^n$.

44. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = ax^n$, $g(y) = by^m$.

45. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}.$

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = by^m$.

46. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = bx^m$.

47. $2y'_x y'''_{xxx} - (y''_{xx})^2 = \lambda x^n(y'_x)^2 + ay^2 + 2by + c.$

This is a special case of equation 3.5.4.1 with $f(x) = -\lambda x^n$.

48. $xy'_x y'''_{xxx} - 3x(y''_{xx})^2 + 3y'_x y''_{xx} = axy^n (y'_x)^4 + by^m (y'_x)^5$.

This is a special case of equation 3.5.4.10 with $f(y) = ay^n$, $g(y) = by^m$.

49. $y'''_{xxx} = ax^{-2n-5}(xy'_x - y)^n (y''_{xx})^3$.

This is a special case of equation 3.5.4.13 with $f(\xi) = a\xi^n$.

50. $y'''_{xxx} = ax^{-4n-5}(xy'_x - y)^n (y''_{xx})^3$.

This is a special case of equation 3.5.4.12 with $f(\xi) = a\xi^n$.

51. $y'''_{xxx} = \left[ax^{-5} + bx^3(xy'_x - y)^n\right](y''_{xx})^3$.

This is a special case of equation 3.5.4.11 with $f(\xi) = b\xi^n$.

52. $y'''_{xxx} = \left[ax(y'_x)^n + by(y'_x)^m + c(y'_x)^k\right](y''_{xx})^3 + s(y'_x)^l (y''_{xx})^2$.

This is a special case of equation 3.5.4.14 with $f(\xi) = a\xi^n$, $g(\xi) = b\xi^m$, $h(\xi) = c\xi^k$, $\varphi(\xi) = s\xi^l$.

53. $xy'''_{xxx} + y''_{xx} = ax^n (xy'_x - y)^m (y''_{xx})^n$.

This is a special case of equation 3.5.5.5 with $f(\xi) = a\xi^m$, $g(\xi) = \xi^n$.

54. $y'''_{xxx} = Ax^n (y'_x)^m (xy'_x - y)^l (y''_{xx})^k$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to the equation $w'''_{ttt} = -At^m w^l (w'_t)^n (w''_{tt})^{3-k}$ which is discussed in Section 3.2.

55. $y'''_{xxx} = ax(xy'_x - y)^n (y''_{xx})^2 + bx(xy'_x - y)^m (y''_{xx})^k$.

This is a special case of equation 3.5.4.16 with $f(\xi) = a\xi^n$, $g(\xi) = b\xi^m$.

56. $yy'''_{xxx} = y'_x y''_{xx} + ay^{-n-m}(y'_x)^n (y''_{xx})^m \left[yy''_{xx} - (y'_x)^2\right]$.

This is a special case of equation 3.5.5.7 with $f(\xi) = a\xi^n$, $g(\xi) = \xi^m$.

57. $(y'''_{xxx})^2 = a(x^2 y''_{xx} - 2xy'_x + 2y) + by''_{xx} + c$.

Differentiating with respect to x, we obtain

$$y'''_{xxx}(2y''''_{xxxx} - ax^2 - b) = 0$$

Equating the second factor to zero and integrating, we find the solution

$$y = \frac{ax^6}{720} + \frac{bx^4}{48} + C_3 x^3 + C_2 x^2 + C_1 x + C_0,$$

The integration constants C_i and parameters a, b, and c are related by

$$36C_3^2 = 2aC_0 + 2bC_2 + c.$$

This constraint is obtained by substituting the above solution into the original equation. In addition, to the first factor corresponds the solution $y = \widetilde{C}_2 x^2 + \widetilde{C}_1 x + \widetilde{C}_0$, where constants $\widetilde{C}_i$ are related by $2a\widetilde{C}_0 + 2b\widetilde{C}_2 + c = 0$.

3.4.2. Equations Containing Exponential Functions

1. $y'''_{xxx} = a(y + be^x + c)^n - be^x.$

The substitution $w = y + be^x + c$ leads to the equation $w'''_{xxx} = aw^n$ whose solvable cases are outlined in Section 3.2.

2. $y'''_{xxx} = ae^{\lambda y} y'_x + be^{\mu y} (y'_x)^3.$

This is a special case of equation 3.5.2.4 with $f(y) = ae^{\lambda y}$, $g(y) = be^{\mu y}$.

3. $y'''_{xxx} = ae^{\lambda y} y'_x + by^n (y'_x)^3.$

This is a special case of equation 3.5.2.4 with $f(y) = ae^{\lambda y}$, $g(y) = by^n$.

4. $y'''_{xxx} = ay^n y'_x + be^{\lambda y} (y'_x)^3.$

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = be^{\lambda y}$.

5. $y'''_{xxx} = be^{\lambda y} (y'_x)^3 + a(y'_x)^{-5}.$

This is a special case of equation 3.5.2.9 with $f(y) = be^{\lambda y}$.

6. $y'''_{xxx} = 2\lambda^2 (y'_x)^3 + ae^{\lambda m y} (y'_x)^{m-5}.$

This is a special case of equation 3.5.2.23 with $f(\xi) = a\xi^{m-6}$.

7. $y'''_{xxx} = -3y''_{xx} + ae^{mx} y^m y'_x + ae^{mx} y^{m+1} + 2y.$

This is a special case of equation 3.5.3.3 with $f(\xi) = a\xi^m$.

8. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2 (y'_x)^3 = ae^{\beta x - \lambda y}.$

This is a special case of equation 3.5.3.2 with $f(x) = ae^{\beta x}$.

9. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2 (y'_x)^3 = ax^n e^{-\lambda y}.$

This is a special case of equation 3.5.3.2 with $f(x) = ax^n$.

10. $xy'''_{xxx} + (1 - ax) y''_{xx} = be^{2ax} (xy'_x - y)^n.$

This is a special case of equation 3.5.3.7 with $f(\xi) = b\xi^n$.

11. $yy'''_{xxx} + 3y'_x y''_{xx} = ae^{\lambda x}.$

Solution: $y^2 = C_2 x^2 + C_1 x + C_0 + 2a\lambda^{-3} e^{\lambda x}.$

12. $yy'''_{xxx} + 3y'_x y''_{xx} + ae^{\lambda x} yy'_x = be^{\mu x}.$

This is a special case of equation 3.5.3.19 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$.

13. $yy'''_{xxx} + 3y'_x y''_{xx} + ae^{\lambda x} yy'_x = bx^n.$

This is a special case of equation 3.5.3.19 with $f(x) = ae^{\lambda x}$, $g(x) = bx^n$.

14. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = be^{\lambda x}$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = be^{\lambda x}$.

15. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = be^{\lambda x}$.

This is a special case of equation 3.5.3.20 with $f(x) = be^{\lambda x}$.

16. $yy'''_{xxx} + (3y'_x + 2ay)y''_{xx} + 2a(y'_x)^2 + a^2 yy'_x = be^{\lambda x}$.

This is a special case of equation 3.5.3.33 with $f(x) = e^{ax}$, $g(x) = be^{(\lambda+a)x}$.

17. $yy'''_{xxx} + (3y'_x + ae^{\lambda x}y)y''_{xx} + ae^{\lambda x}(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = ae^{\lambda x}$.

18. $(y + a)y'''_{xxx} + by'_x y''_{xx} + ce^{\lambda y}y'_x = 0$.

This is a special case of equation 3.5.3.25 with $f(y) = ce^{\lambda y}$.

19. $(y + ax + b)y'''_{xxx} + 3(y'_x + a)y''_{xx} = ke^{\lambda x}$.

Solution: $(y + ax + b)^2 = C_2x^2 + C_1x + C_0 + 2k\lambda^{-3}e^{\lambda x}$.

20. $y^2y'''_{xxx} - 3yy'_x y''_{xx} + 2(y'_x)^3 = ae^{\lambda x}y^3$.

Solution: $\ln|y| = C_2x^2 + C_1x + C_0 + a\lambda^{-3}e^{\lambda x}$.

21. $y^2y'''_{xxx} + 3myy'_x y''_{xx} + m(m - 1)(y'_x)^3 = ae^{\lambda x}y^{2-m}$.

This is a special case of equation 3.5.3.30 with $f(x) = ae^{\lambda x}$, $n = m + 1$.

22. $x^2yy'''_{xxx} + x(3xy'_x + 2ay)y''_{xx} + 2ax(y'_x)^2 + a(a - 1)yy'_x = be^{\lambda x}$.

This is a special case of equation 3.5.3.33 with $f(x) = x^a$, $g(x) = bx^{a-2}e^{\lambda x}$.

23. $2y'_x y'''_{xxx} - (y''_{xx})^2 = ke^{\lambda x}(y'_x)^2 + ay^2 + 2by + c$.

This is a special case of equation 3.5.4.1 with $f(x) = -ke^{\lambda x}$.

24. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + be^{\mu y}(y'_x)^4$.

This is a special case of equation 3.5.4.3 with $f(x) = ae^{\lambda x}$, $g(y) = be^{\mu y}$.

25. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + by^m(y'_x)^4$.

This is a special case of equation 3.5.4.3 with $f(x) = ae^{\lambda x}$, $g(y) = by^m$.

26. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + be^{\mu y}(y'_x)^4$.

This is a special case of equation 3.5.4.3 with $f(x) = ax^n$, $g(y) = be^{\mu y}$.

27. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + be^{\mu x}y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$.

28. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda x}(y'_x)^2 + bx^n y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ae^{\lambda x}$, $g(x) = bx^n$.

29. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + be^{\lambda x}y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = be^{\lambda x}$.

30. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda y}(y'_x)^4 + bx^{-1}e^{\mu y}(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ae^{\lambda y}$, $g(y) = be^{\mu y}$.

31. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ae^{\lambda y}(y'_x)^4 + bx^{-1}y^n(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ae^{\lambda y}$, $g(y) = by^n$.

32. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}e^{\lambda y}(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = be^{\lambda y}$.

3.4.3. Equations Containing Hyperbolic Functions

1. $y'''_{xxx} = a\cosh^n(\lambda y)y'_x + b\cosh(\mu y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\cosh^n(\lambda y)$, $g(y) = b\cosh(\mu y)$.

2. $y'''_{xxx} = ay^n y'_x + b\cosh(\mu y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = b\cosh(\mu y)$.

3. $y'''_{xxx} = a\sinh^n(\lambda y)y'_x + b\sinh(\mu y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\sinh^n(\lambda y)$, $g(y) = b\sinh(\mu y)$.

4. $y'''_{xxx} = b\cosh(\lambda y)(y'_x)^3 + a(y'_x)^{-5}$.

This is a special case of equation 3.5.2.9 with $f(y) = b\cosh(\lambda y)$.

5. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\cosh\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 3.5.2.24 with $f(\xi) = a\xi^{2m}$.

6. $y'''_{xxx} = \frac{1}{2}\lambda^2(y'_x)^3 + a(\sinh\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 3.5.2.25 with $f(\xi) = a\xi^{2m}$.

7. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cosh(\lambda x)$.

Solution: $y^2 = C_2x^2 + C_1x + C_0 + 2a\lambda^{-3}\sinh(\lambda x)$.

8. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sinh(\lambda x)$.

Solution: $y^2 = C_2x^2 + C_1x + C_0 + 2a\lambda^{-3}\cosh(\lambda x)$.

9. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cosh^n(\lambda x)$.

This is a special case of equation 3.5.3.17 with $f(x) = a\cosh^n(\lambda x)$.

10. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sinh^n(\lambda x)$.

This is a special case of equation 3.5.3.17 with $f(x) = a\sinh^n(\lambda x)$.

11. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tanh^n(\lambda x)$.

This is a special case of equation 3.5.3.17 with $f(x) = a\tanh^n(\lambda x)$.

12. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\cosh^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\cosh^m(\lambda x)$.

13. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\sinh^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\sinh^m(\lambda x)$.

14. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\tanh^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\tanh^m(\lambda x)$.

15. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\cosh^n(\lambda x)$.

This is a special case of equation 3.5.3.20 with $f(x) = b\cosh^n(\lambda x)$.

16. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\sinh^n(\lambda x)$.

This is a special case of equation 3.5.3.20 with $f(x) = b\sinh^n(\lambda x)$.

17. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\tanh^n(\lambda x)$.

This is a special case of equation 3.5.3.20 with $f(x) = b\tanh^n(\lambda x)$.

18. $yy'''_{xxx} + (3y'_x + ay\cosh^n x)y''_{xx} + a\cosh^n x(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = a\cosh^n x$.

19. $yy'''_{xxx} + (3y'_x + ay\sinh^n x)y''_{xx} + a\sinh^n x(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = a\sinh^n x$.

20. $yy'''_{xxx} + (3y'_x + ay\tanh^n x)y''_{xx} + a\tanh^n x(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = a\tanh^n x$.

21. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\cosh^n(\lambda y)y'_x = 0$.

This is a special case of equation 3.5.3.25 with $f(y) = c\cosh^n(\lambda y)$.

22. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\sinh^n(\lambda y)y'_x = 0$.

This is a special case of equation 3.5.3.25 with $f(y) = c\sinh^n(\lambda y)$.

23. $(y+a)y'''_{xxx} + by'_x y''_{xx} + c\tanh^n(\lambda y)y'_x = 0.$

This is a special case of equation 3.5.3.25 with $f(y) = c\tanh^n(\lambda y)$.

24. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\cosh^k(\lambda x)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x) = a\cosh^k(\lambda x)$, $n = m+1$.

25. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\sinh^k(\lambda x)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x) = a\sinh^k(\lambda x)$, $n = m+1$.

26. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\tanh^k(\lambda x)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x) = a\tanh^k(\lambda x)$, $n = m+1$.

27. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = a\cosh^n(\lambda x)$, $g(y) = by^m$.

28. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cosh^m(\lambda y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = ax^n$, $g(y) = b\cosh^m(\lambda y)$.

29. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + b\tanh^m(\mu y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = a\tanh^n(\lambda x)$, $g(y) = b\tanh^m(\mu y)$.

30. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = a\tanh^n(\lambda x)$, $g(y) = by^m$.

31. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tanh^m(\lambda y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = ax^n$, $g(y) = b\tanh^m(\lambda y)$.

32. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + b\cosh^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\cosh^n(\lambda x)$, $g(x) = b\cosh^m(\mu x)$.

33. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\cosh^n(\lambda x)$, $g(x) = bx^m$.

34. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\cosh^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = b\cosh^m(\lambda x)$.

35. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + b\tanh^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\tanh^n(\lambda x)$, $g(x) = b\tanh^m(\mu x)$.

36. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\tanh^n(\lambda x)$, $g(x) = bx^m$.

37. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tanh^m(\lambda x)y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = b\tanh^m(\lambda x)$.

38. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda y)(y'_x)^4 + bx^{-1}\cosh^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\cosh^n(\lambda y)$, $g(y) = b\cosh^m(\mu y)$.

39. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cosh^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\cosh^n(\lambda y)$, $g(y) = by^m$.

40. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\cosh^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = b\cosh^m(\lambda y)$.

41. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda y)(y'_x)^4 + bx^{-1}\tanh^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\tanh^n(\lambda y)$, $g(y) = b\tanh^m(\mu y)$.

42. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tanh^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\tanh^n(\lambda y)$, $g(y) = by^m$.

43. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\tanh^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = b\tanh^m(\lambda y)$.

3.4.4. Equations Containing Logarithmic Functions

1. $y'''_{xxx} = a\ln^n(\lambda y)y'_x + b\ln^m(\mu y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\ln^n(\lambda y)$, $g(y) = b\ln^m(\mu y)$.

2. $y'''_{xxx} = a\ln^n(\lambda y)y'_x + by^m(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\ln^n(\lambda y)$, $g(y) = by^m$.

3. $y'''_{xxx} = ay^n y'_x + b\ln^m(\lambda y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = b\ln^m(\lambda y)$.

4. $y'''_{xxx} = ay^{-5/2}(2\ln y'_x - \ln y)$.

This is a special case of equation 3.5.2.11 with $f(\xi) = 2a\ln\xi$.

5. $y'''_{xxx} = ay^{-5/4}(4\ln y'_x - \ln y)$.

This is a special case of equation 3.5.2.12 with $f(\xi) = 4a\ln\xi$.

6. $x^3 y'''_{xxx} = a(\ln y - \ln x)(xy'_x - y)$.

This is a special case of equation 3.5.2.2 with $f(\xi) = a\ln\xi$.

7. $x^5 y'''_{xxx} = a(\ln y - 2\ln x)(xy'_x - 2y).$

This is a special case of equation 3.5.2.3 with $f(\xi) = a\ln\xi$.

8. $y'''_{xxx} = -3y''_{xx} + a(x + \ln y)^n (y'_x + y) + 2y.$

This is a special case of equation 3.5.3.3 with $f(\xi) = a\ln^n \xi$.

9. $xy'''_{xxx} = b(xy'_x - y + a\ln x)y''_{xx}.$

This is a special case of equation 3.5.3.8 with $f(\xi) = b\xi$.

10. $yy'''_{xxx} + 3y'_x y''_{xx} = a\ln^n(bx).$

This is a special case of equation 3.5.3.17 with $f(x) = a\ln^n(bx)$.

11. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\ln^m(\lambda x).$

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\ln^m(\lambda x)$.

12. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\ln^n(\lambda x).$

This is a special case of equation 3.5.3.20 with $f(x) = b\ln^n(\lambda x)$.

13. $(y + a)y'''_{xxx} + by'_x y''_{xx} + c\ln^n(\lambda y)y'_x = 0.$

This is a special case of equation 3.5.3.25 with $f(y) = c\ln^n(\lambda y)$.

14. $y^2 y'''_{xxx} + 3myy'_x y''_{xx} + m(m-1)(y'_x)^3 = a\ln^k(bx)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x) = a\ln^k(bx)$, $n = m + 1$.

15. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^4(\ln y'_x - 2\ln y).$

This is a special case of equation 3.5.4.6 with $f(\xi) = a\ln\xi$.

16. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + b\ln^m(\mu y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = a\ln^n(\lambda x)$, $g(y) = b\ln^m(\mu y)$.

17. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = a\ln^n(\lambda x)$, $g(y) = by^m$.

18. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\ln^m(\lambda y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x) = ax^n$, $g(y) = b\ln^m(\lambda y)$.

19. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda x)(y'_x)^2 + b\ln^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\ln^n(\lambda x)$, $g(x) = b\ln^m(\mu x)$.

20. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n x\,(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x) = a\ln^n x$, $g(x) = bx^m$.

21. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n (y'_x)^2 + b\ln^m(\lambda x) y^{-1} (y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = b\ln^m(\lambda x)$.

22. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda y)(y'_x)^4 + bx^{-1}\ln^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\ln^n(\lambda y)$, $g(y) = b\ln^m(\mu y)$.

23. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n (y'_x)^4 + bx^{-1}\ln^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = b\ln^m(\lambda y)$.

24. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\ln^n(\lambda y)(y'_x)^4 + bx^{-1} y^m (y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\ln^n(\lambda y)$, $g(y) = by^m$.

