Chapter 4
Fourth Order Differential Equations

4.1. Linear Equations

4.1.1. Preliminary Comments

1. A nonhomogeneous linear equation of the fourth order has the form

$$f_4 y''''_{xxxx} + f_3 y'''_{xxx} + f_2 y''_{xx} + f_1 y'_x + f_0 y = g(x), \qquad f_k = f_k(x). \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution of the corresponding homogeneous equation (with $g \equiv 0$). Then, the substitution

$$y = y_0(x) \int z(x)\,dx \tag{2}$$

leads to a linear equation of the third order:

$$f_4 y_0 z''' + (4f_4 y_0' + f_3 y_0) z'' + (6f_4 y_0'' + 3f_3 y_0' + f_2 y_0) z' + (4f_4 y_0''' + 3f_3 y_0'' + 2f_2 y_0' + f_1 y_0) z = g, \tag{3}$$

where prime denotes differentiation with respect to x.

2. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly-independent particular solutions of equation (1) with $g \equiv 0$. Then, the substitution

$$y = y_1 \int y_2 w\,dx - y_2 \int y_1 w\,dx \tag{4}$$

yields a second order linear equation:

$$f_4 \Delta_1 w'' + (3f_4 \Delta_2 + f_3 \Delta_1) w' + [f_4(3\Delta_3 + 2\varepsilon) + 2f_3 \Delta_2 + f_2 \Delta_1] w = g, \tag{5}$$

where

$$\Delta_1 = y_1' y_2 - y_1 y_2', \quad \Delta_2 = y_1'' y_2 - y_1 y_2'', \quad \Delta_3 = y_1''' y_2 - y_1 y_2''', \quad \varepsilon = y_1'' y_2' - y_1' y_2''.$$

4.1.2. Equations Containing Power Functions

1. $\quad \boldsymbol{y''''_{xxxx} + ay = 0}$.

1°. Solution with $a = 0$:

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3.$$

2°. Solution with $a = 4k^4 > 0$:

$$y = C_1 \cosh kx \cos kx + C_2 \cosh kx \sin kx + C_3 \sinh kx \cos kx + C_4 \sinh kx \sin kx.$$

3°. Solution with $a = -k^4 < 0$:

$$y = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx.$$

2. $\boldsymbol{y''''_{xxxx} + \lambda y = ax^3 + bx^2 + cx + s, \qquad \lambda \neq 0.}$

Solution: $y = \dfrac{1}{\lambda}(ax^3 + bx^2 + cx + s) + w(x)$, where $w(x)$ is the general solution of the equation 4.1.2.1: $w''''_{xxxx} + \lambda w = 0$.

3. $\boldsymbol{y''''_{xxxx} = axy + b.}$

This is a special case of equation 5.1.2.4 with $n = 4$.

4. $\boldsymbol{y''''_{xxxx} = ax^m y.}$

For $m = -2, -4, -6, -8$, and -9, see equations 4.1.2.34, 4.1.2.42, 4.1.2.47, 4.1.2.48, and 4.1.2.53, respectively.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the similar form: $w''''_{tttt} = at^{-m-8}w$.

5. $\boldsymbol{y''''_{xxxx} + ay'_x + by = 0.}$

This is a special case of equation 4.1.2.24.

6. $\boldsymbol{y''''_{xxxx} + 2ay'_x - a^2x^2y = 0.}$

This is a special case of equation 4.1.2.13 with $n = 1$.

7. $\boldsymbol{y''''_{xxxx} + 4axy'_x + (2a - a^2x^4)y = 0.}$

This is a special case of equation 4.1.2.13 with $n = 2$.

8. $\boldsymbol{y''''_{xxxx} + ax(2b - 3a - a^2x^2)y'_x + b(2a - b + a^2x^2)y = 0.}$

The substitution $w = y''_{xx} - axy'_x + by$ leads to a second order linear equation of the form 2.1.2.28: $w''_{xx} + axw'_x + (2a - b + a^2x^2)w = 0$.

9. $\boldsymbol{y''''_{xxxx} + ax^m y'_x - 3ax^{m-1}y = 0.}$

Particular solution: $y_0 = x^3$.

The substitution $z = xy'_x - 3y$ leads to a third order equation of the form 3.1.2.7: $z'''_{xxx} + ax^m z = 0$.

10. $\boldsymbol{y''''_{xxxx} + ax^m y'_x + amx^{m-1}y = 0.}$

Integrating yields a third order equation: $y'''_{xxx} + ax^m y = C$.

11. $\boldsymbol{y''''_{xxxx} + ax^m y'_x + a(m+3)x^{m-1}y = 0.}$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 4.1.2.10: $w''''_{tttt} + bt^n w'_t + bnt^{n-1}w = 0$, where $b = -a$, $n = -m - 6$.

12. $y''''_{xxxx} + bx^m y'_x - a(a^3 + bx^m)y = 0.$

This is a special case of equation 4.1.5.1 with $f = bx^m$.

13. $y''''_{xxxx} + 2anx^{n-1}y'_x + a[n(n-1)x^{n-2} - ax^{2n}]y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a second order equation of the form 2.1.2.7: $w''_{xx} - ax^n w = 0$.

14. $y''''_{xxxx} + (ax^n + b^3)y'_x + abx^n y = 0.$

Particular solution: $y_0 = e^{-bx}$.

15. $y''''_{xxxx} + (ax^{n+1} + bx^n)y'_x - ax^n y = 0.$

Particular solution: $y_0 = ax + b$.

16. $y''''_{xxxx} + 2ay''_{xx} + a^2 y = 0.$

1°. Solution with $a = k^2 > 0$:

$$y = (C_1 + C_2 x)\cos(kx) + (C_3 + C_4 x)\sin(kx).$$

2°. Solution with $a = -k^2 < 0$:

$$y = (C_1 + C_2 x)\exp(kx) + (C_3 + C_4 x)\exp(-kx).$$

17. $y''''_{xxxx} + (a + b)y''_{xx} + aby = 0.$

The case of $a = b$ is given in 4.1.2.16. Let $a \neq b$.

1°. Solution with $a = \alpha^2 > 0,\ b = \beta^2 > 0$:

$$y = C_1\cos(\alpha x) + C_2\sin(\alpha x) + C_3\cos(\beta x) + C_4\sin(\beta x).$$

2°. Solution with $a = \alpha^2 > 0,\ b = -\beta^2 < 0$:

$$y = C_1\cos(\alpha x) + C_2\sin(\alpha x) + C_3\exp(\beta x) + C_4\exp(-\beta x).$$

3°. Solution with $a = -\alpha^2 < 0,\ b = \beta^2 > 0$:

$$y = C_1\exp(\alpha x) + C_2\exp(-\alpha x) + C_3\cos(\beta x) + C_4\sin(\beta x).$$

4°. Solution with $a = -\alpha^2 < 0,\ b = -\beta^2 < 0$:

$$y = C_1\exp(\alpha x) + C_2\exp(-\alpha x) + C_3\exp(\beta x) + C_4\exp(-\beta x).$$

18. $\boldsymbol{y''''_{xxxx} - 2a^2 y''_{xx} + a^4 y - \lambda(ax - b)(y''_{xx} - a^2 y) = 0}$.

This equation is met with in the turbulence theory. Assuming

$$z(x) = y''_{xx} - a^2 y, \tag{1}$$

yields a second order linear equation of the form 2.1.2.12:

$$z''_{xx} - a^2 z - \lambda(ax - b)z = 0. \tag{2}$$

Given the boundary conditions

$$y(0) = y'_x(0) = 0, \quad y(1) = y'_x(1) = 0, \tag{3}$$

we obtain

$$2ay = e^{ax} \int_0^x e^{-ax} z\, dx - e^{-ax} \int_0^x e^{ax} z\, dx$$

The latter is the solution of equation (1) that satisfies the first pair of the boundary conditions (3). In order to satisfy the second pair of the boundary conditions, the solution $z(x)$ of equation (2) must meet the requirements

$$\int_0^1 e^{-ax} z\, dx = \int_0^1 e^{ax} z\, dx = 0.$$

19. $\boldsymbol{y''''_{xxxx} + ax^n y''_{xx} + b(ax^n - b)y = 0}$.

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

The substitution $w = y''_{xx} + by$ leads to a second order linear equation: $w''_{xx} + (ax^n - b)w = 0$.

20. $\boldsymbol{y''''_{xxxx} + ax^{n+1} y''_{xx} - 4ax^n y'_x + 6ax^{n-1} y = 0}$.

Particular solutions: $y_1 = x^2$, $y_2 = x^3$.

The substitution $w = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second order linear equation of the form 2.1.2.7: $w''_{xx} + ax^{n+1} w = 0$.

21. $\boldsymbol{y''''_{xxxx} + 10ax^n y''_{xx} + 10anx^{n-1} y'_x + [3an(n-1)x^{n-2} + 9a^2 x^{2n}]y = 0}$.

This is a special case of equation 4.1.5.26 with $f = ax^n$.

