Supplement 2
Some Special Functions

In Supplement 2, n is a positive integer, unless otherwise specified.

2.1. Gamma-function

The gamma-function $\Gamma(z)$ is an analitic function of z everywhere, except for the points $z = 0, -1, -2, \ldots$

For $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For $-(n+1) < \operatorname{Re} z < -n$, where n is an integer,

$$\Gamma(z) = \int_0^\infty \left[e^{-t} - \sum_{m=0}^{n} \frac{(-1)^m}{m!} \right] t^{z-1}\, dt.$$

The gamma-function possesses the following properties:

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(n+1) = n!, \qquad \Gamma(1) = \Gamma(2) = 1.$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \Gamma\Big(\frac{1}{2}+z\Big)\Gamma\Big(\frac{1}{2}-z\Big) = \frac{\pi}{\cos(\pi z)},$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big), \qquad \Gamma(3z) = \frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\Big(z+\frac{1}{3}\Big)\Gamma\Big(z+\frac{2}{3}\Big),$$

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\Big(z+\frac{k}{n}\Big),$$

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}, \qquad \Gamma\Big(n+\frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$
$$\Gamma\Big(-\frac{1}{2}\Big) = -2\sqrt{\pi}, \; \Gamma\Big(n-\frac{1}{2}\Big) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!},$$

$$\frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n, \qquad \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)} = C_z^w.$$

2.2. Bessel functions J_ν and Y_ν

2.2.1. Basic Formulae

The Bessel functions of the first and second kind J_ν and Y_ν (function Y_ν is also called the Neumann function) are solutions of the Bessel equation 2.1.2.121 and are defined by the formulae

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}.$$

The formulae for $Y_\nu(x)$ is valid for $\nu \neq 0, \pm1, \pm2, \ldots$; (see below for the integral representation of $Y_\nu(x)$, as well as equation 2.1.2.121 for $\nu = 0, \pm1, \pm2, \ldots$).

Function $Z_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$ is reffered to as the cylindric function.

The Bessel functions possess the following properties:

$$2\nu Z_\nu(x) = x[Z_{\nu-1}(x) + Z_{\nu+1}(x)],$$

$$\frac{d}{dx} Z_\nu = \frac{1}{2}[Z_{\nu-1}(x) - Z_{\nu+1}(x) = \pm\frac{1}{x}[\nu Z_\nu(x) - Z_{\nu\pm1}],$$

$$\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x), \qquad \frac{d}{dx}[x^{-\nu} Z_\nu(x)] = -x^{-\nu} Z_{\nu+1}(x),$$

$$\left(\frac{d}{x\,dx}\right)^n [x^\nu J_\nu(x)] = x^{\nu-n} J_{\nu-n}(x), \qquad \left(\frac{d}{x\,dx}\right)^n [x^{-\nu} J_\nu(x)] = (-1)^n x^{-\nu-n} J_{\nu+n}(x),$$

$$J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x) \qquad n = 0,\ 1,\ 2,\ \ldots$$

2.2.2. Bessel functions for $\nu = \pm n \pm \frac{1}{2}$; $n = 0, 1, 2, \ldots$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{1}{x}\sin x - \cos x\right), \qquad J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{1}{x}\cos x - \sin x\right),$$

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\sin\left(x - \frac{n\pi}{2}\right) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} \right.$$
$$\left. + \cos\left(x - \frac{n\pi}{2}\right) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$J_{-n-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x + \frac{n\pi}{2}\right) \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+2k)!}{(2k)!\,(n-2k)!\,(2x)^{2k}} \right.$$
$$\left. - \sin\left(x + \frac{n\pi}{2}\right) \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k (n+2k+1)!}{(2k+1)!\,(n-2k-1)!\,(2x)^{2k+1}}\right],$$

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x, \qquad Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x), \qquad Y_{-n-1/2}(x) = (-1)^n J_{n+1/2}(x),$$

2.2.3. Wronskians an Similar Formulae

Notation: $W(f,g) = fg'_x - f'_x g$.

$$W(J_\nu, J_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \qquad W(J_\nu, Y_\nu) = \frac{2}{\pi x},$$

$$J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin(\pi\nu)}{\pi x}, \quad J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -\frac{2}{\pi x}.$$

