Chapter 2
Second Order Differential Equations

2.1. Linear Equations

2.1.1. Preliminary Comments

1. A homogeneous linear equation of the second order has the form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution ($y \not\equiv 0$) of this equation. Then the general solution of equation (1) can be found from the formula

$$y = y_0\Bigl(C_1 + C_2\int \frac{e^{-F}}{y_0^2}\,dx\Bigr), \qquad \text{where} \quad F = \int \frac{f_1}{f_2}\,dx. \tag{2}$$

For specific equations described below in 2.1.2–2.1.8, often only particular solutions are given, while the general solutions can be obtained with formula (2).

2. The substitution $u = y'_x/y$ brings equation (1) to the Riccati equation

$$f_2(x)u'_x + f_2(x)u^2 + f_1(x)u + f_0(x) = 0$$

which is discussed in Section 1.2.

3. Assuming

$$y = u(x)\exp\Bigl(-\frac{1}{2}\int \frac{f_1}{f_2}\,dx\Bigr) \tag{3}$$

yields from equation (1) the canonical (or normal) form

$$u''_{xx} + f(x)u = 0, \qquad \text{where} \quad f = \frac{f_0}{f_2} - \frac{1}{4}\Bigl(\frac{f_1}{f_2}\Bigr)^2 - \frac{1}{2}\Bigl(\frac{f_1}{f_2}\Bigr)'_x. \tag{4}$$

Substitution (3) is a special case of the more general transformation (φ is an arbitrary function):

$$x = \varphi(\xi), \quad y(x) = u(\xi)\sqrt{|\varphi'_\xi(\xi)|}\,\exp\Bigl(-\frac{1}{2}\int \frac{f_1(\varphi)}{f_2(\varphi)}\,d\varphi\Bigr),$$

which reduces the original equation to the canonical form.

4. A nonhomogeneous linear equation of the second order has the form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x). \tag{5}$$

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly-independent ($y_1/y_2 \not\equiv \text{const}$) solutions of the corresponding homogeneous equation with $g \equiv 0$. Then the general solution of equation (5) can be found from the formula

$$y = C_1 y_1 + C_2 y_2 + y_2 \int y_1 \frac{g}{f_2}\frac{dx}{W} - y_1 \int y_2 \frac{g}{f_2}\frac{dx}{W}, \tag{6}$$

where $W = y_1(y_2)'_x - y_2(y_1)'_x$.

Given a nontrivial particular solution $y_1 = y_1(x)$ of the homogeneous equation with $g \equiv 0$, formula (6) can be used for the construction of the general solution of equation (5) with the second solution $y_2 = y_2(x)$ taken in the form

$$y_2 = y_1 \int \frac{e^{-F}}{y_1^2}\,dx, \qquad \text{where} \quad F = \int \frac{f_1}{f_2}\,dx, \quad W = e^{-F}, \tag{7}$$

In Subsections 2.1.2–2.1.8, mainly homogeneous equations are given; the corresponding nonhomogeneous equations may be solved by means of the formulae (6) and (7).

2.1.2. Equations Containing Power Functions

1. $\boldsymbol{y''_{xx} + ay = 0}$.

Solution:

$$y = \begin{cases} C_1 \sinh(x\sqrt{|a|}\,) + C_2 \cosh(x\sqrt{|a|}\,) & \text{if } a < 0, \\ C_1 + C_2 x & \text{if } a = 0, \\ C_1 \sin(x\sqrt{a}\,) + C_2 \cos(x\sqrt{a}\,) & \text{if } a > 0. \end{cases}$$

2. $\boldsymbol{y''_{xx} - (ax+b)y = 0, \qquad a \neq 0}$.

The substitution $\xi = a^{-2/3}(ax+b)$ leads to the Airy equation

$$y''_{\xi\xi} - \xi y = 0, \tag{1}$$

which is often met with in various applications. The solution of equation (1) can be written as

$$y = C_1 \operatorname{Ai}(\xi) + C_2 \operatorname{Bi}(\xi),$$

where $\operatorname{Ai}(\xi)$ and $\operatorname{Bi}(\xi)$ are the Airy functions of the first and second kind, respectively.