22. $\boldsymbol{y''''_{xxxx} + (ax^n + b)y''_{xx} + abx^n y = 0}$.

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\,\bigr)$.

The substitution $w = y''_{xx} + by$ leads to a second order linear equation of the form 2.1.2.7: $w''_{xx} + ax^n w = 0$.

23. $\boldsymbol{y''''_{xxxx} + ay''_{xx} + bx^n y'_x + bnx^{n-1} ny = sx^m}$.

Intergating yields a third order equation: $y'''_{xxx} + ay''_{xx} + bx^n y = s \int x^m\, dx + C$.

24. $y''''_{xxxx} + a_3 y''_{xx} + a_2 y'_x + a_0 y = 0.$

For $a_0 = 0$, the substitution $w(x) = y'_x$ leads to a third order equation. Let $a_0 \neq 0$ and $P(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial.

1°. Let P be factorizable, so that

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4),$$

where λ_1, λ_2, λ_3, and λ_4 are real numbers. The following cases are possible:

a) λ_i are all different, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

b) $\lambda_1 = \lambda_2$; λ_3 and λ_4 are different and not equal to λ_1, then

$$y = (C_1 + C_2 x) e^{\lambda_1 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

c) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2) e^{\lambda_1 x} + C_4 e^{\lambda_4 x};$$

d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2 + C_4 x^3) e^{\lambda_1 x}.$$

2°. Let

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 + 2b_1\lambda + b_0),$$

where λ_1 and λ_2 are real numbers, and $b_1^2 - b_0 < 0$. If

a) $\lambda_1 \neq \lambda_2$, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \qquad \mu = \sqrt{b_0 - b_1^2};$$

b) $\lambda_1 = \lambda_2$, then

$$y = (C_1 + C_2 x) e^{\lambda_1 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \qquad \mu = \sqrt{b_0 - b_1^2}.$$

3°. Let us assume that

$$P(\lambda) = (\lambda^2 + 2b_1\lambda + b_0)(\lambda^2 + 2\beta_1\lambda + \beta_0),$$

where $b_1^2 - b_0 < 0$ and $\beta_1^2 - \beta_0 < 0$. If

a) $(b_1 - \beta_1)^2 + (b_0 - \beta_0)^2 \neq 0$, then

$$y = e^{-b_1 x}[C_1 \cos(\mu x) + C_2 \sin(\mu x)] + e^{-\beta_1 x}[C_3 \cos(\nu x) + C_4 \sin(\nu x)],$$

where $\mu = \sqrt{b_0 - b_1^2}$, $\nu = \sqrt{\beta_0 - \beta_1^2}$;

b) $b_1 = \beta_1$ and $b_0 = \beta_0$, then

$$y = e^{-b_1 x}[(C_1 + C_2 x) \cos(\mu x) + (C_3 + C_4 x) \sin(\mu x)], \qquad \mu = \sqrt{b_0 - b_1^2}.$$

25. $y''''_{xxxx} + 4axy'''_{xxx} + 6a^2x^2y''_{xx} + 4a^3x^3y'_x + a^4x^4y = 0.$

Solution:

$$y = \sum_{i=1}^{4} C_i \exp(\lambda_i x - \tfrac{1}{2}ax^2),$$

where λ_i are the roots of the biquadratic equation $\lambda^4 - 6a\lambda^2 + 3a^2 = 0$.

26. $y''''_{xxxx} + (ax + b)y'''_{xxx} + [b(a + c)x + c]y''_{xx} + b^2cxy'_x - b^2cy = 0.$

Particular solutions: $y_1 = x, \quad y_2 = e^{-bx}$.

27. $y''''_{xxxx} = ax^ny'''_{xxx} + by'_x - abx^ny.$

Particular solutions: $y_k = \exp(\lambda_k x)$ $(k = 1,\ 2,\ 3)$, where λ_k are the roots of the cubic equation $\lambda^3 - b = 0$.

28. $y''''_{xxxx} + ax^{n+3}y'''_{xxx} - 3ax^{n+2}y''_{xx} + 6ax^{n+1}y'_x - 6ax^ny = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3$.

The substitution $w = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first order linear equation: $w'_x + ax^{n+3}w = 0$.

29. $y''''_{xxxx} + ax^ny'''_{xxx} + bx^{m+1}y''_{xx} - 2bx^my'_x + 2bx^{m-1}y = 0.$

Particular solutions: $y_1 = x, \quad y_2 = x^2$.

The substitution $w = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second order linear equation: $xw''_{xx} + (ax^{n+1} - 2)w'_x + bx^{m+2}w = 0$.

30. $y''''_{xxxx} + ax^ny'''_{xxx} + bx^my''_{xx} + acx^ny'_x + c(bx^m - c)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr), \quad y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr), \quad y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second order linear equation: $w''_{xx} + ax^nw'_x + (bx^m - c)w = 0$.

31. $y''''_{xxxx} + ax^ny'''_{xxx} + (bx^m + c)y''_{xx} + acx^ny'_x + bcx^my = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr), \quad y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr), \quad y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second order linear equation: $w''_{xx} + ax^nw'_x + bx^mw = 0$.

32. $xy''''_{xxxx} + 4y'''_{xxx} + axy = 0.$

The substitution $w(x) = xy$ leads to a constant coefficient equation of the form 4.1.2.1: $w''''_{xxxx} + aw = 0$.

33. $xy''''_{xxxx} - 4my'''_{xxx} + axy = 0, \qquad m = 1,\ 2,\ 3,\ \ldots$

Solution:

$$y = x^{4m+3}\left(\frac{1}{x^3}\frac{d}{dx}\right)^m\left(\frac{w}{x^3}\right),$$

where $w = w(x)$ is the general solution of the constant coefficient equation 4.1.2.1 $w''''_{xxxx} + aw = 0$.

34. $x^2 y''''_{xxxx} = ay.$

This is a special case of equation 5.1.2.23 with $n = 2$.

35. $x^2 y''''_{xxxx} - 2(ax^2 + 6)y''_{xx} + a(ax^2 + 4)y = 0.$

Particular solutions: $y_1 = x^{-1/2} I_{1/2}(x\sqrt{a})$, $y_2 = x^{-1/2} K_{1/2}(x\sqrt{a})$, where $I_{1/2}$ and $K_{1/2}$ are modified Bessel functions.

36. $x^2 y''''_{xxxx} + 6xy'''_{xxx} + 6y''_{xx} - \lambda^2 y = 0.$

The equation of transverse vibrations of a pointed bar.

Solution:

$$y = \frac{1}{\sqrt{x}} \big[C_1 J_1(2\sqrt{\lambda x}\,) + C_2 Y_1(2\sqrt{\lambda x}\,) + C_3 I_1(2\sqrt{\lambda x}\,) + C_4 K_1(2\sqrt{\lambda x}\,)\big],$$

where J_1 and Y_1 are Bessel functions, I_1 and K_1 are modified Bessel functions.

37. $x^2 y''''_{xxxx} + 2(a + 2)xy'''_{xxx} + (a + 1)(a + 2)y''_{xx} - b^4 y = 0.$

Solution:

$$y = x^{-a/2}[C_1 J_a(\xi) + C_2 Y_a(\xi) + C_3 I_a(\xi) + C_4 K_a(\xi)],$$

where $\xi = 2b\sqrt{x}$, J_a and Y_a are Bessel functions, I_a and K_a are modified Bessel functions.

38. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} + ax^2 y = 0.$

The substitution $w(x) = x^2 y$ leads to a constant coefficient equation of the form 4.1.2.1: $w''''_{xxxx} + aw = 0$.

39. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^3 y + b.$

The substitution $w(x) = x^2 y$ leads to an equation of the form 4.1.2.3: $w''''_{xxxx} = axw + b$.

40. $x^2 y''''_{xxxx} + axy'''_{xxx} + (bx^{n+1} + c)y''_{xx} + (a-4)bx^n y'_x + b(c - 2a + 6)x^{n-1} y = 0.$

The substitution $w(x) = x^2 y''_{xx} + (a - 4)xy'_x + (c - 2a + 6)y$ leads to a first order equation of the form 2.1.2.7: $w''_{xx} + bx^{n-1} w = 0$.

41. $x^3 y''''_{xxxx} + 2x^2 y'''_{xxx} - xy''_{xx} + y'_x - a^4 x^3 y = 0.$

Solution:

$$y = C_1 J_0(ax) + C_2 Y_0(ax) + C_3 I_0(ax) + C_4 K_0(ax)],$$

where J_0 and Y_0 are Bessel functions, I_0 and K_0 are modified Bessel functions.

42. $x^4 y''''_{xxxx} = ay.$

Solution:

$$y = C_1 x^{k_1} + C_2 x^{k_2} + C_3 x^{k_3} + C_4 x^{k_4},$$

where $k_{1,2} = \frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{a+1}}$, $k_{3,4} = \frac{3}{2} \pm \sqrt{\frac{5}{4} - \sqrt{a+1}}$.

43. $\boldsymbol{x^4 y''''_{xxxx} + A_3 x^3 y'''_{xxx} + A_2 x^2 y''_{xx} + A_1 x y'_x + A_0 y = 0}$.