2.2.4. Integral Representation

Functions J_ν and Y_ν may be expressed in terms of definite integrals (with $x > 0$):

$$\pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu \int_0^\infty \exp(-x\sinh t - \nu t)\,dt,$$
$$\pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)e^{-x\sinh t}\,dt.$$

For $|\nu| < \frac{1}{2}$, $x > 0$,

$$J_\nu(x) = \frac{2^{1-\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\sin(xt)\,dt}{(t^2-1)^{\nu+1/2}}, \quad Y_\nu(x) = -\frac{2^{1-\nu}x^{-\nu}}{\pi^{1/2}\Gamma(\frac{1}{2}-\nu)}\int_1^\infty \frac{\cos(xt)\,dt}{(t^2-1)^{\nu+1/2}}.$$

For $\nu = 0$, $x > 0$,

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\,dt, \qquad Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt.$$

2.2.5. Integrals with Bessel Functions on Closed Intervals

$$\int_0^x x^\lambda J_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\Gamma(\nu+1)}F\Big(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ -\frac{x^2}{4}\Big),$$

where $\operatorname{Re}(\lambda+\nu) > -1$, $F(a,b,c;x)$ is the hypergeometric series (see equation 2.1.2.158),

$$\int_0^x x^\lambda Y_\nu(x)\,dx = \frac{\cos(\nu\pi)\Gamma(-\nu)}{2^\nu\pi(\lambda+\nu+1)}x^{\lambda+\nu+1}F\Big(\frac{\lambda+\nu+1}{2},\ \nu+1,\frac{\lambda+\nu+3}{2},\ -\frac{x^2}{4}\Big)$$
$$- \frac{2^\nu\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}F\Big(\frac{\lambda-\nu+1}{2},\ 1-\nu,\frac{\lambda-\nu+3}{2},\ -\frac{x^2}{4}\Big),$$

where $\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1$.

2.2.6. Asymptotic Expansion, as $|x| \to \infty$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\bigg\{\cos\Big(\frac{4x-2\nu\pi-\pi}{4}\Big)\bigg[\sum_{m=0}^{M-1}(-1)^m(\nu,2m)(2x)^{-2m} + O(|x|^{-2M})\bigg]$$
$$- \sin\Big(\frac{4x-2\nu\pi-\pi}{4}\Big)\bigg[\sum_{m=0}^{M-1}(-1)^m(\nu,2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1})\bigg]\bigg\},$$

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \left\{ \sin\left(\frac{4x - 2\nu\pi - \pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m (\nu, 2m)(2x)^{-2m} + O(|x|^{-2M}) \right] \right.$$

$$\left. + \cos\left(\frac{4x - 2\nu\pi - \pi}{4}\right) \left[\sum_{m=0}^{M-1} (-1)^m (\nu, 2m+1)(2x)^{-2m-1} + O(|x|^{-2M-1}) \right] \right\},$$

where $(\nu, m) = \frac{1}{2^{2m} m!}(4\nu^2 - 1)(4\nu^2 - 3^2) \ldots [4\nu^2 - (2m-1)^2] = \frac{\Gamma(\frac{1}{2} + \nu + m)}{m!\,\Gamma(\frac{1}{2} + \nu - m)}.$

2.3. Modified Bessel Functions I_ν and K_ν

2.3.1. Basic Formulae

The modified Bessel functions of the first and second kind I_ν and K_ν (function K_ν is also called the Basset function) are solutions the modified Bessel equation 2.1.2.122 and are defined by the formulae

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu + k + 1)}, \qquad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu} - I_\nu}{\sin \pi\nu},$$

(see equation 2.1.2.122 with $\nu = 0,\ 1,\ 2,\ \ldots$ for the representation of $K_\nu(x)$).

The modified Bessel functions possess the following properties:

$$K_{-\nu}(x) = K_\nu(x); \qquad I_{-n}(x) = (-1)^n I_n(x), \quad n = 0,\ 1,\ 2,\ \ldots$$

$$2\nu I_\nu(x) = x[I_{\nu-1}(x) - I_{\nu+1}(x)], \qquad 2\nu K_\nu(x) = -x[K_{\nu-1}(x) - K_{\nu+1}(x)],$$

$$\frac{d}{dx} I_\nu(x) = \frac{1}{2}[I_{\nu-1}(x) + I_{\nu+1}(x)], \qquad \frac{d}{dx} K_\nu(x) = -\frac{1}{2}[K_{\nu-1}(x) + K_{\nu+1}(x)],$$