The Airy functions admits the following integral representation:

$$\operatorname{Ai}(\xi) = \frac{1}{\pi}\int_0^\infty \cos\Bigl(\frac{1}{3}t^3 + \xi t\Bigr)\,dt,$$

$$\operatorname{Bi}(\xi) = \frac{1}{\pi}\int_0^\infty \Bigl[\exp\Bigl(-\frac{1}{3}t^3 + \xi t\Bigr) + \sin\Bigl(\frac{1}{3}t^3 + \xi t\Bigr)\Bigr]\,dt.$$

The Airy functions can be expressed in terms of the Bessel functions and the modified Bessel functions of the order 1/3 with the formulae

$$\begin{aligned}
\operatorname{Ai}(\xi) &= \frac{1}{3}\sqrt{\xi}\,\bigl[I_{-1/3}(z) - I_{1/3}(z)\bigr] = \frac{1}{\pi}\sqrt{\frac{\xi}{3}}\,K_{1/3}(z),\\
\operatorname{Bi}(\xi) &= \sqrt{\frac{\xi}{3}}\,\bigl[I_{-1/3}(z) + I_{1/3}(z)\bigr],\\
\operatorname{Ai}(-\xi) &= \frac{1}{3}\sqrt{\xi}\,\bigl[J_{-1/3}(z) + J_{1/3}(z)\bigr],\\
\operatorname{Bi}(-\xi) &= \sqrt{\frac{\xi}{3}}\,\bigl[J_{-1/3}(z) - J_{1/3}(z)\bigr],
\end{aligned}$$

where $z = \frac{2}{3}\xi^{3/2}$.

For large values of ξ, the leading terms of the asymptotic expansions of the Airy functions are

$$\begin{aligned}
\operatorname{Ai}(\xi) &= \frac{1}{2\sqrt{\pi}}\xi^{-1/4}\exp(-z),\\
\operatorname{Ai}(-\xi) &= \frac{1}{\sqrt{\pi}}\xi^{-1/4}\sin\Bigl(z + \frac{\pi}{4}\Bigr),\\
\operatorname{Bi}(\xi) &= \frac{1}{\sqrt{\pi}}\xi^{-1/4}\exp(z),\\
\operatorname{Bi}(-\xi) &= \frac{1}{\sqrt{\pi}}\xi^{-1/4}\cos\Bigl(z + \frac{\pi}{4}\Bigr).
\end{aligned}$$

The Airy equation (1) is a special case of the equation 2.1.2.7 with $n = 1$.

3. $\boldsymbol{y''_{xx} - (a^2x^2 + a)y = 0.}$

Particular solution: $y_0 = \exp\Bigl(\dfrac{ax^2}{2}\Bigr)$.

4. $\boldsymbol{y''_{xx} - (ax^2 + b)y = 0.}$

The transformation $z = x^2\sqrt{a}$, $u = e^{z/2}y$ leads to the degenerate hypergeometric equation 2.1.2.65:

$$zu''_{zz} + \Bigl(\frac{1}{2} - z\Bigr)u'_z - \frac{1}{4}\Bigl(\frac{b}{\sqrt{a}} + 1\Bigr)u = 0.$$

5. $\boldsymbol{y''_{xx} + a^3x(2 - ax)y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a^2x}{2} + ax\Bigr)$.

6. $\boldsymbol{y''_{xx} - (ax^2 + bx + c)y = 0.}$

The substitution $\xi = x + \dfrac{b}{2a}$ leads to an equation of the form 2.1.2.4:

$$y''_{\xi\xi} - \Bigl(a\xi^2 + c - \frac{b^2}{4a}\Bigr)y = 0.$$

7. $y''_{xx} - ax^n y = 0.$

1°. For $n = -2$, this is the Euler equation 2.1.2.118, while for $n = -4$, this is the equation 2.1.2.198 (in both cases the solution is expressed in terms of elementary function).

2°. Assume $2/(n+2) = 2m+1$, where m is an integer. Then the solution is

$$y = x(x^{1-2q}D)^{m+1}\left[C_1 \exp\left(\frac{\sqrt{a}}{q}x^q\right) + C_2 \exp\left(-\frac{\sqrt{a}}{q}x^q\right)\right] \quad \text{if } m \geq 0,$$

$$y = x(x^{1-2q}D)^{-m}\left[C_1 \exp\left(\frac{\sqrt{a}}{q}x^q\right) + C_2 \exp\left(-\frac{\sqrt{a}}{q}x^q\right)\right] \quad \text{if } m < 0,$$

where $D = \dfrac{d}{dx}$, $q = \dfrac{n+2}{2} = \dfrac{1}{2m+1}$.