The Euler equation.

The substitution $t = \ln|x|$ leads to a constant coefficient equation of the form 4.1.2.24:

$$y''''_{tttt} + (A_3 - 6)y'''_{ttt} + (11 - 3A_3 + A_2)y''_{tt} + (2A_3 - A_2 + A_1 - 6)y'_t + A_0 y = 0.$$

44. $\boldsymbol{x^4 y''''_{xxxx} - 2n(n+1)x^2 y''_{xx} + 4n(n+1)x y'_x + [ax^4 + n(n+1)(n+3)(n-2)]y = 0,}$ where n is a positive integer.

Solution:

$$y = x^{-n} \sum_{\nu=1}^{4} C_\nu \exp(\lambda_\nu x) P_\nu(x), \qquad a \neq 0,$$

where λ_ν are four different roots of the equation $\lambda^4 + a = 0$, and P_ν is some definite polynomial of the degree $\leq 4n$. For $a = 0$, we have the Euler equation 4.1.2.43.

45. $\boldsymbol{x^4 y''''_{xxxx} + 2(2-n)x^3 y'''_{xxx} + (1-n)(2-n)x^2 y''_{xx} - a^4 x^{2n} y = 0}$.

Solution:

$$y = \sqrt{x}\,[C_1 J_{1/n}(\xi) + C_2 Y_{1/n}(\xi) + C_3 I_{1/n}(\xi) + C_4 K_{1/n}(\xi)],$$

where $\xi = \dfrac{2a}{n} x^{n/2}$, J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

46. $\boldsymbol{x^4 y''''_{xxxx} + 6x^3 y'''_{xxx} + [4x^4 + (7 - a^2 - b^2)x^2]y''_{xx}}$
$\boldsymbol{\qquad + x(16x^2 + 1 - a^2 - b^2)y'_x + (8x^2 + a^2 b^2)y = 0}$.

Solution with $ab \neq 0$:

$$y = C_1 J_\mu(x) J_\nu(x) + C_2 J_\mu(x) Y_\nu(x) + C_3 Y_\mu(x) J_\nu(x) + C_4 Y_\mu(x) Y_\nu(x),$$

where J_μ and Y_μ are Bessel functions, $2\mu = a + b$, $2\nu = a - b$.

47. $\boldsymbol{x^6 y''''_{xxxx} = ay}$.

This is a special case of equation 5.1.2.24 with $n = 2$.

48. $\boldsymbol{x^8 y''''_{xxxx} = ay}$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ yields a constant coefficient equation: $w''''_{tttt} = aw$.

49. $\boldsymbol{x^8 y''''_{xxxx} + 4x^7 y'''_{xxx} = ay}$.

The substitution $w(x) = xy$ leads to an equation of the form 4.1.2.48: $x^8 w''''_{xxxx} = aw$.

50. $\boldsymbol{(ax+b)^4 (cx+d)^4 y''''_{xxxx} = ky}$.

The transformation

$$\xi = \ln \frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^3}$$

leads to a constant coefficient equation.

51. $(ax^2+bx+c)^4 y''''_{xxxx} = ky$.

The transformation

$$\xi = \int \frac{dx}{ax^2+bx+c}, \qquad w = \frac{y}{(ax^2+bx+c)^{3/2}}$$

leads to a constant coefficient equation:

$$w''''_{\xi\xi\xi\xi} - \tfrac{5}{2} D w''_{\xi\xi} + \left(\tfrac{9}{16} D^2 - k\right) w = 0, \qquad \text{where} \quad D = b^2 - 4ac.$$

52. $(ax+b)^2 (cx+d)^6 y''''_{xxxx} = ky$.

The transformation

$$\xi = \frac{ax+b}{cx+d}, \qquad w = \frac{y}{(cx+d)^3}$$

leads to an equation of the form 4.1.2.34:

$$\xi^2 w''''_{\xi\xi\xi\xi} = k\Delta^{-4} w, \qquad \text{where} \quad \Delta = ad - bc.$$

53. $x^9 y''''_{xxxx} = ay + bx^4$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 4.1.2.3: $w''''_{tttt} = atw + b$.

54. $(ax+b)^9 y''''_{xxxx} = (cx+d)y$.

The transformation

$$\xi = \frac{cx+d}{ax+b}, \qquad w = \frac{y}{(ax+b)^3}$$

leads to an equation of the form 4.1.2.3:

$$w''''_{\xi\xi\xi\xi} = \Delta^{-4} \xi w, \qquad \text{where} \quad \Delta = ad - bc.$$

4.1.3. Equations Containing Exponential, Hyperbolic, and Logarithmic Functions

1. $y''''_{xxxx} + a^3 y'_x + be^{ax}(a^2 - be^{ax})y = 0$.

The substitution $w = y''_{xx} + ay'_x + be^{ax}y$ leads to a second order linear equation of the form 2.1.3.10: $w''_{xx} - aw'_x + (a^2 - be^{ax})w = 0$.

2. $y''''_{xxxx} + ae^{\lambda x} y'_x - (abe^{\lambda x} + b^4)y = 0$.

Particular solution: $y_0 = e^{bx}$.

3. $y''''_{xxxx} + 2a\lambda e^{\lambda x} y'_x + a(\lambda^2 e^{\lambda x} - ae^{2\lambda x})y = 0$.

The substitution $w = y''_{xx} + ae^{\lambda x}y$ leads to a second order linear equation of the form 2.1.3.1: $w''_{xx} - ae^{\lambda x}w = 0$.

4. $y''''_{xxxx} + (ae^{\lambda x} + b^3)y'_x + abe^{\lambda x}y = 0$.

Particular solution: $y_0 = e^{-bx}$.

5. $\boldsymbol{y''''_{xxxx} + (ax+b)e^{\lambda x} y'_x - a e^{\lambda x} y = 0.}$

Particular solution: $y_0 = ax + b$.

6. $\boldsymbol{y''''_{xxxx} + a e^{\lambda x} y''_{xx} - b(a e^{\lambda x} + b) y = 0.}$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\,\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\,\bigr)$.

The substitution $w = y''_{xx} - by$ leads to a second order linear equation of the form 2.1.3.10: $w''_{xx} + (a e^{\lambda x} + b) w = 0$.

7. $\boldsymbol{y''''_{xxxx} + (a + b e^{\lambda x}) y''_{xx} + ab e^{\lambda x} y = 0.}$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second order linear equation of the form 2.1.3.1: $w''_{xx} + b e^{\lambda x} w = 0$.

8. $\boldsymbol{y''''_{xxxx} + 10 a e^{\lambda x} y''_{xx} + 10 a\lambda e^{\lambda x} y'_x + (3a\lambda^2 e^{\lambda x} + 9a^2 e^{2\lambda x}) y = 0.}$

This is a special case of equation 4.1.5.26 with $f(x) = a e^{\lambda x}$.

9. $\boldsymbol{y''''_{xxxx} + a y'''_{xxx} + b e^{\lambda x} y'_x + ab e^{\lambda x} y = 0.}$

Particular solution: $y_0 = e^{-ax}$.

10. $\boldsymbol{y''''_{xxxx} = a e^{\lambda x} y'''_{xxx} + b y'_x - ab e^{\lambda x} y.}$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where β_k are the roots of the cubic equation $\beta^3 - b = 0$.

11. $\boldsymbol{y''''_{xxxx} + a e^{\lambda x} y'''_{xxx} + b e^{\mu x} y''_{xx} + ac e^{\lambda x} y'_x + c(b e^{\mu x} - c) y = 0.}$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second order linear equation:

$$w''_{xx} + a e^{\lambda x} w'_x + c(b e^{\mu x} - c) w = 0.$$

12. $\boldsymbol{y''''_{xxxx} + a e^{\lambda x} y'''_{xxx} + (b e^{\mu x} + c) y''_{xx} + ac e^{\lambda x} y'_x + bc e^{\mu x} y = 0.}$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\,\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\,\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second order linear equation: $w''_{xx} + a e^{\lambda x} w'_x + b e^{\mu x} w = 0$.

13. $\boldsymbol{y''''_{xxxx} + a x^3 e^{\lambda x} y'''_{xxx} - 3a x^2 e^{\lambda x} y''_{xx} + 6 a x e^{\lambda x} y'_x - 6 a e^{\lambda x} y = 0.}$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$.

The substitution $w = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6x y'_x - 6y$ leads to a first order linear equation: $w'_x + a x^3 e^{\lambda x} w = 0$.

14. $(ae^x + b)y''''_{xxxx} = ae^x y.$

Particular solution: $y_0 = ae^x + b.$

15. $(ax^m + be^x + c)y''''_{xxxx} = be^x y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^x + c.$

16. $(ax^m e^x + b)y''''_{xxxx} = by, \qquad m = 0,\ 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^{-x}.$

17. $y''''_{xxxx} + b\exp(\lambda x^n)\, y''_{xx} + a[b\exp(\lambda x^n) - a]y = 0.$

This is a special case of equation 4.1.5.5 with $f(x) = b\exp(\lambda x^n)$.