2.3.2. Modified Bessel Functions for $\nu = \pm n \pm \frac{1}{2}$; $n = 0,\ 1,\ 2,\ \ldots$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x,$$

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{1}{x} \sinh x + \cosh x\right), \qquad I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{1}{x} \cosh x + \sinh x\right),$$

$$I_{n+1/2}(x) = \frac{1}{\sqrt{2\pi x}} \left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} - (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k} \right],$$

$$I_{-n-1/2}(x) = \frac{1}{\sqrt{2\pi x}} \left[e^x \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!\,(n-k)!\,(2x)^k} + (-1)^n e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k} \right],$$

$$K_{\pm 1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}, \qquad K_{\pm 3/2}(x) = \frac{\pi}{2x}\left(1 + \frac{1}{x}\right) e^{-x},$$

$$K_{n+1/2}(x) = K_{-n-1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k}.$$

2.3.3. Wronskians and Similar Formulae

Notation: $W(f,g) = fg'_x - f'_x g$.

$$W(I_\nu, I_{-\nu}) = -\frac{2}{\pi x}\sin(\pi\nu), \qquad W(I_\nu, K_\nu) = -\frac{1}{x},$$

$$I_\nu(x)I_{-\nu+1}(x) - I_{-\nu}(x)I_{\nu-1}(x) = -\frac{2\sin(\pi\nu)}{\pi x}, \quad I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = \frac{1}{x}.$$

2.3.4. Integral Representation

Functions I_ν and K_ν may be expressed in terms of definite integrals (with $x>0$, $\nu>-\frac{1}{2}$):

$$I_\nu(x) = \frac{x^\nu}{\pi^{1/2}2^\nu\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}\exp(-xt)(1-t^2)^{\nu-1/2}\,dt,$$

$$K_\nu(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\nu t)\,dt.$$

2.3.5. Integrals with Modified Bessel Functions on Closed Intervals

$$\int_0^x x^\lambda I_\nu(x)\,dx = \frac{x^{\lambda+\nu+1}}{2^\nu(\lambda+\nu+1)\Gamma(\nu+1)}F\Big(\frac{\lambda+\nu+1}{2},\ \frac{\lambda+\nu+3}{2},\ \nu+1;\ \frac{x^2}{4}\Big),$$

where $\operatorname{Re}(\lambda+\nu) > -1$, $F(a,b,c;x)$ is the hypergeometric series (see equation 2.1.2.158),

$$\int_0^x x^\lambda K_\nu(x)\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{\lambda-\nu+1}x^{\lambda-\nu+1}F\Big(\frac{\lambda-\nu+1}{2},\ 1-\nu,\ \frac{\lambda-\nu+3}{2},\ \frac{x^2}{4}\Big)$$
$$+\frac{2^{-\nu-1}\Gamma(-\nu)}{\lambda+\nu+1}x^{\lambda+\nu+1}F\Big(\frac{\lambda+\nu+1}{2},\ 1+\nu,\ \frac{\lambda+\nu+3}{2},\ \frac{x^2}{4}\Big),$$

where $\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1$.

2.3.6. Asymptotic Expansion, as $x \to \infty$

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\Big[1+\sum_{m=1}^{M}(-1)^m\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m} + O(x^{-M-1})\Big],$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\Big[1+\sum_{m=1}^{M}\frac{(4\nu^2-1)(4\nu^2-3^2)\dots[4\nu^2-(2m-1)^2]}{m!\,(8x)^m} + O(x^{-M-1})\Big].$$

2.4. Degenerate Hypergeometric Functions

2.4.1. Definitions

The degenerate hypergeometric functions $\Phi(a,b;x)$ and $\Psi(a,b;x)$ are solutions of the degenerate hypergeometric equation 2.1.2.65.

If $b \neq 0, -1, -2, -3, \ldots$, function $\Phi(a, b; x)$ is expressed in terms of Kummer's series:

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$, $(a)_0 = 1$. Function $\Psi(a, b; x)$ is defined as follows:

$$\Psi(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \Phi(a, b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \Phi(a-b+1,\ 2-b;\ x).$$

Table 2.1 represents some special cases where Φ is expressed in terms of simpler functions.