3.4.5. Equations Containing Trigonometric Functions

1. $y'''_{xxx} = ay^n y'_x + b\cos(\lambda y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = b\cos(\lambda y)$.

2. $y'''_{xxx} = ay^n y'_x + b\sin(\lambda y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = b\sin(\lambda y)$.

3. $y'''_{xxx} = ay^n y'_x + b\tan(\lambda y)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = ay^n$, $g(y) = b\tan(\lambda y)$.

4. $y'''_{xxx} = a\cos^n(\lambda y) y'_x + by^m (y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\cos^n(\lambda y)$, $g(y) = by^m$.

5. $y'''_{xxx} = a\sin^n(\lambda y) y'_x + by^m (y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\sin^n(\lambda y)$, $g(y) = by^m$.

6. $y'''_{xxx} = a\tan^n(\lambda y) y'_x + by^m (y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\tan^n(\lambda y)$, $g(y) = by^m$.

7. $y'''_{xxx} = a\cos^n(\lambda y + \varepsilon) y'_x + b\cos(\mu y + \delta)(y'_x)^3$.

This is a special case of equation 3.5.2.4 with $f(y) = a\cos^n(\lambda y + \varepsilon)$, $g(y) = b\cos(\mu y + \delta)$.

8. $y'''_{xxx} = b\cos^n(\lambda y)(y'_x)^3 + a(y'_x)^{-5}$.

This is a special case of equation 3.5.2.9 with $f(y) = b\cos^n(\lambda y)$.

9. $y'''_{xxx} = b\tan^n(\lambda y)(y'_x)^3 + a(y'_x)^{-5}$.

This is a special case of equation 3.5.2.9 with $f(y) = b\tan^n(\lambda y)$.

10. $y'''_{xxx} = -\frac{1}{2}\lambda^2(y'_x)^3 + a(\cos\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 3.5.2.28 with $f(\xi) = a\xi^{2m}$.

11. $y'''_{xxx} = -\frac{1}{2}\lambda^2(y'_x)^3 + a(\sin\lambda y)^{-m-3}(y'_x)^{2m+1}$.

This is a special case of equation 3.5.2.29 with $f(\xi) = a\xi^{2m}$.

12. $yy'''_{xxx} + 3y'_x y''_{xx} = a\cos(\lambda x)$.

Solution: $y^2 = C_2x^2 + C_1x + C_0 - 2a\lambda^{-3}\sin(\lambda x)$.

13. $yy'''_{xxx} + 3y'_x y''_{xx} = a\sin^n(\lambda x)$.

This is a special case of equation 3.5.3.17 with $f(x) = a\sin^n(\lambda x)$.

14. $yy'''_{xxx} + 3y'_x y''_{xx} = a\tan^n(\lambda x)$.

This is a special case of equation 3.5.3.17 with $f(x) = a\tan^n(\lambda x)$.

15. $yy'''_{xxx} + 3y'_x y''_{xx} + a\sin(\lambda x)yy'_x = b\sin^n(\mu x)$.

This is a special case of equation 3.5.3.19 with $f(x) = a\sin(\lambda x)$, $g(x) = b\sin^n(\mu x)$.

16. $yy'''_{xxx} + 3y'_x y''_{xx} + a\sin(\lambda x)yy'_x = bx^n$.

This is a special case of equation 3.5.3.19 with $f(x) = a\sin(\lambda x)$, $g(x) = bx^n$.

17. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\cos^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\cos^m(\lambda x)$.

18. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\sin^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\sin^m(\lambda x)$.

19. $yy'''_{xxx} + 3y'_x y''_{xx} + ax^n yy'_x = b\tan^m(\lambda x)$.

This is a special case of equation 3.5.3.19 with $f(x) = ax^n$, $g(x) = b\tan^m(\lambda x)$.

20. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\cos^n(\lambda x)$.

This is a special case of equation 3.5.3.20 with $f(x) = b\cos^n(\lambda x)$.

21. $yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = b\tan^n(\lambda x)$.

This is a special case of equation 3.5.3.20 with $f(x) = b\tan^n(\lambda x)$.

22. $yy'''_{xxx} + (3y'_x + ay\sin^n x)y''_{xx} + a\sin^n x(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = a\sin^n x$.

23. $yy'''_{xxx} + (3y'_x + ay\tan^n x)y''_{xx} + a\tan^n x(y'_x)^2 = 0$.

This is a special case of equation 3.5.3.21 with $f(x) = a\tan^n x$.

24. $(y+a)y'''_{xxx}+by'_x y''_{xx}+c\cos^n(\lambda y)y'_x=0.$

This is a special case of equation 3.5.3.25 with $f(y)=c\cos^n(\lambda y)$.

25. $(y+a)y'''_{xxx}+by'_x y''_{xx}+c\tan^n(\lambda y)y'_x=0.$

This is a special case of equation 3.5.3.25 with $f(y)=c\tan^n(\lambda y)$.

25. $y^2y'''_{xxx}+3myy'_x y''_{xx}+m(m-1)(y'_x)^3=a\cos^k(\lambda x)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x)=a\cos^k(\lambda x)$, $n=m+1$.

26. $y^2y'''_{xxx}+3myy'_x y''_{xx}+m(m-1)(y'_x)^3=a\tan^k(\lambda x)y^{2-m}.$

This is a special case of equation 3.5.3.30 with $f(x)=a\tan^k(\lambda x)$, $n=m+1$.

27. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\cos(\lambda x+\varepsilon)(y'_x)^2+b\cos(\mu y+\delta)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=a\cos(\lambda x+\varepsilon)$, $g(y)=b\cos(\mu y+\delta)$.

28. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\cos(\lambda x)(y'_x)^2+by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=a\cos(\lambda x)$, $g(y)=by^m$.

29. $2y'_x y'''_{xxx}-3(y''_{xx})^2=ax^n(y'_x)^2+b\cos(\lambda y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=ax^n$, $g(y)=b\cos(\lambda y)$.

30. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\tan^n(\lambda x)(y'_x)^2+b\tan^m(\mu y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=a\tan^n(\lambda x)$, $g(y)=b\tan^m(\mu y)$.

31. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\tan^n(\lambda x)(y'_x)^2+by^m(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=a\tan^n(\lambda x)$, $g(y)=by^m$.

32. $2y'_x y'''_{xxx}-3(y''_{xx})^2=ax^n(y'_x)^2+b\tan^m(\lambda y)(y'_x)^4.$

This is a special case of equation 3.5.4.3 with $f(x)=ax^n$, $g(y)=b\tan^m(\lambda y)$.

33. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\sin^n(\lambda x)(y'_x)^2+b\sin^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x)=a\sin^n(\lambda x)$, $g(x)=b\sin^m(\mu x)$.

34. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\cos^n(\lambda x)(y'_x)^2+bx^my^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x)=a\cos^n(\lambda x)$, $g(x)=bx^m$.

35. $2y'_x y'''_{xxx}-3(y''_{xx})^2=ax^n(y'_x)^2+b\cos^m(\lambda x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x)=ax^n$, $g(x)=b\cos^m(\lambda x)$.

36. $2y'_x y'''_{xxx}-3(y''_{xx})^2=a\tan^n(\lambda x)(y'_x)^2+b\tan^m(\mu x)y^{-1}(y'_x)^{5/2}.$

This is a special case of equation 3.5.4.4 with $f(x)=a\tan^n(\lambda x)$, $g(x)=b\tan^m(\mu x)$.

37. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda x)(y'_x)^2 + bx^m y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = a\tan^n(\lambda x)$, $g(x) = bx^m$.

38. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ax^n(y'_x)^2 + b\tan^m(\lambda x)y^{-1}(y'_x)^{5/2}$.

This is a special case of equation 3.5.4.4 with $f(x) = ax^n$, $g(x) = b\tan^m(\lambda x)$.

39. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda y)(y'_x)^4 + bx^{-1}\cos^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\cos^n(\lambda y)$, $g(y) = b\cos^m(\mu y)$.

40. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\cos^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\cos^n(\lambda y)$, $g(y) = by^m$.

41. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\cos^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = b\cos^m(\lambda y)$.

42. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda y)(y'_x)^4 + bx^{-1}\tan^m(\mu y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\tan^n(\lambda y)$, $g(y) = b\tan^m(\mu y)$.

43. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = a\tan^n(\lambda y)(y'_x)^4 + bx^{-1}y^m(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = a\tan^n(\lambda y)$, $g(y) = by^m$.

44. $2y'_x y'''_{xxx} - 3(y''_{xx})^2 = ay^n(y'_x)^4 + bx^{-1}\tan^m(\lambda y)(y'_x)^{7/2}$.

This is a special case of equation 3.5.4.5 with $f(y) = ay^n$, $g(y) = b\tan^m(\lambda y)$.

3.5. Nonlinear Equations Containing Arbitrary Functions

3.5.1. Equations of the Form $F(x, y)y'''_{xxx} + G(x, y) = 0$

1. $y'''_{xxx} = f(y)$.

The substitution $w(y) = (y'_x)^2$ leads to a second order equation: $w''_{yy} = \pm 2f(y)w^{-1/2}$. In particular, with $f(y) = ay^n$ the obtained equation is the Emden—Fowler equation which is discussed in Section 2.3.

2. $y'''_{xxx} = f(x)y^{-1}$.

Having integrated the equation, we have

$$yy''_{xx} - \frac{1}{2}(y'_x)^2 = \int f(x)\,dx + C.$$

The substitution $y = w^2$ reduces the latter equation to the form

$$w''_{xx} = \frac{1}{2}\left[\int f(x)\,dx + C\right]w^{-3}.$$

3. $\boldsymbol{y'''_{xxx} = x^{-3} f(y)}$.