18. $y''''_{xxxx} + [a + b\exp(\lambda x^n)]y''_{xx} + ab\exp(\lambda x^n)\, y = 0.$

This is a special case of equation 4.1.5.6 with $f(x) = b\exp(\lambda x^n)$.

19. $y''''_{xxxx} + b\sinh^n(\lambda x)\, y''_{xx} + a[b\sinh^n(\lambda x) - a]y = 0.$

This is a special case of equation 4.1.5.5 with $f(x) = b\sinh^n(\lambda x)$.

20. $y''''_{xxxx} + [a + b\sinh^n(\lambda x)]y''_{xx} + ab\sinh^n(\lambda x)\, y = 0.$

This is a special case of equation 4.1.5.6 with $f(x) = b\sinh^n(\lambda x)$.

21. $y''''_{xxxx} + b\cosh^n(\lambda x)\, y''_{xx} + a[b\cosh^n(\lambda x) - a]y = 0.$

This is a special case of equation 4.1.5.5 with $f(x) = b\cosh^n(\lambda x)$.

22. $y''''_{xxxx} + [a + b\cosh^n(\lambda x)]y''_{xx} + ab\cosh^n(\lambda x)\, y = 0.$

This is a special case of equation 4.1.5.6 with $f(x) = b\cosh^n(\lambda x)$.

23. $x^2 y''''_{xxxx} + 2axy'_x - a[1 + ax^2\ln^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\ln(bx)\, y$ leads to a second order linear equation: $w''_{xx} - a\ln(bx)w = 0$.

24. $y''''_{xxxx} + a\ln^n(\lambda x)(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y) = 0.$

This is a special case of equation 4.1.5.15 with $f(x) = a\ln^n(\lambda x)$.

4.1.4. Equation Containing Trigonometric Functions

1. $y''''_{xxxx} + 2ab\cos(bx)\, y'_x - a[b^2\sin(bx) + a\sin^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\sin(bx)\, y$ leads to a second order linear equation of the form 2.1.6.3: $w''_{xx} - a\sin(bx)w = 0$.

2. $y''''_{xxxx} + a\sin^n(\lambda x)\, y'_x + b[a\sin^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}.$

3. $\boldsymbol{y''''_{xxxx} + [a\sin^n(\lambda x) + b^3]y'_x + ab\sin^n(\lambda x)\,y = 0.}$

Particular solution: $y_0 = e^{-bx}$.

4. $\boldsymbol{y''''_{xxxx} + a\tan^n(\lambda x)\,y'_x + b[a\tan^n(\lambda x) - b^3]y = 0.}$

Particular solution: $y_0 = e^{-bx}$.

5. $\boldsymbol{y''''_{xxxx} + [a\tan^n(\lambda x) + b^3]y'_x + ab\tan^n(\lambda x)\,y = 0.}$

Particular solution: $y_0 = e^{-bx}$.

6. $\boldsymbol{y''''_{xxxx} + a\sin^n(\lambda x)\,y''_{xx} + b[a\sin^n(\lambda x) - b]y = 0.}$

The substitution $w = y''_{xx} + by$ leads to a second order linear equation: $w''_{xx} + [a\sin^n(\lambda x) - b]w = 0$.

7. $\boldsymbol{y''''_{xxxx} + [a + b\sin^n(\lambda x)]y''_{xx} + ab\sin^n(\lambda x)\,y = 0.}$

The substitution $w = y''_{xx} + ay$ leads to a second order linear equation: $w''_{xx} + b\sin^n(\lambda x)\,w = 0$.

8. $\boldsymbol{y''''_{xxxx} + b\tan^n(\lambda x)\,y''_{xx} + a[b\tan^n(\lambda x) - a]y = 0.}$

This is a special case of equation 4.1.5.5 with $f(x) = b\tan^n(\lambda x)$.

9. $\boldsymbol{y''''_{xxxx} + [a + b\tan^n(\lambda x)]y''_{xx} + ab\tan^n(\lambda x)\,y = 0.}$

This is a special case of equation 4.1.5.6 with $f(x) = b\tan^n(\lambda x)$.

10. $\boldsymbol{y''''_{xxxx} + a\sin^n(\lambda x)\,(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.}$

This is a special case of equation 4.1.5.15 with $f(x) = a\sin^n(\lambda x)$.

11. $\boldsymbol{y''''_{xxxx} = a\sin^n(\lambda x)\,y'''_{xxx} + by'_x - ab\sin^n(\lambda x)\,y.}$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where β_k are the roots of the cubic equation $\beta^3 - b = 0$.

12. $\boldsymbol{y''''_{xxxx} = a\tan^n(\lambda x)\,y'''_{xxx} + by'_x - ab\tan^n(\lambda x)\,y.}$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where β_k are the roots of the cubic equation $\beta^3 - b = 0$.

13. $\boldsymbol{x^2y''''_{xxxx} + a\sin^n(\lambda x)\,(x^2y''_{xx} - 4xy'_x + 6y) = 0.}$

The substitution $w = x^2y''_{xx} - 4xy'_x + 6y$ leads to a second order linear equation: $w''_{xx} + a\sin^n(\lambda x)\,w = 0$.

14. $\sin^4 x\, y''''_{xxxx} + 2\sin^3 x \cos x\, y'''_{xxx} + \sin^2 x\,(\sin^2 x - 3)\, y''_{xx}$
$\quad + \sin x \cos x\,(2\sin^2 x + 3)\, y'_x + (a^4 \sin^4 x - 3)y = 0.$

The equation of a loaded rigid spherical shell.
If $a^4 = 1 - \lambda^2$ then the equation can be written as

$$\mathbf{LL}(y) - \lambda^2 y = 0, \qquad \text{where} \quad \mathbf{L} \equiv \frac{d^2}{dx^2} + \cot x \frac{d}{dx} - \cot^2 x.$$

This equation falls into two second order equations:

$$\mathbf{L}(y) + \lambda y = 0, \qquad \mathbf{L}(y) - \lambda y = 0,$$

which differ only in the sign of parameter λ. The transformation $\xi = \sin^2 x$, $w = y/\sin x$ reduces the latter equations to the hypergeometric equations 2.1.2.158:

$$\xi(\xi - 1)w''_{\xi\xi} + (\tfrac{5}{2}\xi - 2)w'_\xi + \tfrac{1}{4}(1 \mp \lambda)w = 0.$$

15. $(a\cos x + b)y''''_{xxxx} = a\cos x\, y.$

Particular solution: $y_0 = a\cos x + b$.

16. $(ax^m + b\cos x)y''''_{xxxx} = b\cos x\, y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\cos x$.

17. $(a\sin x + b)y''''_{xxxx} = a\sin x\, y.$

Particular solution: $y_0 = a\sin x + b$.

18. $(ax^m + b\sin x)y''''_{xxxx} = b\sin x\, y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\sin x$.

4.1.5. Equations containing arbitrary functions

Notation: f, g, and h are arbitrary functions of x; a, b, and c are parameters.

1. $y''''_{xxxx} + fy'_x - a(f + a^3)y = 0.$

Particular solution: $y_0 = e^{ax}$.

2. $y''''_{xxxx} + (f + a^3)y'_x + afy = 0.$

Particular solution: $y_0 = e^{-ax}$.

3. $y''''_{xxxx} + xfy'_x - 3fy = 0.$

Particular solution: $y_0 = x^3$.
The substitution $z = xy'_x - 3y$ leads to a third order equation: $z'''_{xxx} + xfz = 0$.

4. $y''''_{xxxx} + (ax + b)fy'_x - afy = 0.$

Particular solution: $y_0 = ax + b$.

5. $\boldsymbol{y''''_{xxxx} + fy''_{xx} + a(f-a)y = 0}$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second order equation: $w''_{xx} + (f-a)w = 0$.

6. $\boldsymbol{y''''_{xxxx} + (f+a)y''_{xx} + afy = 0}$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second order equation: $w''_{xx} + fw = 0$.

7. $\boldsymbol{y''''_{xxxx} + f(x)(x^2 y''_{xx} - 4xy'_x + 6y) = 0}$.

Particular solutions: $y_1 = x^2$, $y_2 = x^3$.

The substitution $w = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second order linear equation: $w''_{xx} + x^2 fw = 0$.

8. $\boldsymbol{y''''_{xxxx} + (ax^2 + bx + c)fy''_{xx} - 2afy = 0}$.

Particular solution: $y_0 = ax^2 + bx + c$.

9. $\boldsymbol{y''''_{xxxx} + fy'''_{xxx} + xgy'_x - 2gy = 0}$.

Particular solution: $y_0 = x^2$.

10. $\boldsymbol{y''''_{xxxx} + fy'''_{xxx} - 2a^2 y''_{xx} - a^2 fy'_x + a^4 y = 0}$.

Particular solutions: $y_1 = e^{-ax}$, $y_2 = e^{ax}$.