2.4.2. Basic Properties

Kummer's transformation:

$$\Phi(a, b; x) = e^x \Phi(b-a, b; -x), \qquad \Psi(a, b; x) = x^{1-b} \Psi(1+a-b, 2-b; x).$$

Linear relations for function Φ:

$$\begin{aligned}
&(b-a)\Phi(a-1, b; x) + (2a-b+x)\Phi(a, b; x) - a\Phi(a+1, b; x) = 0,\\
&b(b-1)\Phi(a, b-1; x) - b(b-1+x)\Phi(a, b; x) + (b-a)x\Phi(a, b+1; x) = 0,\\
&(a-b+1)\Phi(a, b; x) - a\Phi(a+1, b; x) + (b-1)\Phi(a, b-1; x) = 0,\\
&b\Phi(a, b; x) - b\Phi(a-1, b; x) - x\Phi(a, b+1; x) = 0,\\
&b(a+x)\Phi(a, b; x) - (b-a)x\Phi(a, b+1; x) - ab\Phi(a+1, b; x) = 0,\\
&(a-1+x)\Phi(a, b; x) + (b-a)\Phi(a-1, b; x) - (b-1)\Phi(a, b-1; x) = 0,
\end{aligned}$$

Linear relations for function Ψ:

$$\begin{aligned}
&\Psi(a-1, b; x) - (2a-b+x)\Psi(a, b; x) + a(a-b+1)\Psi(a+1, b; x) = 0,\\
&(b-a-1)\Psi(a, b-1; x) - (b-1+x)\Psi(a, b; x) + x\Psi(a, b+1; x) = 0,\\
&\Psi(a, b; x) - a\Psi(a+1, b; x) - \Psi(a, b-1; x) = 0,\\
&(b-a)\Psi(a, b; x) - x\Psi(a, b+1; x) + \Psi(a-1, b; x) = 0,\\
&(a+x)\Psi(a, b; x) + a(b-a-1)\Psi(a+1, b; x) - x\Psi(a, b+1; x) = 0,\\
&(a-1+x)\Psi(a, b; x) - \Psi(a-1, b; x) + (a-c+1)\Psi(a, b-1; x) = 0.
\end{aligned}$$

Differentiation formulae:

$$\frac{d}{dx}\Phi(a, b; x) = \frac{a}{b}\Phi(a+1, b+1; x), \qquad \frac{d}{dx}\Psi(a, b; x) = -a\Psi(a+1, b+1; x),$$
$$\frac{d^n}{dx^n}\Phi(a, b; x) = \frac{(a)_n}{(b)_n}\Phi(a+n, b+n; x), \qquad \frac{d^n}{dx^n}\Psi(a, b; x) = (-1)^n (a)_n \Psi(a+n, b+n; x),$$

Wronskian:

$$W(\Phi, \Psi) = \Phi\Psi'_x - \Phi'_x\Psi = -\frac{\Gamma(b)}{\Gamma(a)} x^{-b} e^x.$$

2.4.3. Integral Representation

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1}\, dt, \qquad (\text{if } b > a > 0)$$

$$\Psi(a,b;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1}\, dt, \qquad (\text{if } a > 0, x > 0)$$

where $\Gamma(a)$ is the gamma-function.

2.4.4. Integrals with Degenerate Hypergeometric Functions

$$\int \Phi(a,b;x)\, dx = \frac{b-1}{a-1}\Psi(a-1,b-1;x)+C, \qquad \int \Phi(a,b;x)\, dx = \frac{1}{1-a}\Psi(a-1,b-1;x)+C,$$

$$\int x^n \Phi(a,b;x)\, dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1}(1-b)_k x^{n-k+1}}{(1-a)_k (n-k+1)!} \Phi(a-k, b-k; x) + C,$$

$$\int x^n \Psi(a,b;x)\, dx = n! \sum_{k=1}^{n+1} \frac{(-1)^{k+1} x^{n-k+1}}{(1-a)_k (n-k+1)!} \Phi(a-k, b-k; x) + C.$$

2.4.5. Asymptotic Expansion, as $|x| \to \infty$

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[\sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n!} x^{-n} + O(x^{-N-1}) \right], \qquad x > 0,$$

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \left[\sum_{n=0}^{N} \frac{(a)_n (a-b+1)_n}{n!} (-x)^{-n} + O(|x|^{-N-1}) \right], \qquad x < 0,$$

$$\Psi(a,b;x) = x^{-a} \left[\sum_{n=0}^{N} (-1)^n \frac{(a)_n (a-b+1)_n}{n!} x^{-n} + O(|x|^{-N-1}) \right], \qquad -\infty < x < +\infty.$$

2.5. Legendre Functions

2.5.1. Definitions

The associated Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ of the first and second kind are linearly-independent solutions the Legendre equation (see 2.1.2.212):

$$(1-z^2) y''_{zz} - 2z y'_z + [\nu(\nu+1) - \mu^2 (1-z^2)^{-1}] y = 0,$$

where parameters ν, μ and variable z may be arbitrary real or complex numbers.