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 3.5.5.9: $y'''_{ttt} - 3y''_{tt} + 2y'_t = f(y)$.

4. $\boldsymbol{y'''_{xxx} = x^{-2} f(yx^{-1})}$.

The transformation $t = \ln|x|$, $w = yx^{-1}$ leads to an autonomous equation of the form 3.5.5.9: $w'''_{ttt} - w'_t = f(w)$.

5. $\boldsymbol{y'''_{xxx} = x^{-4} f(yx^{-2})}$.

The transformation $t = x^{-1}$, $w = yx^{-2}$ leads to an autonomous equation of the form 3.5.1.1: $w'''_{ttt} = -f(w)$.

6. $\boldsymbol{y'''_{xxx} = f(y + ax^3 + bx^2 + cx + k)}$.

The substitution $w = y + ax^3 + bx^2 + cx + k$ leads to an autonomous equation of the form 3.5.1.1: $w'''_{xxx} = f(w) + 6a$.

7. $\boldsymbol{y'''_{xxx} = f(y + ae^{\lambda x}) - a\lambda^3 e^{\lambda x}}$.

The substitution $w = y + ae^{\lambda x}$ leads to an autonomous equation of the form 3.5.1.1: $w'''_{xxx} = f(w)$.

8. $\boldsymbol{(y + ax^2 + bx + c) y'''_{xxx} = f(x)}$.

The substitution $w = y + ax^2 + bx + c$ leads to an autonomous equation of the form 3.5.1.2: $ww'''_{xxx} = f(x)$.

9. $\boldsymbol{x(x-a)^3 y'''_{xxx} = f(yx^{-2}), \qquad a \neq 0}$.

The transformation $\xi = \ln\left|\dfrac{x-a}{x}\right|$, $w = \dfrac{y}{x^2}$ leads to an autonomous equation of the form 3.5.5.9: $w'''_{\xi\xi\xi} - 3w''_{\xi\xi} + 2w'_\xi = a^{-3} f(w)$.

10. $\boldsymbol{(ax^2 + bx + c)^2 y'''_{xxx} = f\left(\dfrac{y}{ax^2 + bx + c}\right)}$.

The transformation

$$\xi = \int \frac{dx}{ax^2 + bx + c}, \qquad w = \frac{y}{ax^2 + bx + c}$$

leads to an autonomous equation of the form 3.5.5.9: $w'''_{\xi\xi\xi} + (4ac - b^2) w'_\xi = f(w)$.

11. $\boldsymbol{(ay + be^x) y'''_{xxx} + be^x y = f(x)}$.

Integrating, we obtain

$$(ay + be^x) y''_{xx} - \tfrac{1}{2} a (y'_x)^2 - be^x y'_x + be^x y = \int f(x)\, dx + C.$$

12. $\boldsymbol{y'''_{xxx} = F(x,\ y)}$.

The transformation $x = \dfrac{1}{t}$, $y = \dfrac{w}{t^2}$ leads to an equation of the analogous form: $w'''_{ttt} = -t^{-4} F\left(\dfrac{1}{t},\ \dfrac{w}{t^2}\right)$.

3.5.2. Equations of the Form $F(x,y,y'_x)y'''_{xxx}+G(x,y,y'_x)=0$

1. $y'''_{xxx}=f(y)y'_x.$

Solution:

$$C_3\pm x=\int\left[C_2y+C_1+2\int F(y)\,dy\right]^{-1/2}dy,\qquad\text{where}\quad F(y)=\int f(y)\,dy.$$

2. $y'''_{xxx}=x^{-3}f\left(\frac{y}{x}\right)(xy'_x-y).$

The transformation $z=\dfrac{y}{x}$, $w=\left(y'_x-\dfrac{y}{x}\right)^2$ leads to a second order linear equation: $w''_{zz}=2f(z)+2$.

Integrating the latter twice, we obtain a first order autonomous equation:

$$y'_x=z\pm\left[z^2+C_2z+C_1+2\int_{z_0}^{z}(z-t)f(t)\,dt\right]^{1/2},$$

where $z=y/x$, z_0 is an arbitrary number.

3. $y'''_{xxx}=x^{-5}f\left(\frac{y}{x^2}\right)(xy'_x-2y).$

The transformation $t=\dfrac{1}{x}$, $z=\dfrac{y}{x^2}$ leads to the equation $z'''_{ttt}=f(z)z'_t$. The substitution $w(z)=(z'_t)^2$ next yields a second order linear equation: $w''_{zz}=2f(z)$, whose solution has the form

$$w=C_2z+C_1+2\int_{z_0}^{z}(z-\xi)f(\xi)\,d\xi,\qquad z_0\text{ is arbitrary.}$$

4. $y'''_{xxx}=f(y)y'_x+g(y)(y'_x)^3.$

The substitution $z(y)=(y'_x)^2$ leads to a second order linear equation: $z''_{yy}=2g(y)z+2f(y)$.

5. $y'''_{xxx}=x^{-5}f\left(\frac{y}{x}\right)(xy'_x-y)^3.$

The transformation $z=\dfrac{y}{x}$, $w=\left(y'_x-\dfrac{y}{x}\right)^2$ leads to a second order linear equation: $w''_{zz}=2f(z)w+2$.

6. $y'''_{xxx}=x^{-7}f\left(\frac{y}{x^2}\right)(xy'_x-2y)^3.$

This is a special case of equation 3.5.2.8.

7. $y'''_{xxx}=x^{-3}f\left(\frac{y}{x}\right)(xy'_x-y)+x^{-5}g\left(\frac{y}{x}\right)(xy'_x-y)^3.$

This is a special case of equation 3.5.3.14 with $k=-1$.

The transformation $t=\ln x$, $z=y/x$ followed by the substitution $w(z)=(z'_t)^2$ yields a second order linear equation: $w''_{zz}=2g(z)w+2f(z)+2$.

8. $\boldsymbol{y'''_{xxx} = x^{-5} f\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y) + x^{-7} g\Big(\dfrac{y}{x^2}\Big)(xy'_x - 2y)^3.}$

The transformation $t = \dfrac{1}{x}$, $z = \dfrac{y}{x^2}$ followed by the substitution $w(z) = (z'_t)^2$ leads to a second order linear equation: $w''_{zz} = 2g(z)w + 2f(z)$.

9. $\boldsymbol{y'''_{xxx} = f(y)(y'_x)^3 + a(y'_x)^{-5}.}$

The substitution $z(y) = (y'_x)^2$ leads to Yermakov's equation 2.9.1.12: $z''_{yy} = 2f(y)z + 2az^{-3}$.

10. $\boldsymbol{y'''_{xxx} = f(y'_x).}$

Solution in the parametric form:

$$x = \int_{C_2}^{\tau} \frac{d\tau}{\varphi(\tau)}, \quad y = \int_{C_3}^{\tau} \frac{\tau\, d\tau}{\varphi(\tau)}, \qquad \text{where} \quad \varphi = \pm\Big[C_1 + 2\int f(\tau)\, d\tau\Big]^{1/2}.$$

11. $\boldsymbol{y'''_{xxx} = y^{-5/2} f\Big(\dfrac{y'_x}{\sqrt{y}}\Big).}$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.9.1.5:

$$w''_{yy} = y^{-3} F\Big(\frac{w}{y}\Big), \qquad \text{where} \quad F(\xi) = \pm 2\xi^{-1/2} f(\pm\xi^{1/2}).$$

12. $\boldsymbol{y'''_{xxx} = y^{-5/4} f\Big(\dfrac{y'_x}{y^{1/4}}\Big).}$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.9.1.6:

$$w''_{yy} = y^{-3/2} F\Big(\frac{w}{\sqrt{y}}\Big), \qquad \text{where} \quad F(\xi) = \pm\xi^{-1/2} f(\pm\xi^{1/2}).$$

13. $\boldsymbol{y'''_{xxx} = x f(xy'_x - y).}$

The substitution $z = xy'_x - y$ leads to a second order equation of the form 2.9.2.4 with $n = 1$: $xz''_{xx} = z'_x + x^3 f(z)$.

14. $\boldsymbol{y'''_{xxx} = x^{-2} f\Big(y'_x - \dfrac{y}{x}\Big).}$

The substitution $z = xy'_x - y$ leads to a second order equation: $xz''_{xx} = z'_x + F(z/x)$, which is the special case of the equation 2.9.4.12 with $n = -1$, $m = 1$, $k = -1$, $F(\xi) = \xi f(\xi)$.

15. $\boldsymbol{y'''_{xxx} = x^{-3} f(xy'_x - y).}$

The transformation $t = \ln|x|$, $z = xy'_x - y$ leads to a second order autonomous equation: $z''_{tt} - 2z'_t = f(z)$, which is reduced, with the aid of the substitution $w(z) = \frac{1}{2} z'_t$, the Abel equation $ww'_z - w = \frac{1}{4} f(z)$ (for some functions f, the solutions of the latter equation are given in Subsection 1.3.1).

16. $y'''_{xxx} = x^{-4} f\Big(y'_x - 2\dfrac{y}{x}\Big).$

The transformation $t = \dfrac{1}{x}$, $z = \dfrac{y}{x^2}$ yields $z'''_{ttt} = -f(-z'_t)$. The substitution $w = -z'_t$ leads to a second order equation of the form 2.9.1.1: $w''_{tt} = f(w)$.