11. $\boldsymbol{y''''_{xxxx} + fy'''_{xxx} + gy''_{xx} + afy'_x + a(g-a)y = 0}$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second order linear equation: $w''_{xx} + fw'_x + (g-a)w = 0$.

12. $\boldsymbol{y''''_{xxxx} + fy'''_{xxx} + (g+a)y''_{xx} + afy'_x + agy = 0}$.

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second order equation: $w''_{xx} + fw'_x + gw = 0$.

13. $\boldsymbol{y''''_{xxxx} + fy'''_{xxx} + gy''_{xx} + xhy'_x - hy = 0}$.

Particular solution: $y_0 = x$.

14. $\boldsymbol{y''''_{xxxx} + f(x)y'''_{xxx} + g(x)(x^2 y''_{xx} - 2xy'_x + 2y) = 0}$.

Particular solutions: $y_1 = x$, $y_2 = x^2$.

The substitution $z = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second order equation: $xz''_{xx} + (xf-2)z'_x + x^3 gz = 0$.

15. $\boldsymbol{y''''_{xxxx} + f(x)(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y) = 0}$.

Particular solutions: $y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3$.

The substitution $w = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y$ leads to a first order linear equation: $w'_x + x^3 f w = 0$.

16. $\boldsymbol{y''''_{xxxx} = f y'''_{xxx} + a y'_x - a f y = 0}$.

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ 3)$, where λ_k are the roots of the cubic equation $\lambda^3 - a = 0$.

17. $\boldsymbol{y''''_{xxxx} = (f - a) y'''_{xxx} + (af - b) y''_{xx} + (bf - c) y'_x + c f y = 0}$.

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ 3)$, where λ_k are the roots of the cubic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

18. $\boldsymbol{y''''_{xxxx} + (f + a) y'''_{xxx} + (af + g + axg) y''_{xx} + a^2 x g y'_x - a^2 g y = 0}$.

Particular solutions: $y_1 = x, \quad y_2 = e^{-ax}$.

19. $\boldsymbol{y''''_{xxxx} + (f_3 + a) y'''_{xxx} + (f_2 + a f_3) y''_{xx} + (f_1 + a f_2) y'_x + a f_1 y = 0}$,
where $f_k = f_k(x)$ $(k = 1,\ 2,\ 3)$.

Particular solution: $y_0 = e^{-ax}$.

20. $\boldsymbol{x y''''_{xxxx} + 4 y'''_{xxx} + a x y = f(x)}$.

The substitution $w(x) = xy$ leads to a nonhomogeneous constant-coefficient linear equation: $w''''_{xxxx} + aw = f(x)$.

21. $\boldsymbol{x y''''_{xxxx} + x f y'_x - [(x + 1) f + x + 4] y = 0}$.

Particular solution: $y_0 = x e^x$.

22. $\boldsymbol{x^2 y''''_{xxxx} + a x y'''_{xxx} + (x^2 f + b) y''_{xx} + (a - 4) x f y'_x + (b - 2a + 6) f y = 0}$.

The substitution $w = x^2 y''_{xx} + (a - 4) x y'_x + (b - 2a + 6) y$ leads to a second order equation: $w''_{xx} + f w = 0$.

23. $\boldsymbol{x^4 y''''_{xxxx} + a x^3 y'''_{xxx} + x f y'_x + (a - 3) f y = 0}$.

Particular solution: $y_0 = x^{3-a}$.

24. $\boldsymbol{y''''_{xxxx} + f y'_x + f'_x y = g}$.

Integrating yields $y'''_{xxx} + f y = \int g\,dx + C$.

25. $\boldsymbol{y''''_{xxxx} + 2 f'_x y'_x + (f''_{xx} - f^2) y = 0}$.

The substitution $w = y''_{xx} + f y$ leads to a second order equation: $w''_{xx} - f w = 0$.

26. $\boldsymbol{y''''_{xxxx} + 10 f y'''_{xxx} + 10 f'_x y'_x + (3 f''_{xx} + 9 f^2) y = 0}$.

Solution:

$$y = C_1 w_1^3 + C_2 w_1^2 w_2 + C_3 w_1 w_2^2 + C_4 w_2^3,$$

where w_1 and w_2 are nontrivial linearly-independent solutions of the second order equation $w''_{xx} + f w = 0$.

27. $\boldsymbol{y''''_{xxxx} + (f+g)y''_{xx} + 2f'_x y'_x + (f''_{xx} + fg)y = 0}$.

The substitution $w = y''_{xx} + fy$ leads to a second order equation: $w''_{xx} + gw = 0$.

28. $\boldsymbol{y''''_{xxxx} + 6fy'''_{xxx} + (4f'_x + 11f^2 + 10g)y''_{xx} + (f''_{xx} + 7ff'_x + 6f^3 + 30fg + 10g'_x)y'_x}$
$\boldsymbol{+\, 3(2f'_x g + 5fg'_x + 6f^2 g + g''_{xx} + 3g^2)y = 0}$.

Solution:

$$y = C_1 w_1^3 + C_2 w_1^2 w_2 + C_3 w_1 w_2^2 + C_4 w_2^3,$$

where w_1 and w_2 form a fundamental set of solutions of the second order equation $w''_{xx} + fw'_x + gw = 0$.

29. $\boldsymbol{(fy''_{xx})''_{xx} = 0}$.

The equation of transverse vibrations of a bar.

Solution: $\quad y = C_1 + C_2 x + \displaystyle\int_{x_0}^{x} \frac{x-t}{f(t)}(C_3 + C_4 t)\,dt.$

30. $\boldsymbol{y''''_{xxxx} + fy'_x + (f\tan x - 1)y = 0}$.

Particular solution: $\quad y_0 = \cos x$.

31. $\boldsymbol{y''''_{xxxx} + fy'_x - (1 + f\cot x)y = 0}$.

Particular solution: $\quad y_0 = \sin x$.

32. $\boldsymbol{y''''_{xxxx} = f(x)y}$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the similar form: $w''''_{tttt} = t^{-8} f(1/t)w$.

33. $\boldsymbol{y''''_{xxxx} = f\Big(\dfrac{ax+b}{cx+d}\Big)\dfrac{y}{(cx+d)^8}}$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^3}$ leads to a simpler equation:

$$w''''_{\xi\xi\xi\xi} = \Delta^{-4} f(\xi)w, \qquad \text{where} \quad \Delta = ad - bc.$$

34. $\boldsymbol{y''''_{xxxx} + f(x)y'_x + g(x)y + h(x) = 0}$.

The transformation $x = t^{-1}, y = wt^{-3}$ leads to an equation of the similar form:

$$w''''_{tttt} - t^{-6} f\Big(\frac{1}{t}\Big)w'_t + \Big[3t^{-7} f\Big(\frac{1}{t}\Big) + t^{-8} g\Big(\frac{1}{t}\Big)\Big]w + t^{-5} h\Big(\frac{1}{t}\Big) = 0.$$

4.1.6. Asymptotic Solutions

This subsection presents asymptotic solutions, as $\varepsilon \to 0$ ($\varepsilon > 0$), of some fourth-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being a real number.

1. Consider the equation

$$\varepsilon^4 y''''_{xxxx} - f(x)y = 0 \tag{1}$$

on a closed interval $a \le x \le b$. With the condition $f > 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by the formulae

$$y_1 = [f(x)]^{-3/8} \exp\biggl\{-\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\}, \quad y_2 = [f(x)]^{-3/8} \exp\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\},$$

$$y_3 = [f(x)]^{-3/8} \cos\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\}, \qquad y_4 = [f(x)]^{-3/8} \sin\biggl\{\frac{1}{\varepsilon}\int [f(x)]^{1/4}\,dx\biggr\}.$$

2. Now consider the "biquadratic" equation

$$\varepsilon^4 y''''_{xxxx} - 2\varepsilon^2 g(x) y''_{xx} - f(x)y = 0. \tag{2}$$

Introduce the notation

$$D(x) = [g(x)]^2 + f(x).$$

In the region where the conditions $f(x) \neq 0$ and $D(x) \neq 0$ are satisfied, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2) are described by the formulae

$$y_k = [\lambda_k(x)]^{-1/2}[D(x)]^{-1/4} \exp\biggl\{\frac{1}{\varepsilon}\int \lambda_k(x)\,dx - \frac{1}{2}\int \frac{[\lambda_k(x)]'_x}{\sqrt{D(x)}}\,dx\biggr\}; \qquad k = 1,\ 2,\ 3,\ 4.$$

where

$$\lambda_1(x) = \sqrt{g(x) + \sqrt{D(x)}}, \qquad \lambda_2(x) = -\sqrt{g(x) + \sqrt{D(x)}},$$

$$\lambda_3(x) = \sqrt{g(x) - \sqrt{D(x)}}, \qquad \lambda_4(x) = -\sqrt{g(x) - \sqrt{D(x)}}.$$

4.2. Nonlinear Equations

4.2.1. Equation Containing Power Functions

1. $\boldsymbol{y''''_{xxxx} = Ay^{-5/3}}$.