3°. For any n, the solution is expressed in terms of Bessel functions and modified Bessel functions of the first or second kind (see 2.1.2.122):

$$y = \sqrt{x}\left[C_1 J_{\frac{1}{2q}}\left(\frac{\sqrt{-a}}{q}x^q\right) + C_2 Y_{\frac{1}{2q}}\left(\frac{\sqrt{-a}}{q}x^q\right)\right], \quad \text{if } a < 0,$$

$$y = \sqrt{x}\left[C_1 I_{\frac{1}{2q}}\left(\frac{\sqrt{a}}{q}x^q\right) + C_2 K_{\frac{1}{2q}}\left(\frac{\sqrt{a}}{q}x^q\right)\right], \quad \text{if } a > 0,$$

where $q = \frac{1}{2}(n+2)$.

8. $y''_{xx} - a(ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\left(\dfrac{a}{n+1}x^{n+1}\right)$.

9. $y''_{xx} - ax^{n-2}(ax^n + n + 1)y = 0.$

Particular solution: $y_0 = x\exp\left(\dfrac{ax^n}{n}\right)$.

10. $y''_{xx} + (ax^{2n} + bx^{n-1})y = 0.$

The substitution $\xi = x^{n+1}$ leads to an equation of the form 2.1.2.103:

$$(n+1)^2\xi y''_{\xi\xi} + n(n+1)y'_\xi + (a\xi + b)y = 0.$$

11. $y''_{xx} + ay'_x + by = 0.$

The equation of damping oscillation.

1°. Solution with $\lambda^2 = a^2 - 4b > 0$:

$$y = C_1 \exp\left(\frac{\lambda - a}{2}x\right) + C_2 \exp\left(\frac{-\lambda - a}{2}x\right).$$

2°. Solution with $\lambda^2 = 4b - a^2 > 0$:

$$y = \exp\left(-\frac{ax}{2}\right)\left(C_1 \sin\frac{\lambda x}{2} + C_2 \cos\frac{\lambda x}{2}\right).$$

3°. Solution with $a^2 = 4b$:

$$y = \exp\left(-\frac{ax}{2}\right)(C_1 x + C_2).$$

12. $\boldsymbol{y''_{xx} + ay'_x + (bx + c)y = 0}$.

This is a special case of equation 2.1.2.103.

13. $\boldsymbol{y''_{xx} + ay'_x - (bx^2 + c)y = 0}$.

The substitution $y = u\exp\left(\frac{1}{2}x^2\sqrt{b}\right)$ leads to an equation of the form 2.1.2.103: $u''_{xx} + (2\sqrt{b}\,x + a)u'_x + (a\sqrt{b}\,x - c + \sqrt{b}\,)u = 0$.

14. $\boldsymbol{y''_{xx} + ay'_x + b(-bx^2 + ax + 1)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{bx^2}{2}\Bigr)$.

15. $\boldsymbol{y''_{xx} + ay'_x + bx(-bx^3 + ax + 2)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{bx^3}{3}\Bigr)$.

16. $\boldsymbol{y''_{xx} + ay'_x + b(-bx^{2n} + ax^n + nx^{n-1})y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{n+1}x^{n+1}\Bigr)$.

17. $\boldsymbol{y''_{xx} + ay'_x + b(-bx^{2n} - ax^n + nx^{n-1})y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{n+1}x^{n+1} - ax\Bigr)$.

18. $\boldsymbol{y''_{xx} + xy'_x + (n + 1)y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots}$

Solution: $y = \dfrac{d^n}{dx^n}\biggl\{\exp\Bigl(-\dfrac{x^2}{2}\Bigr)\Bigl[C_1 + C_2\displaystyle\int\exp\Bigl(\dfrac{x^2}{2}\Bigr)\,dx\Bigr]\biggr\}$.

19. $\boldsymbol{y''_{xx} - 2xy'_x + 2ny = 0, \qquad n = 1,\ 2,\ 3,\ \ldots}$

Solution: $y = \exp(x^2)\dfrac{d^n}{dx^n}\biggl\{\exp(-x^2)\Bigl[C_1 + C_2\displaystyle\int\exp(x^2)\,dx\Bigr]\biggr\}$.

20. $\boldsymbol{y''_{xx} + 2axy'_x + (bx^4 + a^2x^2 + cx + a)y = 0}$.

This is a special case of equation 2.1.2.46 with $n = 1$, $m = 2$.

21. $\boldsymbol{y''_{xx} + (ax + b)y'_x + ay = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{2}x^2 - bx\Bigr)$.

22. $\boldsymbol{y''_{xx} + (ax + b)y'_x - ay = 0}$.