17. $y'''_{xxx} = (ay^2 + by + c)^{-5/4} f\bigg(\dfrac{y'_x}{(ay^2 + by + c)^{1/4}}\bigg).$

The substitution $w(y) = (y'_x)^2$ leads to an equation of the form 2.9.1.9:

$$w''_{yy} = w^{-3} F\bigg(\frac{w}{\sqrt{ay^2 + by + c}}\bigg), \qquad \text{where} \quad F(\xi) = \pm 2\xi^{5/2} f(\pm\xi^{1/2}).$$

18. $y'''_{xxx} = x(ay^2 + bx^2 y + cx^4)^{-5/4} f\bigg(\dfrac{xy'_x - 2y}{(ay^2 + bx^2 y + cx^4)^{1/4}}\bigg).$

The transformation $t = \dfrac{1}{x}$, $z = \dfrac{y}{x^2}$ leads to an equation of the form 3.5.2.17:

$$z'''_{ttt} = -(az^2 + bz + c)^{-5/4} f\bigg(\frac{-z'_t}{(az^2 + bz + c)^{1/4}}\bigg).$$

19. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + x^{-3/4} y^{-5/4} f\bigg(\dfrac{xy'_x - y}{x^{3/4} y^{1/4}}\bigg).$

The transformation $t = \ln x$, $z = y/x$ followed by the substitution $w(z) = (z'_t)^2$ leads to a second order equation of the form 2.9.1.6:

$$w''_{zz} = z^{-3/2} F(wz^{-1/2}), \qquad \text{where} \quad F(\xi) = \pm\xi^{-1/2} f\big(\pm\sqrt{\xi}\,\big).$$

20. $y'''_{xxx} = -x^{-2} y'_x + x^{-3} y + x^{1/2} y^{-5/2} f\bigg(\dfrac{xy'_x - y}{\sqrt{xy}}\bigg).$

The transformation $t = \ln x$, $z = y/x$ followed by the substitution $w(z) = (z'_t)^2$ leads to a second order equation of the form 2.9.1.5:

$$w''_{zz} = z^{-3} F(w/z), \qquad \text{where} \quad F(\xi) = \pm 2\xi^{-1/2} f\big(\pm\sqrt{\xi}\,\big).$$

21. $y'''_{xxx} = f(y,\ y'_x).$

This is a special case of the equation 3.5.5.9. The transformation

$$x = \int [\varphi(\tau)]^{-3/2} d\tau, \quad y = [\varphi(\tau)]^{-1} \tag{1}$$

leads to an analogous equation with respect to function $\varphi = \varphi(\tau)$:

$$\varphi'''_{\tau\tau\tau} = -\varphi^{-5/2} f(\varphi^{-1},\ -\varphi^{-1/2}\varphi'_\tau).$$

Note two important cases of transforming equations of a special form:

$$y'''_{xxx} = f(y) \quad \xrightarrow{\text{transformation (1)}} \quad \varphi'''_{\tau\tau\tau} = -\varphi^{-5/2} f(\varphi^{-1}),$$

$$y'''_{xxx} = Ay^n \quad \xrightarrow{\text{transformation (1)}} \quad \varphi'''_{\tau\tau\tau} = -A\varphi^{-n-5/2}.$$

22. $y'''_{xxx} = \dfrac{1}{x^4} f\Big(\dfrac{y}{x^2},\ y'_x - 2\dfrac{y}{x}\Big).$

The transformation $t = \dfrac{1}{x}$, $z = \dfrac{y}{x^2}$ leads to an equation of the form 3.5.5.9: $z'''_{ttt} = -f(z,\ -z'_t)$, which admits, with the aid of the substitution $w(z) = (z'_t)^2$, lowering of its order: $w''_{zz} = \mp 2w^{-1/2} f(z,\ \mp w^{1/2})$.

23. $y'''_{xxx} = 2\lambda^2 (y'_x)^3 + e^{6\lambda y} f(e^{\lambda y} y'_x) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = e^{-2\lambda y}$.

24. $y'''_{xxx} = \frac{1}{2}\lambda^2 (y'_x)^3 + (\cosh \lambda y)^{-3} f\bigg(\dfrac{y'_x}{\sqrt{\cosh \lambda y}}\bigg) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \cosh \lambda y$.

25. $y'''_{xxx} = \frac{1}{2}\lambda^2 (y'_x)^3 + (\sinh \lambda y)^{-3} f\bigg(\dfrac{y'_x}{\sqrt{\sinh \lambda y}}\bigg) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \sinh \lambda y$.

26. $y'''_{xxx} = (\sinh y)^{-2} (y'_x)^3 + (\tanh y)^3 f\big(y'_x \sqrt{\tanh y}\,\big) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \coth y$.

27. $y'''_{xxx} = -(\cosh y)^{-2} (y'_x)^3 + (\coth y)^3 f\big(y'_x \sqrt{\coth y}\,\big) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \tanh y$.

28. $y'''_{xxx} = -\frac{1}{2}\lambda^2 (y'_x)^3 + (\cos \lambda y)^{-3} f\bigg(\dfrac{y'_x}{\sqrt{\cos \lambda y}}\bigg) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \cos \lambda y$.

29. $y'''_{xxx} = -\frac{1}{2}\lambda^2 (y'_x)^3 + (\sin \lambda y)^{-3} f\bigg(\dfrac{y'_x}{\sqrt{\sin \lambda y}}\bigg) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \sin \lambda y$.

30. $y'''_{xxx} = (\sin y)^{-2} (y'_x)^3 + (\tan y)^3 f\big(y'_x \sqrt{\tan y}\,\big) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \cot y$.

31. $y'''_{xxx} = (\cos y)^{-2} (y'_x)^3 + (\cot y)^3 f\big(y'_x \sqrt{\cot y}\,\big) y'_x.$

This is a special case of equation 3.5.2.32 with $\psi(y) = \tan y$.

32. $y'''_{xxx} = \dfrac{1}{2} \dfrac{\psi''_{yy}}{\psi} (y'_x)^3 + \psi^{-3} f\bigg(\dfrac{y'_x}{\sqrt{\psi}}\bigg) y'_x, \qquad \psi = \psi(x).$

The substitution $z = (y'_x)^2$ leads to a second order equation of the form 2.9.1.14:

$$z''_{yy} = \frac{\psi''_{yy}}{\psi} z + 2\psi^{-3} f\bigg(\pm\sqrt{\frac{z}{\psi}}\bigg).$$

33. $[ay + f(x)]y'''_{xxx} + g(y)y'_x + f'''_{xxx}(x)y + h(x) = 0.$

The equation admits the first integral:

$$[ay + f(x)]y''_{xx} - \tfrac{1}{2}a(y'_x)^2 - f'_x(x)y'_x + f''_{xx}(x)y + \int g(y)\,dy + \int h(x)\,dx = C.$$

34. $y'''_{xxx} = F(x, y, y'_x).$

Let $F \neq \varphi(x)y'_x + \psi(x)y + \chi(x)$, i.e., the equation is a nonlinear one. Then, its order can be lowered by one if the right-hand side of the equation has the following form:

$$F(x,y,y'_x) = f^{-2}E\biggl\{\Phi(u,w) + \int \bigl[2ff'''_{xxx}w + (f^2 f''''_{xxxx} + 2ff'_x f'''_{xxx})u \\ - (2f'''_{xxx}g + fg'''_{xxx})E^{-1} - (2ff'_x f'''_{xxx} + f^2 f''''_{xxxx} - 2kff'''_{xxx})V\bigr]\,dx\biggr\}, \qquad (1)$$

where

$$E = \exp\biggl(-k\int f^{-1}\,dx\biggr), \quad V = \int f^{-2}gE^{-1}\,dx,$$
$$u = f^{-1}E^{-1}y + V, \quad w = f^{-1}E^{-1}(fy'_x - f'_x y + g) - kV;$$

$\Phi = \Phi(u, w)$, $f = f(x)$, $g = g(x)$ are arbitrary functions, k is an arbitrary constant.

The integral in (1) may always be expressed in terms of E and V. The following cases are possible:

1) For $f'''_{xxx} \neq 0$,

$$F = f^{-2}E\Phi(u,\ w) + f^{-2}\bigl[2ff''_{xx} - (f'_x)^2\bigr]y'_x + f^{-3}\bigl[f^2 f'''_{xxx} - 2ff'_x f''_{xx} + (f'_x)^3\bigr]y \\ + f^{-3}\bigl\{\bigl[ff''_{xx} - (f'_x)^2\bigr]g + f(f'_x g'_x - fg''_{xx}) + k(fg'_x - f'_x g - kg)\bigr\} + k^3 f^{-2}EV.$$

2) For $f = ax^2 + bx + c$, $a \neq 0$,

$$F = f^{-2}E\Phi(u,w) + f^{-3}\bigl[ff'_x g'_x - ff''_{xx}g - f^2 g''_{xx} + k(fg'_x - f'_x g - kf)\bigr] + k(k^2 + \Delta)V,$$

where $\Delta = 4ac - b^2$.