Multiply both sides of the equation by $y^{5/3}$ and differentiate the resulting expression with respect to x. We have

$$3yy^{(5)}_x + 5y'_x y''''_{xxxx} = 0.$$

Integrating the latter equation three times, we obtain a chain of equalities:

$$3yy''''_{xxxx} + 2y'_x y'''_{xxx} - (y''_{xx})^2 = 2C_2, \tag{1}$$

$$3yy'''_{xxx} - y'_x y''_{xx} = 2C_2 x + C_1, \tag{2}$$

$$3yy''_{xx} - 2(y'_x)^2 = C_2 x^2 + C_1 x + C_0, \tag{3}$$

where C_0, C_1, and C_2 are arbitrary constants. By eliminating the highest derivatives from (1)–(3) with the help of the original equation, we obtain a first order equation:

$$(2Py'_x - 3P'_x y)^2 = 9(C_1^2 - 4C_0C_2)y^2 - 2P^3 + 54APy^{4/3},$$

where $P = C_2x^2 + C_1x + C_0$. The substitution $y = (P/w)^{3/2}$ leads to an equation with separation of variables whereof integration finally yields

$$\int \bigl[9(C_1^2 - 4C_0C_2) + 54Aw - 2w^3\bigr]^{-1/2} \frac{dw}{w} \pm \int \frac{dx}{3P} = C_3.$$

2. $\boldsymbol{y''''_{xxxx} = Ay^m}$.

By integrating, we obtain ($m \neq -1$)

$$2y'_x y'''_{xxx} - (y''_{xx})^2 = \frac{2A}{m+1} y^{m+1} + \frac{4}{3}C,$$

where C is an arbitrary constant. The substitution $w(y) = (y'_x)^{3/2}$ leads to a second order equation:

$$w''_{yy} = \Bigl(\frac{3A}{2m+2} y^{m+1} + C\Bigr) w^{-5/3}.$$

The value $C = 0$ corresponds to the Emden—Fowler equation whose integrable cases are specified in Section 2.3 for some values of m (to those cases correspond three-parameter families of particular solutions of the original equation).

3. $\boldsymbol{y''''_{xxxx} = Ax^{-3m-5}y^m}$.

The transformation $x = t^{-1}$, $y = t^{-3}w(t)$ leads to an equation of the form 4.2.1.2: $w''''_{xxxx} = Aw^m$.

4. $\boldsymbol{y''''_{xxxx} = Ax^{-\frac{3m+5}{2}} y^m}$.

This is a special case of equation 4.2.3.3 with $f(w) = Aw^m$.

5. $\boldsymbol{y''''_{xxxx} = (ay + bx^k)^m, \qquad k = 0,\ 1,\ 2,\ 3}$.

The substitution $aw = ay + bx^k$ leads to an equation of the form 4.2.1.2: $w''''_{xxxx} = a^m w^m$.

6. $\boldsymbol{x^{3m+1}(ax+b)^4 y''''_{xxxx} = cy^m}$.

This is a special case of equation 4.2.3.5 with $f(w) = cw^m$.

7. $\boldsymbol{y''''_{xxxx} = (ax^2 + bx + c)^{-\frac{3m+5}{2}} y^m}$.

This is a special case of equation 4.2.3.6 with $f(w) = w^m$.

8. $\boldsymbol{y''''_{xxxx} - \frac{5}{2}ay''_{xx} + \frac{9}{16}a^2y = by^{-5/3}}$.

The transformation $\xi = e^{x\sqrt{a}}$, $w(\xi) = \xi^{3/2}y$ leads to an equation of the form 4.2.1.1: $w''''_{\xi\xi\xi\xi} = a^{-2}bw^{-5/3}$.

9. $\boldsymbol{xy''''_{xxxx} + 4y'''_{xxx} = Ax^{-5/3}y^{-5/3}}$.

The substitution $w(x) = xy$ leads to an equation of the form 4.2.1.1: $w''''_{xxxx} = Aw^{-5/3}$.

10. $\boldsymbol{xy''''_{xxxx} + 2y'''_{xxx} = a(xy'_x - y)^m}$.

The substitution $w(x) = xy'_x - y$ leads to a third order equation: $w'''_{xxx} = aw^m$ (Section 3.2 presents its solutions for $m = -\frac{7}{2}, -\frac{5}{2}, -2, -\frac{4}{3}, -\frac{7}{6}, -\frac{1}{2}$, 0, and 1).

11. $\boldsymbol{x^2y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^{-10/3}y^{-5/3}}$.

The substitution $w(x) = x^2y$ leads to an equation of the form 4.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

12. $\boldsymbol{x^4y''''_{xxxx} + 6x^3y'''_{xxx} + 7x^2y''_{xx} + xy'_x = ay^{-5/3}}$.

The substitution $t = \ln|x|$ leads to an equation of the form 4.2.1.1: $y''''_{xxxx} = ay^{-5/3}$.

13. $\boldsymbol{yy''''_{xxxx} = ay'_xy'''_{xxx}}$.

Having integrated this equation, we obtain the third order equation $y'''_{xxx} = Cy^a$ whose solvable cases are specified in Section 3.2.

14. $\boldsymbol{yy''''_{xxxx} + 4y'_xy'''_{xxx} + 3(y''_{xx})^2 = ax^n}$.

This is a special case of equation 4.2.3.22 with $f(x) = ax^n$.

15. $\boldsymbol{yy''''_{xxxx} + 4y'_xy'''_{xxx} + 3(y''_{xx})^2 = ay^{-10/3}}$.

The substitution $w = y^2$ leads to an equation of the form 4.2.1.1: $w''''_{xxxx} = 2aw^{-5/3}$.

16. $\boldsymbol{yy''''_{xxxx} + \frac{3}{2}y'_xy'''_{xxx} + \frac{1}{2}(y''_{xx})^2 = (ax+b)y^{-1/2}}$.

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a constant-coefficient fifth-order linear equation: $2x_t^{(5)} = ax + b$.

17. $\boldsymbol{y^3y''''_{xxxx} = 4y^2y'_xy'''_{xxx} + 3y^2(y''_{xx})^2 - 6(y'_x)^4}$.

This is a special case of equation 4.2.3.27 with $f \equiv 0$.

Solution in the parametric form:

$$x = \pm \int \frac{dx}{\sqrt{2\xi^4 + C_2\xi + C_1}} + C_3, \qquad y = C_4 \exp\left(\pm \int \frac{\xi\, d\xi}{\sqrt{2\xi^4 + C_2\xi + C_1}}\right).$$

18. $\boldsymbol{y''_{xx}y''''_{xxxx} = a(y'''_{xxx})^2}$.

Solution:

$$y = \begin{cases} C_0 + C_1x + (C_2 + C_3x)^{\frac{3-2a}{1-a}} & \text{if } a \neq 1, \\ C_0 + C_1x + C_2\exp(C_3x) & \text{if } a = 1. \end{cases}$$

19. $y''_{xx}y''''_{xxxx} - \frac{1}{2}(y'''_{xxx})^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma.$

Differentiating with respect to x yields

$$y''_{xx}\bigl(y^{(5)}_x - \alpha x - \beta\bigr) = 0.$$

By equating the expression in the parentheses to zero and integrating it, we find the solution:

$$y = \alpha\frac{x^6}{6!} + \beta\frac{x^5}{5!} + C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0.$$

The constants C_k and parameters α, β, and γ are related by the constraint

$$48C_2C_4 - 18C_3^2 = -\alpha C_0 + \beta C_1 + \gamma$$

obtained by means of substituting the solution into the original equation. In addition, there exists the solution

$$y = \widetilde{C}_1x + \widetilde{C}_0, \qquad \text{where} \quad \alpha\widetilde{C}_0 - \beta\widetilde{C}_1 - \gamma = 0.$$

20. $y''''_{xxxx} = ay^ky'_x(y'''_{xxx})^s.$

This is a special case of equation 4.2.3.29 with $f(y) = Ay^k$, $g(w) = w^s$. For $k = -1$ and $s = 1$, see equation 4.2.1.13.

The first integral has the form:

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - \frac{A}{k+1}y^{k+1} = C \quad \text{if } k \neq -1,\ s \neq 1; \tag{1}$$

$$\ln y'''_{xxx} - \frac{A}{k+1}y^{k+1} = C \quad \text{if } k \neq -1,\ s = 1; \tag{2}$$

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - A\ln y = C \quad \text{if } k = -1,\ s = 1. \tag{3}$$

For $C = 0$, equality (1) is changing to the equation

$$y'''_{xxx} = \left[\frac{A(1-s)}{k+1}\right]^{\frac{1}{1-s}} y^{\frac{k+1}{1-s}}$$

which is discussed in Section 3.2 (the solutions given there generate 3-parametric families of particular solutions of the original equation for $k = (1-s)\beta - 1$, where $\beta = -\frac{7}{2}, -\frac{5}{2}, -2, -\frac{4}{3}, -\frac{7}{6}, -\frac{1}{2}, 0$, and 1).