Particular solution: $y_0 = ax + b$.

23. $\boldsymbol{y''_{xx} + (ax + b)y'_x + c(ax + b - c)y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

24. $y''_{xx} + (ax + 2b)y'_x + (abx - a + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

25. $y''_{xx} + (ax + b)y'_x + (cx + d)y = 0.$

This is a special case of equation 2.1.2.103.

26. $y''_{xx} + (ax + b)y'_x + c[(a - c)x^2 + bx + 1]y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{cx^2}{2}\right)$.

27. $y''_{xx} + 2(ax + b)y'_x + (a^2x^2 + 2abx + c)y = 0.$

The substitution $u = y\exp\left(\frac{1}{2}ax^2 + bx\right)$ leads to an equation of the form 2.1.2.1: $u''_{xx} + (c - a - b^2)u = 0$.

28. $y''_{xx} + (ax + b)y'_x + (\alpha x^2 + \beta x + \gamma)y = 0.$

Assuming $y = u\exp(sx^2)$, where s is a root of the quadratic equation $4s^2 + 2as + \alpha = 0$, yields an equation of the form 2.1.2.103:

$$u''_{xx} + [(a + 4s)x + b]u'_x + [(\beta + 2bs)x + \gamma + 2s]u = 0.$$

29. $y''_{xx} + (ax + b)y'_x + c(-cx^{2n} + ax^{n+1} + bx^n + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{c}{n+1}x^{n+1}\right)$.

30. $y''_{xx} + a(x^2 - b^2)y'_x - a(x + b)y = 0.$

Particular solution: $y_0 = x - b$.

31. $y''_{xx} + (ax^2 + b)y'_x + c(ax^2 + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

32. $y''_{xx} + (ax^2 + 2b)y'_x + (abx^2 - ax + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

33. $y''_{xx} + (2x^2 + a)y'_x + (x^4 + ax^2 + 2x + b)y = 0.$

The substitution $u = y\exp\left(\frac{1}{3}x^3\right)$ leads to a constant coefficient equation: $u''_{xx} + au'_x + bu = 0$.

34. $y''_{xx} + (ax^2 + bx)y'_x + (3ax + 2b)y = 0.$

Particular solution: $y_0 = x\exp\left(-\frac{1}{3}ax^3 - \frac{1}{2}bx^2\right)$.

35. $y''_{xx} + (abx^2 + bx + 2a)y'_x + a^2(bx^2 + 1)y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

36. $y''_{xx} + (ax^2 + bx + c)y'_x + x(abx^2 + bc + 2a)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\frac{a}{3}x^3 - cx\Bigr)$.

37. $y''_{xx} + (ax^2 + bx + c)y'_x + (abx^3 + acx^2 + b)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\frac{b}{2}x^2 - cx\Bigr)$.

38. $y''_{xx} + (ax^3 + 2b)y'_x + (abx^3 - ax^2 + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

39. $y''_{xx} + (ax^3 + bx)y'_x + 2(2ax^2 + b)y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\frac{a}{4}x^4 - \frac{b}{2}x^2\Bigr)$.

40. $y''_{xx} + (abx^3 + bx^2 + 2a)y'_x + a^2(bx^3 + 1)y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

41. $y''_{xx} + ax^n y'_x = 0.$

This equation is encountered in the theory of diffusion boundary layer.

Solution: $y = C_1 + C_2 \int \exp\Bigl(-\frac{ax^{n+1}}{n+1}\Bigr)\,dx$.

42. $y''_{xx} + ax^n y'_x + bx^{n-1}y = 0.$

For $n = -1$, we obtain the Euler equation 2.1.2.118. For $n \neq -1$, the substitution $z = x^{n+1}$ leads to an equation of the form 2.1.2.103:

$$(n+1)^2 z y''_{zz} + (n+1)(az + n)y'_z + by = 0.$$

43. $y''_{xx} + 2ax^n y'_x + a(ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\frac{a}{n+1}x^{n+1}\Bigr)$.

44. $y''_{xx} + ax^n y'_x + (bx^{2n} + cx^{n-1})y = 0.$

The substitution $\xi = x^{n+1}$ leads to an equation of the form 2.1.2.103:

$$(n+1)^2 \xi y''_{\xi\xi} + (n+1)(a\xi + n)y'_\xi + (b\xi + c)y = 0.$$

45. $y''_{xx} + ax^n y'_x - b(ax^{n+m} + bx^{2m} + mx^{m-1}y = 0).$

Particular solution: $y_0 = \exp\Bigl(\frac{b}{m+1}x^{m+1}\Bigr)$.