3) For $f = ax + b$, $a \neq k$, $a \neq -\frac{1}{2}k$,

$$F = f^{-k/a-2}\Phi(u,w) - f^{-3}\bigl[f^2 g''_{xx} - (a+k)fg'_x + k(a+k)g\bigr] - k(a^2 - k^2)f^{-k/a-2}U,$$

where

$$u = f^{k/a-1}y + U, \quad w = f^{k/a}y'_x - f^{k/a-1}g + (a-k)U, \quad U = \int f^{k/a-2}g\,dx.$$

4) For $f = kx + b$,

$$F = f^{-3}\bigl[\Phi(u,w) - f^2 g''_{xx} + 2kfg'_x - 2k^2\bigr],$$

where

$$u = y + W, \qquad w = fy'_x + g, \qquad W = \int f^{-1}g\,dx.$$

5) For $f = -\frac{1}{2}kx + b$,

$$F = \Phi(u, w) + \frac{(kx - 2b)^2 g''_{xx} + k(kx - 2b)g'_x + 2k^2 g}{(kx - 2b)^3} + 6kQ,$$

where

$$u = (kx-2b)^{-3}y - 2Q, \quad w = (kx-2b)^{-2}y'_x - 2(kx-2b)^{-3}g - 6kQ, \quad Q = \int \frac{g\,dx}{(kx-2b)^4}.$$

6) For $f = ax + b, \;\; g = c = \text{const}, \;\; k = 0$,

$$F = \frac{1}{(ax+b)^2}\Phi\Bigl(\frac{ay - c}{ax + b},\; y'_x\Bigr).$$

7) For $f = kx + b, \;\; g = 0$,

$$F = \frac{1}{(kx+b)^3}\Phi\bigl(y,\; (kx+b)y'_x\bigr).$$

In all these cases, the transformation

$$t = \int f^{-1}dx, \qquad u = f^{-1}E^{-1}y + V$$

leads to an autonomous equation:

$$u'''_{ttt} - 3ku''_{tt} + 3k^2u'_t - k^3u = \Phi(u,\; u'_t - ku),$$

which is reducible, with the aid of the substitution $z(u) = u'_t$, to a second order equation:

$$z^2 z''_{uu} + z(z'_u)^2 - 3kzz'_u + 3k^2z - k^3u = \Phi(u,\; z - ku).$$

Remark. The original equation can be reduced, with the aid of point transformations, to an autonomous form *only* for function F of the form (1).

3.5.3. Equations of the Form $F(x, y, y'_x)y'''_{xxx} + G(x, y, y'_x)y''_{xx} + H(x, y, y'_x) = 0$

1. $y'''_{xxx} + ay''_{xx} + by'_x + cy = e^{\lambda x}f(ye^{-\lambda x})$.

The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation of the form 3.5.5.9:

$$w'''_{xxx} + (3\lambda + a)w''_{xx} + (3\lambda^2 + 2a\lambda + b)w'_x + (\lambda^3 + a\lambda^2 + b\lambda + c)w = f(w).$$

2. $y'''_{xxx} + 3\lambda y'_x y''_{xx} + \lambda^2(y'_x)^3 = f(x)e^{-\lambda y}$.

Solution:

$$e^{\lambda y} = C_2x^2 + C_1x + C_0 + \frac{\lambda}{2}\int_{x_0}^{x}(x-t)^2 f(t)\,dt,$$

where x_0 is an arbitrary number.

3. $\boldsymbol{y'''_{xxx} = -3y''_{xx} + 2y + f(e^x y)(y'_x + y).}$

The transformation $z = e^x y$, $w = e^{2x}(y'_x + y)^2$ leads to a second order linear equation: $w''_{zz} = 2f(z) + 6$. Integrating the latter, we find the solution:

$$\int \frac{dz}{\sqrt{3z^2 + C_2 z + C_1 + 2\Phi(z)}} = \pm x + C_3,$$

where

$$z = e^x y, \quad \Phi(z) = \int \left[\int f(z)\, dz \right] dz.$$

4. $\boldsymbol{xy'''_{xxx} + 3y''_{xx} = f(xy).}$

The substitution $w(x) = xy$ leads to an autonomous equation of the form 3.5.1.1: $w'''_{xxx} = f(w)$.

5. $\boldsymbol{xy'''_{xxx} = f(xy'_x - y)y''_{xx}.}$

The substitution $z = xy'_x - y$ leads to a second order equation of the form 2.9.2.7: $xz''_{xx} = [f(z) + 1]z'_x$.

6. $\boldsymbol{xy'''_{xxx} + (1-a)y''_{xx} = x^{2a} f(xy'_x - y).}$

The substitution $z = xy'_x - y$ leads to a second order equation of the form 2.9.2.4: $xz''_{xx} = az'_x + x^{2a+1} f(z)$.

7. $\boldsymbol{xy'''_{xxx} + (1-ax)y''_{xx} = e^{2ax} f(xy'_x - y).}$

The substitution $z = xy'_x - y$ leads to a second order equation of the form 2.9.2.17: $z''_{xx} - az'_x = e^{2ax} f(z)$.

8. $\boldsymbol{xy'''_{xxx} = f(xy'_x - y + a\ln x)y''_{xx}.}$

The substitution $z = xy'_x - y$ leads to a second order equation of the form 2.9.2.22: $xz''_{xx} = \left[f\big(\ln(x^a e^z)\big) + 1 \right] z'_x$.

9. $\boldsymbol{x^2 y'''_{xxx} + 6xy''_{xx} + 6y'_x = f(x^2 y).}$

The substitution $w(x) = x^2 y$ leads to an autonomous equation of the form 3.5.1.1: $w'''_{xxx} = f(w)$.

10. $\boldsymbol{x^3 y'''_{xxx} + x^2 y''_{xx} = f(xy'_x - y).}$

The transformation $t = \ln|x|$, $z = xy'_x - y$ leads to an autonomous equation of the form 2.9.6.2: $z''_{tt} - z'_t = f(z)$, which is reduced, with the aid of the substitution $w = z'_t$, to the Abel equation $ww'_z - w = f(w)$ (see Subsection 1.3.1).

11. $\boldsymbol{x^3 y'''_{xxx} + ax^2 y''_{xx} + bxy'_x = f(y).}$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 3.5.5.9: $y'''_{ttt} + (a-3)y''_{tt} + (b-a+2)y'_t = f(y)$.

12. $\boldsymbol{x^3 y'''_{xxx} + a x^2 y''_{xx} + b x y'_x = f(x^m e^{\lambda y})}$.

The transformation $t = \ln x$, $\lambda w = \lambda y + mt$ leads to an autonomous equation of the form 3.5.5.9:

$$w'''_{ttt} + (a-3)w''_{tt} + (b-a+2)w'_t = f(e^{\lambda w}) + \frac{m}{\lambda}(b-a+2).$$

13. $\boldsymbol{x^3 y'''_{xxx} = -\frac{3}{2} x^2 y''_{xx} + f\left(\frac{y}{\sqrt{x}}\right)(2xy'_x - y)}$.

The transformation $t = \dfrac{y}{\sqrt{x}}$, $z = \dfrac{1}{x}\left(xy'_x - \frac{1}{2}y\right)^2$ leads to a second order linear equation $2z''_{tt} = 8f(t) + 1$ whose solution has the form

$$z = \tfrac{1}{4}t^2 + C_2 t + C_1 + 4\int_{t_0}^{t} (t-\xi) f(\xi)\, d\xi, \qquad t_0 \text{ is an arbitrary number.}$$

Passing on to variables x, $t = yx^{-1/2}$, we obtain an equation with separation of variables.

14. $\boldsymbol{x^3 y'''_{xxx} = -3(k+1)x^2 y''_{xx} + k(k+1)(2k+1)y}$
$\boldsymbol{\qquad + f(x^k y)(xy'_x + ky) + x^{2k} g(x^k y)(xy'_x + ky)^3}$.

The transformation $t = \ln x$, $z = x^k y$ followed by the substitution $w(z) = (z'_t)^2$ yields a second order linear equation: $w''_{zz} = 2g(z)w + 2f(z) + 6k^2 + 6k + 2$.

15. $\boldsymbol{x^4 y'''_{xxx} + x^3 y''_{xx} = f\left(y'_x - \frac{y}{x}\right)}$.

The substitution $w(x) = xy'_x - y$ leads to an equation of the form 2.9.1.5: $w''_{xx} = x^{-3} f(w/x)$.

16. $\boldsymbol{x^4 y'''_{xxx} = -\frac{3}{2} x^3 y''_{xx} + f\left(\frac{y}{\sqrt{x}}\right)(2xy'_x - y)^3}$.

The transformation $t = \dfrac{y}{\sqrt{x}}$, $z = \dfrac{1}{x}(xy'_x - \frac{1}{2}y)^2$ leads to a second order linear equation: $2z''_{tt} = 16f(t)z + \frac{1}{2}$.

17. $\boldsymbol{y y'''_{xxx} + 3y'_x y''_{xx} = f(x)}$.

Solution: $\quad y^2 = C_2 x^2 + C_1 x + C_0 + \displaystyle\int_{x_0}^{x} (x-t)^2 f(t)\, dt$, $\quad x_0$ is an arbitrary number.

18. $\boldsymbol{y(y'''_{xxx} + 3a y''_{xx} + 2a^2 y'_x) = f(x)}$.

Having integrated the equation, we obtain

$$2yy''_{xx} + 2ayy'_x - (y'_x)^2 = e^{-2ax}\left[2\int e^{2ax} f(x)\, dx + C\right].$$

19. $\boldsymbol{y y'''_{xxx} + 3y'_x y''_{xx} + f(x) y y'_x = g(x)}$.

The substitution $w = yy'_x$ leads to a second order linear equation: $w''_{xx} + f(x)w = g(x)$.

20. $\boldsymbol{yy'''_{xxx} + 3y'_x y''_{xx} + a\big[yy''_{xx} + (y'_x)^2\big] = f(x).}$

Solution:

$$y^2 = C_3 e^{-ax} + C_2 x + C_1 + 2\int_{x_0}^{x} (x-t)e^{-at}F(t)\,dt,$$

where $F(t) = \int e^{at} f(t)\,dt$, x_0 is an arbitrary number.