4.2.2. Equations Containing Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $y''''_{xxxx} = ae^{\lambda y}.$

This is a special case of equation 4.2.3.1 with $f(y) = ae^{\lambda y}$.

2. $y''''_{xxxx} = a(y + be^x)^{-5/3} - be^x.$

The substitution $w = y + be^x$ leads to an equation of the form 4.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

3. $y''''_{xxxx} = a(y + be^x)^m - be^x.$

The substitution $w = y + be^x$ leads to an equation of the form 4.2.1.2: $w''''_{xxxx} = aw^m$.

4. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2 y''_{xx} - 4\lambda^3 y'_x + \lambda^4 y = a\exp(\frac{8}{3}\lambda x)y^{-5/3}.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 4.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

5. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2 y''_{xx} - 4\lambda^3 y'_x + \lambda^4 y = ae^{\lambda(1-m)x}y^m.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 4.2.1.2: $w''''_{xxxx} = aw^m$.

6. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ae^{\lambda x}.$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}e^{\lambda x}$.

7. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cosh(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cosh(\lambda x)$.

8. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tanh^m(\lambda x).$

This is a special case of equation 4.2.3.22 with $f(x) = a\tanh^m(\lambda x)$.

9. $y''''_{xxxx} = a\ln^m(by).$

This is a special case of equation 4.2.3.1 with $f(y) = a\ln^m(by)$.

10. $y''''_{xxxx} = ax^{-5}(\ln y - 3\ln x).$

This is a special case of equation 4.2.3.2 with $f(w) = a\ln w$.

11. $y''''_{xxxx} = ax^{-5/2}(2\ln y - 3\ln x).$

This is a special case of equation 4.2.3.3 with $f(w) = 2a\ln w$.

12. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\ln^m(\lambda x).$

This is a special case of equation 4.2.3.22 with $f(x) = a\ln^m(\lambda x)$.

13. $y''''_{xxxx} = a\cos^m(\lambda y).$

This is a special case of equation 4.2.3.1 with $f(y) = a\cos^m(\lambda y)$.

14. $y''''_{xxxx} = a\tan^m(\lambda y).$

This is a special case of equation 4.2.3.1 with $f(y) = a\tan^m(\lambda y)$.

15. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cos(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cos(\lambda x)$.

16. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tan^m(\lambda x).$

This is a special case of equation 4.2.3.22 with $f(x) = a\tan^m(\lambda x)$.

4.2.3. Equations Containing Arbitrary Functions

1. $y''''_{xxxx} = f(y)$.

By integrating, we obtain

$$2y'_x y'''_{xxx} - (y''_{xx})^2 = 2\int f(y)\,dy + 2C.$$

The substitution $w(y) = |y'_x|^{3/2}$ leads to a second order equation:

$$w''_{yy} = \frac{3}{2}\left[\int f(y)\,dy + C\right] w^{-5/3}.$$

2. $y''''_{xxxx} = x^{-5} f(yx^{-3})$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 4.2.3.1: $w''''_{tttt} = f(w)$.

3. $y''''_{xxxx} = x^{-5/2} f(yx^{-3/2})$.

The transformation $x = e^t$, $y = x^{3/2}w$ leads to an equation of the form 4.2.3.14: $w''''_{tttt} - \frac{5}{2}w''_{tt} = -\frac{9}{16}w + f(w)$.

4. $y''''_{xxxx} = f(y + \alpha x^3 + \beta x^2 + \gamma x + \delta)$.

The substitution $w = y + \alpha x^3 + \beta x^2 + \gamma x + \delta$ leads to an equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

5. $x(ax+b)^4 y''''_{xxxx} = f(yx^{-3})$.

The transformation $\xi = \ln\dfrac{ax+b}{x}$, $w = \dfrac{y}{x^3}$ leads to an autonomous equation of the form 4.2.3.34.

6. $y''''_{xxxx} = (ax^2+bx+c)^{-5/2} f\left(\dfrac{y}{(ax^2+bx+c)^{3/2}}\right)$.

1°. The transformation

$$\xi = \int \frac{dx}{ax^2+bx+c}, \qquad w = \frac{y}{(ax^2+bx+c)^{3/2}}$$

leads to an autonomous equation of the form 4.2.3.14 for $w = w(\xi)$:

$$w''''_{xxxx} - \tfrac{5}{2}\Delta w''_{\xi\xi} + \tfrac{9}{16}\Delta^2 w = f(w), \qquad \text{where} \quad \Delta = b^2 - 4ac.$$

Therefore, having integrated the latter equation, we obtain

$$w'_\xi w'''_{\xi\xi\xi} - \tfrac{1}{2}(w''_{\xi\xi})^2 - \tfrac{5}{4}\Delta(w'_\xi)^2 = -\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C.$$

The substitution $z(w) = |w'_\xi|^{3/2}$ leads to a second order equation:

$$z''_{ww} = \tfrac{15}{8}\Delta z^{-1/3} + \tfrac{3}{2}\left[-\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C\right] z^{-5/3}.$$

2°. The first integral of the original equation has the form

$$(Py'_x - \tfrac{3}{2}P'_x y)y'''_{xxx} - \tfrac{1}{2}P(y''_{xx})^2 + \tfrac{1}{2}P'_x y'_x y''_{xx} + 3ayy''_{xx} - 2a(y'_x)^2 = \int f(w)\,dw + C,$$

where $P = ax^2 + bx + c$, $w = yP^{-3/2}$.

7. $\boldsymbol{y''''_{xxxx} = f(y + ae^{x}) - ae^{x}}$.

The substitution $w = y + ae^x$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

8. $\boldsymbol{y''''_{xxxx} = f(y + a\cosh x) - a\cosh x}$.

The substitution $w = y + a\cosh x$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

9. $\boldsymbol{y''''_{xxxx} = f(y + a\sinh x) - a\sinh x}$.

The substitution $w = y + a\sinh x$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

10. $\boldsymbol{y''''_{xxxx} = f(y + a\cos x) - a\cos x}$.

The substitution $w = y + a\cos x$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

11. $\boldsymbol{y''''_{xxxx} = f(y + a\sin x) - a\sin x}$.

The substitution $w = y + a\sin x$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

12. $\boldsymbol{y''''_{xxxx} = f(y)y'_x + g(x)}$.

By integrating, we find

$$y'''_{xxx} = \int f(y)\,dy + \int g(x)\,dx + C.$$

For $g(x) \equiv 0$, the order of this equation can lowered by one with the help of the substitution $w(y) = y'_x$.

13. $\boldsymbol{y''''_{xxxx} = x^{-4} f(xy'_x - y)}$.

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third order autonomous equation of the form 3.5.5.9: $w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w)$.

14. $\boldsymbol{y''''_{xxxx} + ay''_{xx} = f(y)}$.

Having integrated this equation, we obtain

$$2y'_x y'''_{xxx} - \left(y''_{xx}\right)^2 + a(y'_x)^2 = 2\int f(y)\,dy + 2C,$$

where C is an arbitrary constant. The substitution $w(y) = |y'_x|^{3/2}$ leads to a second order equation:

$$w''_{yy} = -\tfrac{3}{4}aw^{-1/3} + \tfrac{3}{2}\left[\int f(y)\,dy + C\right]w^{-5/3}.$$

15. $y''''_{xxxx} = x^{-2} f(xy'_x - y) y''_{xx}$.

The substitution $t = \ln|x|$, $w = xy'_x - y$ leads to a third order equation:

$$w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w) w'_t.$$

Integrating it, we obtain a second order automous equation:

$$w''_{tt} - 5w'_t + 6w = \int f(w)\,dw + C.$$

The substitution $z(w) = \frac{1}{5} w'_t$ leads to the Abel equation of the second kind:

$$zz'_w - z = \tfrac{1}{25}\left[-6w + \int f(w)\,dw + C\right]$$

(see Section 1.3).

16. $y''''_{xxxx} = x^m f(x^2 y''_{xx} - 2xy'_x + 2y)$.

The substitution $w = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second order equation: $xw''_{xx} - 2w'_x = x^{m+3} f(w)$.

For $m = -4$, the substitution $z(w) = \frac{1}{3} xw'_x$ leads to the Abel equation of the second kind: $zz'_w - z = \frac{1}{9} f(w)$ (see Subsection 1.3.1).

17. $y''''_{xxxx} + ay'''_{xxx} + by''_{xx} + cy'_x = e^{\lambda x} f(ye^{-\lambda x})$.

The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation:

$$\begin{aligned} &w''''_{xxxx} + (4\lambda + a) w'''_{xxx} + (6\lambda^2 + 3a\lambda + b) w''_{xx} \\ &\qquad + (4\lambda^3 + 3a\lambda^2 + 2b\lambda + c) w'_x + (\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda) w = f(w), \end{aligned}$$

which can be reduced to a third order equation by means of the substitution $z(w) = w'_x$. For $a = -4\lambda$ and $c = 8\lambda^3 - 2b\lambda$, the above equation coincides, to a precision of the notation, with the equation 4.2.3.14 and can be reduced to a second order equation.