With $|1-z| < 2$, the following formulae may be used:

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\mu/2} F\Big(-\nu,\ 1+\nu,\ 1-\mu,\ \frac{1-z}{2}\Big),$$

$$Q_\nu^\mu(z) = A \left(\frac{z-1}{z+1} \right)^{\frac{\mu}{2}} F\Big(-\nu,\ 1+\nu,\ 1+\mu,\ \frac{1-z}{2}\Big) + B \left(\frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F\Big(-\nu,\ 1+\nu,\ 1-\mu,\ \frac{1-z}{2}\Big),$$

$$A = e^{i\mu\pi} \frac{\Gamma(-\mu)\Gamma(1+\nu+\mu)}{2\Gamma(1+\nu-\mu)}, \quad B = e^{i\mu\pi} \frac{\Gamma(\mu)}{2}, \quad i^2 = -1,$$

where $F(a,b,c;z)$ is the hypergeometric series (see equation 2.1.2.158).

With $|z|>1$, the following formulae may be used:

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\Gamma(-\frac{1}{2}-\nu)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)} z^{-\nu+\mu-1}(z^2-1)^{-\mu/2} F\Big(\frac{1+\nu-\mu}{2}, \frac{2+\nu-\mu}{2}, \frac{2\nu+3}{2}, \frac{1}{z^2}\Big)$$
$$+ \frac{2^\nu\Gamma(\frac{1}{2}+\nu)}{\sqrt{\pi}\,\Gamma(1+\nu-\mu)} z^{\nu+\mu}(z^2-1)^{-\mu/2} F\Big(-\frac{\nu+\mu}{2}, \frac{1-\nu-\mu}{2}, \frac{1-2\nu}{2}, \frac{1}{z^2}\Big),$$

$$Q_\nu^\mu(z) = e^{i\pi\mu} \frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1}(z^2-1)^{\mu/2} F\Big(\frac{2+\nu+\mu}{2}, \frac{1+\nu+\mu}{2}, \frac{2\nu+3}{2}, \frac{1}{z^2}\Big),$$

The functions

$$P_\nu(z) \equiv P_\nu^0(z), \qquad Q_\nu(z) \equiv Q_\nu^0(z)$$

are called the Legendre functions and are solutions of the Legendre equation 2.1.2.148.

The modified associated Legendre functions on the cut $z=x, -1<x<1$, are defined by the formulae

$$P_\nu^\mu(x) = \frac{1}{2}\Big[e^{\frac{1}{2}i\mu\pi}P_\nu^\mu(x+i0) + e^{-\frac{1}{2}i\mu\pi}P_\nu^\mu(x-i0)\Big],$$
$$Q_\nu^\mu(x) = \frac{1}{2}e^{-i\mu\pi}\Big[e^{-\frac{1}{2}i\mu\pi}Q_\nu^\mu(x+i0) + e^{\frac{1}{2}i\mu\pi}Q_\nu^\mu(x-i0)\Big].$$

2.5.2. Trigonometric Expansions

With $-1<x<1$, the modified associated Legendre functions can be expressed in terms of trigonometric series:

$$P_\nu^\mu(\cos\theta) = \frac{2^{\mu+1}}{\sqrt{\pi}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k(1+\nu+\mu)_k}{k!(\nu+\frac{3}{2})_k} \sin[(2k+\nu+\mu+1)\theta],$$
$$Q_\nu^\mu(\cos\theta) = \sqrt{\pi}\,2^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\sin\theta)^\mu \sum_{k=0}^\infty \frac{(\frac{1}{2}+\mu)_k(1+\nu+\mu)_k}{k!(\nu+\frac{3}{2})_k} \cos[(2k+\nu+\mu+1)\theta],$$

where $0<\theta<\pi$.