21. $\boldsymbol{yy'''_{xxx} + [3y'_x + f(x)y]y''_{xx} + f(x)(y'_x)^2 = 0.}$

Solution:

$$y^2 = C_3 x + C_2 + C_1 \int_{x_0}^{x} (x-t)e^{-F(t)}\,dt, \qquad \text{where} \quad F(t) = \int f(t)\,dt.$$

22. $\boldsymbol{yy'''_{xxx} + (3y'_x + 2ay)y''_{xx} + 2a(y'_x)^2 + a^2 yy'_x = f(x).}$

Integrating the equation twice, we obtain a first order equation with separation of variables:

$$e^{ax} yy'_x = C_2 x + C_1 + \int_{x_0}^{x} (x-t)e^{at} f(t)\,dt.$$

23. $\boldsymbol{yy'''_{xxx} + [3y'_x + f(x)y]y''_{xx} + f(x)(y'_x)^2 + g(x)yy'_x + h(x) = 0.}$

The substitution $w = yy'_x$ leads to a second order linear equation: $w''_{xx} + f(x)w'_x + g(x)w + h(x) = 0$.

24. $\boldsymbol{(y+a)y'''_{xxx} + by'_x y''_{xx} = f(x).}$

Having integrated the equation, we obtain

$$(y+a)y''_{xx} + \tfrac{1}{2}(b-1)(y'_x)^2 = \int f(x)\,dx + C.$$

With $b \neq -1$, the substitution $y = w^{\frac{2}{b+1}} - a$ leads to the equation

$$w''_{xx} = \frac{b+1}{2}\left[\int f(x)\,dx + C\right] w^{\frac{b-3}{b+1}}$$

(for $C = 0$ and $f(x) = \lambda x^n$, see Section 2.3).

25. $\boldsymbol{(y+a)y'''_{xxx} + by'_x y''_{xx} + f(y)y'_x = 0.}$

Having integrated the equation, we obtain a second order equation:

$$(y+a)y''_{xx} + \tfrac{1}{2}(b-1)(y'_x)^2 + \int f(y)\,dy = C,$$

which is reduced, with the aid of the substitution $w(y) = (y'_x)^2$, to a first order linear equation:

$$(y+a)w'_y + (b-1)w + 2\int f(y)\,dy = 2C.$$

26. $(y+a)y'''_{xxx}+by'_x y''_{xx}+f(y)y'_x=f(x)$.

Having integrated the equation, we obtain

$$(y+a)y''_{xx}+\tfrac{1}{2}(b-1)(y'_x)^2+\int f(y)\,dy=\int g(x)\,dx+C.$$

27. $(y+ax+b)y'''_{xxx}+3(y'_x+a)y''_{xx}=f(x)$.

Solution:

$$(y+ax+b)^2=C_2x^2+C_1x+C_0+\int_{x_0}^{x}(x-t)^2f(t)\,dt,$$

where x_0 is an arbitrary number.

28. $x(yy'''_{xxx}+3y'_x y''_{xx})+a\bigl[yy''_{xx}+(y'_x)^2\bigr]=f(x)$.

Solution:

$$y^2=C_3x^{2-a}+C_2x+C_1+2\int_{x_0}^{x}(x-t)t^{-a}F(t)\,dt,$$

where $F(t)=\int t^{a-1}f(t)\,dt$, x_0 is an arbitrary number.

29. $y^2y'''_{xxx}-3yy'_x y''_{xx}+2(y'_x)^3=f(x)y^3$.

Solution:

$$\ln|y|=C_2x^2+C_1x+C_0+\tfrac{1}{2}\int_{x_0}^{x}(x-t)^2f(t)\,dt,$$

where x_0 is an arbitrary number.

30. $y^2y'''_{xxx}+3(n-1)yy'_x y''_{xx}+(n-1)(n-2)(y'_x)^3=f(x)y^{3-n}, \qquad n\neq 0$.

Solution:

$$y^n=C_2x^2+C_1x+C_0+\frac{n}{2}\int_{x_0}^{x}(x-t)^2f(t)\,dt,$$

where x_0 is an arbitrary number.

31. $y^2y'''_{xxx}=-3y^2y''_{xx}+2y^3+y^2f(e^xy)(y'_x+y)+g(e^xy)(y'_x+y)^3$.

The substitution $z(x)=e^xy$, followed by lowering of the equation order and the substitution $w(z)=(z'_x)^2$, leads to a second order linear equation: $w''_{zz}=2z^{-2}g(z)w+2f(z)+6$.

32. $y\bigl(fy'''_{xxx}+\frac{3}{2}f'_x y''_{xx}+\frac{1}{2}f''_{xx}y'_x\bigr)=g(x), \qquad f=f(x)$.

Having integrated the equation, we obtain

$$2fyy''_{xx}+f'_xyy'_x-f(y'_x)^2=2\int g(x)\,dx+C.$$

33. $fyy'''_{xxx}+(3fy'_x+2f'_xy)y''_{xx}+2f'_x(y'_x)^2+f''_{xx}yy'_x=g, \qquad f=f(x),\ g=g(x)$.

Integrating the equation twice, we obtain a first order linear equation with separation of variables:

$$f(x)yy'_x=C_2x+C_1+\int_{x_0}^{x}(x-t)g(t)\,dt.$$

3.5.4. Equations of the Form $F(x,y,y'_x)y'''_{xxx}+\sum_\alpha G_\alpha(x,y,y'_x)(y''_{xx})^\alpha=0$

1. $2y'_x y'''_{xxx}-(y''_{xx})^2+f(x)(y'_x)^2=ay^2+2by+c.$

Differentiating both sides of the equation with respect to x and dividing by y'_x, we arrive at a fourth order linear equation: $y''''_{xxxx}+fy''_{xx}+\frac{1}{2}f'_x y'_x=ay+b$.

2. $2y'_x y'''_{xxx}-(y''_{xx})^2-\lambda y'_x y''_{xx}+F(x)(y'_x)^2=e^{\lambda x}(ay^2+2by+c).$

Multiplying both sides by $e^{-\lambda x}$, we arrive at an equation of the form 3.5.4.17 with $f(x)=e^{-\lambda x}$, $g(x)=e^{-\lambda x}F(x)$.

3. $2y'_x y'''_{xxx}-3(y''_{xx})^2=f(x)(y'_x)^2+g(y)(y'_x)^4.$

Solution:

$$\int\frac{dy}{u^2(y)}=\int\frac{dx}{w^2(x)}+C,$$

where $u=u(y)$ and $w=w(x)$ are the general solutions of the second order linear equations

$$4u''_{yy}-g(y)u=0\quad\text{and}\quad 4w''_{xx}+f(x)w=0.$$

4. $2y'_x y'''_{xxx}-3(y''_{xx})^2=f(x)(y'_x)^2+g(x)y^{-1}(y'_x)^{5/2}.$

The substitution $w(x)=\dfrac{y}{\sqrt{y'_x}}$ leads to a nonhomogeneous second-order linear equation: $4w''_{xx}+f(x)w+g(x)=0$.

5. $2y'_x y'''_{xxx}-3(y''_{xx})^2=f(y)(y'_x)^4+x^{-1}g(y)(y'_x)^{7/2}.$

Taking y as the independent variable, we obtain an equation of the form 3.5.4.4 for $x=x(y)$:

$$2x'_y x'''_{yyy}-3(x''_{yy})^2=-f(y)(x'_y)^2-g(y)x^{-1}(x'_y)^{5/2}.$$

6. $2y'_x y'''_{xxx}-3(y''_{xx})^2=y^4 f\left(\dfrac{y'_x}{y^2}\right).$

The substitution $w(x)=\dfrac{y}{\sqrt{y'_x}}$ leads to the second order autonomous equation of the form 2.9.1.1: $w''_{xx}=F(w)$, where $F(w)=-\frac{1}{4}w^5 f(w^{-2})$.

7. $2y'_x y'''_{xxx}-3(y''_{xx})^2=x^{-4}y^4 f\left(\dfrac{xy'_x}{y^2}\right).$

The substitution $w(x)=\dfrac{y}{\sqrt{y'_x}}$ leads to the second order equation of the form 2.9.1.6: $w''_{xx}=x^{-3/2}F(wx^{-1/2})$, where $F(\xi)=-\frac{1}{4}\xi^5 f(\xi^{-2})$.

8. $2y'_x y'''_{xxx}-3(y''_{xx})^2=x^{-8}y^4 f\left(\dfrac{x^2y'_x}{y^2}\right).$

The substitution $w(x)=\dfrac{y}{\sqrt{y'_x}}$ leads to the second order equation of the form 2.9.1.5: $w''_{xx}=x^{-3}F(wx^{-1})$, where $F(\xi)=-\frac{1}{4}\xi^5 f(\xi^{-2})$.

9. $2xy'_x y'''_{xxx} - x(y''_{xx})^2 + ny'_x y''_{xx} + F(x)(y'_x)^2 = x^{1-n}(ay^2 + 2by + c).$

Multiplying both sides by x^{n-1}, we arrive at an equation of the form 3.5.4.17 with $f(x) = x^n$, $g(x) = x^{n-1}F(x)$.

10. $xy'_x y'''_{xxx} - 3x(y''_{xx})^2 + 3y'_x y''_{xx} = xf(y)(y'_x)^4 + g(y)(y'_x)^5.$

Taking y as the independent variable, we obtain an equation of the form 3.5.3.23 for $x = x(y)$:

$$xx'''_{yyy} + 3x'_y x''_{yy} = -g(y) - f(y)xx'_y.$$

11. $y'''_{xxx} = \left[x^3 f(xy'_x - y) + ax^{-5}\right](y''_{xx})^3.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 3.5.2.9: $w'''_{ttt} = -f(w)(w'_t)^3 - a(w'_t)^{-5}$.

12. $y'''_{xxx} = x^{-5} f\left(\dfrac{xy'_x - y}{x^4}\right)(y''_{xx})^3.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 3.5.2.12: $w'''_{ttt} = w^{-5/4}F(w'_t w^{-1/4})$, where $F(\xi) = -\xi^{-5}f(\xi^{-4})$.