18. $xy''''_{xxxx} + 4y'''_{xxx} = f(xy)$.

The substitution $w(x) = xy$ leads to an equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

19. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = f(x^2 y)$.

The substitution $w(x) = x^2 y$ leads to an autonomous equation of the form 4.2.3.1: $w''''_{xxxx} = f(w)$.

20. $x^4 y''''_{xxxx} + a_3 x^3 y'''_{xxx} + a_2 x^2 y''_{xx} + a_1 xy'_x = f(y)$.

The substitution $t = \ln|x|$ leads to an autonomous equation:

$$y''''_{tttt} + (a_3 - 6) y'''_{ttt} + (11 - 3a_3 + a_2) y''_{tt} + (2a_3 - a_2 + a_1 - 6) y'_t = f(y),$$

the order of which can lowered with the help of the substitution $w(y) = y'_t$. For $a_3 = 6$ and $a_1 = a_2 - 6$, the latter equation coincides, to a precision of the notation, with the equation 4.2.3.14 and can be reduced to a second order equation.

21. $\boldsymbol{x^4 y''''_{xxxx} + a x^3 y'''_{xxx} + b x^2 y''_{xx} + c x y'_x + s y = x^{-k} f(y x^k)}.$

The transformation $t = \ln x$, $w = y x^k$ leads to an autonomous equation of the form 4.2.3.34.

22. $\boldsymbol{y y''''_{xxxx} + 4 y'_x y'''_{xxx} + 3 (y''_{xx})^2 = f(x)}.$

Solution: $y^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + \frac{1}{3} \int_{x_0}^{x} (x - t)^3 f(t)\, dt.$

23. $\boldsymbol{y y''''_{xxxx} + a y'_x y'''_{xxx} + (a - 1)(y''_{xx})^2 = f(x)}.$

Having integrated this equation, we find

$$y y''_{xx} + \frac{a - 2}{2} (y'_x)^2 = C_1 x + C_0 + \int_{x_0}^{x} (x - t) f(t)\, dt.$$

24. $\boldsymbol{y y''''_{xxxx} + (4 y'_x + f y) y'''_{xxx} + 3 (y''_{xx})^2 + 3 f y'_x y''_{xx} + g = 0, \qquad f = f(x),\ g = g(x)}.$

The substitution $w = (y y'_x)''_{xx}$ leads to a first order linear equation: $w'_x + f w + g = 0$.

Solution:

$$y^2 = C_2 x^2 + C_1 x + C_0 + \int_{x_0}^{x} (x - t)^2 w(t)\, dt,$$

where $w(x) = e^{-F(x)} \bigl[C_3 - \int e^{F(x)} g(x)\, dx\bigr]$, $F(x) = \int f(x)\, dx$; x_0 is any number.

25. $\boldsymbol{y y''''_{xxxx} + (4 y'_x + f y) y'''_{xxx} + 3 (y''_{xx})^2 + (3 f y'_x + g y) y''_{xx} + g (y'_x)^2 + h y y'_x + s = 0},$ where $f = f(x)$, $g = g(x)$, $h = h(x)$, $s = s(x)$.

The substitution $w = y y'_x$ leads to a nonhomogeneous third-order linear equation: $w'''_{xxx} + f w''_{xx} + g w'_x + h w + s = 0$.

26. $\boldsymbol{(y + a x + b) y''''_{xxxx} + 4 (y'_x + a) y'''_{xxx} + 3 (y''_{xx})^2 = f(x)}.$

Solution: $(y + a x + b)^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + \frac{1}{3} \int_{x_0}^{x} (x - t)^3 f(t)\, dt.$

27. $\boldsymbol{y y''''_{xxxx} = 4 y'_x y'''_{xxx} + 3 (y''_{xx})^2 - 6 \frac{(y'_x)^4}{y^2} + \bigl[y y''_{xx} - (y'_x)^2\bigr] f\Bigl(\frac{y'_x}{y}\Bigr)}.$

The transformation $\xi = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y} - \Bigl(\dfrac{y'_x}{y}\Bigr)^2$ leads to a second order linear equation for w^2: $(w^2)''_{\xi\xi} = 24 \xi^2 + 2 f(\xi)$. Integrating yields

$$w^2 = C_2 \xi + C_1 + 2 \xi^4 + 2 \int_{\xi_0}^{\xi} (\xi - t) f(t)\, dt.$$

Taking into account that $\xi'_x = w$, $y'_x = \xi y$, $y'_\xi = \xi y / w$, we find the solution in the parametric form:

$$x = \int \frac{d\xi}{w} + C_3, \qquad y = C_4 \exp\Bigl(\int \frac{\xi\, d\xi}{w}\Bigr),$$

where

$$w = \pm \sqrt{C_2 \xi + C_1 + 2 \xi^4 + 2 \int_{\xi_0}^{\xi} (\xi - t) f(t)\, dt}.$$

28. $y''_{xx}y''''_{xxxx} - 3(y'''_{xxx})^2 = f(xy'_x - y)(y''_{xx})^5.$

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ leads to an equation of the form 4.2.3.1: $u''''_{tttt} = -f(u)$.

29. $y''''_{xxxx} = f(y)y'_x g(y'''_{xxx}).$

By integrating, we obtain a third order autonomous equation:

$$\int \frac{dw}{g(w)} = \int f(y)\,dy + C, \qquad \text{where} \quad w = y'''_{xxx},$$

the order of which can be lowered by means of the substitution $z(y) = y'_x$.

30. $xy''''_{xxxx} + 2y'''_{xxx} = (xy''_{xx})^{-5} f\left(\dfrac{xy''_{xx}}{\sqrt{xy'_x - y}}\right).$

The substitution $w(x) = xy'_x - y$ leads to a third order equation of the form 3.5.2.11:

$$w'''_{xxx} = w^{-5/2} F\left(\frac{w'_x}{\sqrt{w}}\right), \qquad \text{where} \quad F(\xi) = \xi^{-5} f(\xi).$$

31. $x^2y''''_{xxxx} + 2xy'''_{xxx} = f(x^2y''_{xx} - 2xy'_x + 2y)g(x^2y'''_{xxx}).$

The substitution $w(x) = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second order equation of the form 2.9.4.2: $w''_{xx} = f(w)g(w'_x)$.

32. $y''''_{xxxx} = f(x)g(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y).$

The substitution $w(x) = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first order equation with separation of variables: $w'_x = x^3 f(x)g(w)$.

33. $y''''_{xxxx} = f(x,\ y'_x,\ y''_{xx},\ y'''_{xxx}).$

The substitution $w(x) = y'_x$ leads to a third order equation: $w'''_{xxx} = f(x,\ w,\ w'_x,\ w''_{xx})$.

34. $y''''_{xxxx} = f(y,\ y'_x,\ y''_{xx},\ y'''_{xxx}).$

Autonomous equation.

The substitution $w(y) = (y'_x)^2$ leads to a third order equation:

$$ww'''_{yyy} + \tfrac{1}{2}w'_y w''_{yy} = 2f(y,\ \pm\sqrt{w},\ \tfrac{1}{2}w'_y,\ \pm\tfrac{1}{2}\sqrt{w}\,w''_{yy}).$$

35. $y''''_{xxxx} = yf\left(\dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y},\ \dfrac{y'''_{xxx}}{y}\right).$

The transformation $\xi = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y} - \left(\dfrac{y'_x}{y}\right)^2$ leads to a second order equation:

$$w^2w''_{\xi\xi} + w(w'_\xi)^2 + 4\xi ww'_\xi + 3w^2 + 6\xi^2 w + \xi^4 = f(\xi,\ w + \xi^2,\ ww'_\xi + 3\xi w + \xi^3).$$

36. $y''''_{xxxx} = yx^{-4} f\left(x^k y^m,\ \dfrac{xy'_x}{y},\ \dfrac{x^2y''_{xx}}{y},\ \dfrac{x^3y'''_{xxx}}{y}\right).$

The homogeneous equation in the extended sense.

The transformation $t = x^k y^m$, $z = \dfrac{xy'_x}{y}$ leads to a third order equation.

37. $\boldsymbol{y''''_{xxxx} = yx^{-4} f\left(\dfrac{xy'_x}{y},\ \dfrac{x^2 y''_{xx}}{y},\ \dfrac{x^3 y'''_{xxx}}{y}\right)}.$

The transformation $z = \dfrac{xy'_x}{y}$, $w = \dfrac{x^2 y''_{xx}}{y}$ leads to a second order equation.

38. $\boldsymbol{y''''_{xxxx} = x^{-4} f(x^m e^{\alpha y},\ xy'_x,\ x^2 y''_{xx},\ x^3 y'''_{xxx})}.$

The transformation $z = x^m e^{\alpha y}$, $w = xy'_x$ leads to a third order equation.

39. $\boldsymbol{y''''_{xxxx} = yf\left(e^{\alpha x} y^m,\ \dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y},\ \dfrac{y'''_{xxx}}{y}\right)}.$

The transformation $z = e^{\alpha x} y^m$, $w = y'_x/y$ leads to a third order equation.