46. $y''_{xx} + 2ax^n y'_x + (a^2x^{2n} + bx^{2m} + anx^{n-1} + cx^{m-1})y = 0.$

The substitution $w = y\exp\Bigl(\frac{a}{n+1}x^{n+1}\Bigr)$ leads to an equation of the form 2.1.2.10: $w''_{xx} + (bx^{2m} + cx^{m-1})w = 0$.

47. $y''_{xx} + (ax^n + b)y'_x + c(ax^n + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

48. $y''_{xx} + (ax^n + 2b)y'_x + (abx^n - ax^{n-1} + b^2)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

49. $y''_{xx} + (abx^n + bx^{n-1} + 2a)y'_x + a^2(bx^n + 1)y = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

50. $y''_{xx} + (abx^n + 2bx^{n-1} - a^2x)y'_x + a(abx^n + bx^{n-1} - a^2x)y = 0.$

Particular solution: $y_0 = (ax+2)e^{-ax}$.

51. $y''_{xx} + x^n[ax^2 + (ac+b)x + bc]y'_x - x^n(ax+b)y = 0.$

Particular solution: $y_0 = x + c$.

52. $y''_{xx} + (ax^n + bx^m)y'_x - (ax^{n-1} + bx^{m-1})y = 0.$

Particular solution: $y_0 = x$.

53. $y''_{xx} + (ax^n + bx^m)y'_x + (anx^{n-1} + bmx^{m-1})y = 0.$

Integrating, we obtain a first order linear equation: $y'_x + (ax^n + bx^m)y = C$.

54. $y''_{xx} + (ax^n + bx^m)y'_x + [a(n+1)x^{n-1} + b(m+1)x^{m-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n+1}x^{n+1} - \dfrac{b}{m+1}x^{m+1}\Bigr)$.

55. $y''_{xx} + (ax^n + bx^m)y'_x + c(ax^n + bx^m - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

56. $y''_{xx} + (ax^n + bx^m)y'_x + [abx^{m+n} + b(m+1)x^{m-1} - ax^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{b}{m+1}x^{m+1}\Bigr)$.

57. $y''_{xx} + (ax^n + bx^m + c)y'_x + (abx^{m+n} + bcx^m + anx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1} - cx\Bigr)$.

58. $xy''_{xx} + \frac{1}{2}y'_x + ay = 0.$

Solution:

$$y = \begin{cases} C_1\cos\sqrt{4ax} + C_2\sin\sqrt{4ax} & \text{if } ax > 0,\\ C_1\cosh\sqrt{4|ax|} + C_2\sinh\sqrt{4|ax|} & \text{if } ax < 0. \end{cases}$$

59. $xy''_{xx} + ay'_x + (bx + c)y = 0.$

This is a special case of equation 2.1.2.103.

60. $xy''_{xx} + ny'_x + bx^{1-2n}y = 0.$

For $n = 1$, this is the Euler equation 2.1.2.118. For $n \neq 1$, the solution is

$$y = \begin{cases} C_1 \sin\left(\dfrac{\sqrt{b}}{n-1}x^{1-n}\right) + C_2 \cos\left(\dfrac{\sqrt{b}}{n-1}x^{1-n}\right) & \text{if } b > 0, \\ C_1 \exp\left(\dfrac{\sqrt{-b}}{n-1}x^{1-n}\right) + C_2 \exp\left(\dfrac{-\sqrt{-b}}{n-1}x^{1-n}\right) & \text{if } b < 0. \end{cases}$$

61. $xy''_{xx} + (1-3n)y'_x - a^2n^2x^{2n-1}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n)$.

62. $xy''_{xx} + ay'_x + bx^n y = 0.$

If $n = -1$ and $b = 0$, we have the Euler equation 2.1.2.118. If $n \neq -1$ and $b \neq 0$, the solution is expressed in terms of Bessel functions:

$$y = x^{\frac{1-a}{2}}\left[C_1 J_\nu\left(\frac{2\sqrt{b}}{n+1}x^{\frac{n+1}{2}}\right) + C_2 Y_\nu\left(\frac{2\sqrt{b}}{n+1}x^{\frac{n+1}{2}}\right)\right], \quad \nu = \frac{|1-a|}{n+1}.$$

63. $xy''_{xx} + ay'_x + bx^n(-bx^{n+1} + a + n)y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{b}{n+1}x^{n+1}\right)$.