13. $y'''_{xxx} = x^{-5} f\left(\dfrac{xy'_x - y}{x^2}\right)(y''_{xx})^3.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 3.5.2.11: $w'''_{ttt} = w^{-5/2}F(w'_t w^{-1/2})$, where $F(\xi) = -\xi^{-5}f(\xi^{-2})$.

14. $y'''_{xxx} = [xf(y'_x) + yg(y'_x) + h(y'_x)](y''_{xx})^3 + \varphi(y'_x)(y''_{xx})^2.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to a linear equation:

$$w'''_{ttt} = -\varphi(t)w''_{tt} - [f(t) + tg(t)]w'_t + g(t)w - h(t).$$

15. $y'''_{xxx} = f(y)y'_x(y''_{xx})^m.$

Solution with $m \neq 1$:

$$C_3 \pm x = \int \left\{2\int \left[(1-m)F(y) + C_2\right]^{\frac{1}{1-m}} dy + C_1\right\}^{-\frac{1}{2}} dy, \quad F(y) = \int f(y)\,dy.$$

Solution with $m = 1$:

$$C_3 \pm x = \int \left\{C_2 \int e^{F(y)}\,dy + C_1\right\}^{-\frac{1}{2}} dy, \quad F(y) = \int f(y)\,dy.$$

16. $y'''_{xxx} = xf(xy'_x - y)(y''_{xx})^2 + xg(xy'_x - y)(y''_{xx})^k.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to the equation

$$w'''_{ttt} = -f(w)w'_t w''_{tt} - g(w)w'_t(w''_{tt})^{3-k}.$$

Lowering its order with the substitution $z(w) = w''_{tt}$ ($z'_w = w'''_{ttt}/w'_t$), we have the Bernoulli equation $z'_w = -f(w)z - g(w)z^{3-k}$.

17. $2fy'_x y'''_{xxx} - f(y''_{xx})^2 + f'_x y'_x y''_{xx} + g(y'_x)^2 = ay^2 + 2by + c, \quad f = f(x),\ g = g(x).$

Differentiating the both sides of the equation with respect to x and dividing by y'_x, we arrive at a fourth order linear equation:

$$fy''''_{xxxx} + \tfrac{3}{2}f'_x y'''_{xxx} + (g + \tfrac{1}{2}f''_{xx})y''_{xx} + \tfrac{1}{2}g'_x y'_x = ay + b.$$

3.5.5. Equations of the Form $F(x, y, y'_x, y''_{xx})y'''_{xxx} + G(x, y, y'_x, y''_{xx}) = 0$

1. $y'''_{xxx} = f(y''_{xx})$.

Solution in the parametric form:

$$x = \int_{C_1}^{\tau} \frac{d\tau_1}{f(\tau_1)}, \qquad y = \int_{C_2}^{\tau} \frac{d\tau_1}{f(\tau_1)} \int_{C_3}^{\tau_1} \frac{\tau_2\, d\tau_2}{f(\tau_2)}.$$

2. $y'''_{xxx} = f(y)y'_x g(y''_{xx})$.

Integrating the equation and substituting $w(y) = \frac{1}{2}(y'_x)^2$, we arrive at a first order equation:

$$\int \frac{d\xi}{g(\xi)} = \int f(y)\, dy + C, \qquad \text{where} \quad \xi = w'_y.$$

Solving this equation for w'_y, we obtain an equation with separation of variables.

3. $y'''_{xxx} = f(y)g(y'_x)h(y''_{xx})$.

The substitution $w(y) = \frac{1}{2}(y'_x)^2$ leads to a second order equation:

$$w''_{yy} = f(y)\varphi(w)h(w'_y), \qquad \text{where} \quad \varphi(w) = \pm\frac{g\left(\pm\sqrt{2w}\,\right)}{\sqrt{2w}},$$

whose solvable cases for some functions f, g, and h are outlined in Section 2.7.

4. $y'''_{xxx} = xf(xy'_x - y)g(y''_{xx})$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$, where $w = w(t)$, leads to an equation of the form 3.5.5.2: $w'''_{ttt} = -f(w)w'_t g\Big(\dfrac{1}{w''_{tt}}\Big)(w''_{tt})^3$.

5. $xy'''_{xxx} + y''_{xx} = f(xy'_x - y)g(xy''_{xx})$.

The substitution $w(x) = xy'_x - y$ leads to an equation of the form 2.9.4.2: $w''_{xx} = f(w)g(w'_x)$.

6. $y'''_{xxx} = f(x)g(x^2y''_{xx} - 2xy'_x + 2y)$.

The substitution $w(x) = x^2y''_{xx} - 2xy'_x + 2y$ leads to a first order equation with separation of variables: $w'_x = x^2f(x)g(w)$.

7. $y'''_{xxx} = \dfrac{y'_x y''_{xx}}{y} + \left[y''_{xx} - \dfrac{(y'_x)^2}{y}\right] f\left(\dfrac{y'_x}{y}\right) g\left(\dfrac{y''_{xx}}{y}\right)$.

The transformation $t = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y}$ leads to a first order equation with separation of variables: $w'_t = f(t)g(w)$.

8. $y'''_{xxx} = F(x, y'_x, y''_{xx})$.

The substitution $u(x) = y'_x$ leads to a second order equation: $u''_{xx} = F(x, u, u'_x)$.

9. $y'''_{xxx} = F(y, y'_x, y''_{xx})$.

Autonomous equation.

The substitution $w(y) = (y'_x)^2$ leads to a second order equation:

$$w''_{yy} = \pm\frac{2}{\sqrt{w}}F\big(y,\ \pm\sqrt{w},\ \tfrac{1}{2}w'_y\big).$$

10. $y'''_{xxx} = yF\left(\dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y}\right)$.

The transformation $t = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y}$ leads to a first order equation: $(w - t^2)w'_t = -tw + F(t, w)$.

11. $y'''_{xxx} = F(x,\ xy'_x - y,\ xy''_{xx})$.

The substitution $z = xy'_x - y$ leads to a second order equation: $z''_{xx} = x^{-1}z'_x + xF(x, z, z'_x)$.

12. $y'''_{xxx} = x^{-k-3}F(x^k y,\ x^{k+1}y'_x,\ x^{k+2}y''_{xx})$.

Homogeneous equation in the extended sense.

The transformation $t = \ln x$, $z = x^k y$ leads to an autonomous equation:

$$z'''_{ttt} = 3(k+1)z''_{tt} - (3k^2 + 6k + 2)z'_t + k(k+1)(k+2)z \\ + F\big(z,\ z'_t - kz,\ z''_{tt} - (2k+1)z'_t + k(k+1)z\big).$$

The substitution $w(z) = (z'_t)^2$ leads to a second order equation:

$$w''_{zz} = \pm 3(k+1)w^{-1/2}w'_z - 6k^2 - 12k - 4 \pm 2k(k+1)(k+2)zw^{-1/2} \\ \pm 2w^{-1/2}F\big(z,\ \pm w^{1/2} - kz,\ \tfrac{1}{2}w'_z \mp (2k+1)w^{1/2} + k(k+1)z\big).$$

13. $y'''_{xxx} = yx^{-3}F\left(x^k y^m,\ \dfrac{xy'_x}{y},\ \dfrac{x^2 y''_{xx}}{y}\right)$.

Homogeneous equation in the extended sense.

The transformation $t = x^k y^m$, $z = \dfrac{xy'_x}{y}$ leads to a second order equation.

14. $y'''_{xxx} = yx^{-3}F\left(\dfrac{xy'_x}{y},\ \dfrac{x^2 y''_{xx}}{y}\right)$.

This is a special case of equation 3.5.5.13.

The transformation $z = \dfrac{xy'_x}{y}$, $w = \dfrac{x^2 y''_{xx}}{y}$ leads to a first order equation:

$$(w + z - z^2)w'_z = 2w - zw + F(z, w).$$

15. $y'''_{xxx} = F(x, y, y'_x, y''_{xx})$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to the equation

$$w'''_{ttt} = -F\left(w'_t,\ tw'_t - w,\ t,\ \frac{1}{w''_{tt}}\right)(w''_{tt})^3.$$

16. $\boldsymbol{y'''_{xxx} = e^{-\alpha x} F(e^{\alpha x} y,\ e^{\alpha x} y'_x,\ e^{\alpha x} y''_{xx})}$.

The substitution $z = e^{\alpha x} y$ leads to an autonomous equation of the form 3.5.5.9:

$$z'''_{xxx} - 3\alpha z''_{xx} + 3\alpha^2 z'_x - \alpha^3 z = F(z,\ z'_x - \alpha z,\ z''_{xx} - 2\alpha z'_x + \alpha^2 z).$$

17. $\boldsymbol{y'''_{xxx} = y F\left(e^{\alpha x} y,\ \dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y}\right)}$.

Exponential homogeneous equation.

The transformation $z = e^{\alpha x} y$, $w = \dfrac{y'_x}{y}$ leads to a second order equation:

$$z^2(w+\alpha)^2 w''_{zz} + z^2(w+\alpha)(w'_z)^2 + z(w+\alpha)(4w+\alpha)w'_z + w^3 = F\big(z,\ w,\ z(w+\alpha)w'_z + w^2\big).$$

18. $\boldsymbol{y'''_{xxx} = x^{-3} F(x^m e^y,\ x y'_x,\ x^2 y''_{xx})}$.

Exponential homogeneous equation.

The transformation $z = x^m e^y$, $w = x y'_x + m$ leads to a second order equation:

$$z^2 w^2 w''_{zz} + z^2 w (w'_z)^2 + z w^2 w'_z - 3 z w w'_z + 2w - 2m = F(z,\ w - m,\ z w w'_z - w + m).$$