2.5.3. Some Relations

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z), \quad P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\Gamma(\nu-n+1)} P_\nu^{-n}(z) \qquad n=0,\ 1,\ 2,\ \dots$$
$$Q_\nu^\mu(z) = \frac{\pi}{2\sin(\mu\pi)} e^{i\pi\mu}\Big[P_\nu^\mu(z) - \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)} P_\nu^{-\mu}(z)\Big].$$

For $-1<x<1$,

$$P_{\nu+1}^\mu(x) = \frac{2\nu+1}{\nu-\mu+1} xP_\nu^\mu(x) - \frac{\nu+\mu}{\nu-\mu+1} P_{\nu-1}^\mu(x),$$
$$P_{\nu+1}^\mu(x) = P_{\nu-1}^\mu(x) - (2\nu+1)(1-x^2)^{1/2}P_\nu^{\mu-1}(x),$$
$$\frac{d}{dx}P_\nu^\mu(x) = \frac{\nu x}{x^2-1}P_\nu^\mu(x) - \frac{\nu+\mu}{x^2-1}P_{\nu-1}^\mu(x).$$

For $0 < x < 1$,

$$P_\nu^\mu(-x) = P_\nu^\mu(x)\cos[\pi(\nu+\mu)] - 2\pi^{-1}Q_\nu^\mu(x)\sin[\pi(\nu+\mu)],$$
$$Q_\nu^\mu(-x) = -Q_\nu^\mu(x)\cos[\pi(\nu+\mu)] - \tfrac{1}{2}\pi P_\nu^\mu(x)\sin[\pi(\nu+\mu)].$$

Wronskians:

$$W(P_\nu, Q_\nu) = \frac{1}{1-x^2}, \qquad W(P_\nu^\mu, Q_\nu^\mu) = \frac{k}{1-x^2},$$

where $k = 2^{2\mu}\dfrac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}$.

For $n = 0,\ 1,\ 2,\ \ldots,$

$$P_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}P_\nu(x), \qquad Q_\nu^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}Q_\nu(x).$$

The Legendre polynomials $P_n(x)$ and the Legendre function $Q_n(x)$ are given by the formulae

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n,$$
$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

Functions $P_n = P_n(x)$ can be conveniently calculated by the recurrence relations

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2-1), \quad \ldots, \quad P_{n+1} = \frac{2n+1}{n+1}xP_n - \frac{n}{n+1}P_{n-1}.$$

Three leading functions $Q_n = Q_n(x)$ are

$$Q_0 = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad Q_1 = \frac{x}{2}\ln\frac{1+x}{1-x} - 1, \quad Q_2 = \frac{3x^2-1}{4}\ln\frac{1+x}{1-x} - \frac{3}{2}x.$$

2.5.4. Integral Representation

For $n = 0,\ 1,\ 2,\ \ldots$

$$P_\nu^n(z) = \frac{\Gamma(\nu+n+1)}{\pi\Gamma(\nu+1)}\int_0^\pi \left(z+\cos t\sqrt{z^2-1}\right)^\nu \cos(nt)\,dt, \quad \operatorname{Re} z > 0,$$
$$Q_\nu^n(z) = (-1)^n\frac{\Gamma(\nu+n+1)}{2^{\nu+1}\Gamma(\nu+1)}(z^2-1)^{-n/2}\int_0^\pi (z+\cos t)^{n-\nu-1}(\sin t)^{2\nu+1}\,dt, \quad \operatorname{Re}\nu > -1,$$

Note that in the latter formulae $z \neq x, -1 < x < 1$.

2.6. The Weierstrass function $\wp$

2.6.1. Definitions

The Weierstrass function $\wp = \wp(z, g_2, g_3)$ is defined implicitly by the elliptic integral

$$z = \int_\infty^\wp \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}$$

and satisfies the first order differential equation

$$(\wp'_z)^2 = 4\wp^3 - g_2\wp - g_3.$$

2.6.2. Some Properties

Below $\wp(z)$ stands for $\wp(z, g_2, g_3)$.

$$\wp(z) = \wp(-z), \qquad \wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left[\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right]^2.$$

In the vicinity of the point $z = 0$, the Weierstrass function can be expanded into the series

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \frac{3g_2g_3}{6160}z^8 + \cdots .$$