64. $xy''_{xx} + axy'_x + ay = 0.$

Particular solution: $y_0 = xe^{-ax}$.

65. $xy''_{xx} + (b-x)y'_x - ay = 0.$

The degenerate hypergeometric equation.

If $b \neq 0, -1, -2, -3, \ldots$, Kummer's series is a particular solution:

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$, $(a)_0 = 1$. If $b > a > 0$, this solution may be written in terms of the definite integral

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}\,dt,$$

where $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$ is the gamma-function.

If b is not an integer, then the general solution has the form

$$y = C_1\Phi(a, b; x) + C_2 x^{1-b}\Phi(a-b+1,\ 2-b;\ x).$$

TABLE 2.1
Special cases of Kummer's function $\Phi(a, b; x)$.

a	b	z	Φ	Conventional notation
a	a	x	e^x	
1	2	$2x$	$\dfrac{1}{x} e^x \sinh x$	
a	$a+1$	$-x$	$a x^{-a} \gamma(a, x)$	Incomplete gamma-function $\gamma(a, x) = \int_0^x e^{-t} t^{a-1}\, dt$
$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\dfrac{\sqrt{\pi}}{2} \operatorname{erf} x$	Error function $\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$
$-n$	$\frac{1}{2}$	$\frac{x^2}{2}$	$\dfrac{n!}{(2n)!} \left(-\dfrac{1}{2}\right)^{-n} H_{2n}(x)$	Hermite polynomials $H_n = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} \left(e^{-x^2}\right)$, $n = 0,\ 1,\ 2,\ 3,\ \ldots$
$-n$	$\frac{3}{2}$	$\frac{x^2}{2}$	$\dfrac{n!}{(2n+1)!} \left(-\dfrac{1}{2}\right)^{-n} H_{2n+1}(x)$	
$-n$	b	x	$\dfrac{n!}{(b)_n} L_n^{(b-1)}(x)$	Laguerre polynomials $L_n^{(\alpha)}(x) = \dfrac{e^x x^{-\alpha}}{n!} \dfrac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right)$, $\alpha = b-1$, $(b)_n = b(b+1)\ldots(b+n-1)$
$\nu+\frac{1}{2}$	$2\nu+1$	$2x$	$\Gamma(1+\nu) e^x \left(\dfrac{x}{2}\right)^{-\nu} I_\nu(x)$	Modified Bessel functions $I_\nu(x)$
$n+1$	$2n+2$	$2x$	$\Gamma\left(n+\dfrac{3}{2}\right) e^x \left(\dfrac{x}{2}\right)^{-n-\frac{1}{2}} I_{n+\frac{1}{2}}(x)$	

In Table 2.1 are given some special cases where Φ is expressed in terms of simpler functions.

Function Φ possesses the properties

$$\Phi(a, b; x) = e^x \Phi(b-a,\ b;\ -x); \qquad \frac{d^n}{dx^n} \Phi(a, b; x) = \frac{(a)_n}{(b)_n} \Phi(a+n,\ b+n;\ x).$$

The following asymptotic relations hold:

$$\Phi(a, b; x) \to \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left[1 + O\left(\frac{1}{|x|}\right)\right], \qquad \text{as } x \to +\infty,$$

$$\Phi(a, b; x) \to \frac{\Gamma(b)}{\Gamma(b-a)} (-x)^{-a} \left[1 + O\left(\frac{1}{|x|}\right)\right], \qquad \text{as } x \to -\infty.$$

The following function is a solution of the degenerate hypergeometric equation:

$$\Psi(a, b; x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \Phi(a, b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \Phi(a-b+1,\ 2-b;\ x).$$

Calculate the limit as $b \to n$ (n is an integer) to obtain

$$\Psi(a,b;x) = \frac{(-1)^{n-1}}{n!\,\Gamma(a-n)}\biggl\{\Phi(a,n{+}1;x)\ln x + \sum_{r=0}^{\infty}\frac{(a)_r}{(n+1)_r}\bigl[\psi(a+r)-\psi(1+r)-\psi(1+n+r)\bigr]\frac{x^r}{r!}\biggr\} + \frac{(n-1)!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\frac{x^{r-n}}{r!},$$

where $n = 0,\ 1,\ 2,\ \ldots$ (the last sum is omitted for $n=0$), $\psi(z) = [\ln\Gamma(z)]'_z$ is the logarithmic derivative of the gamma-function:

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1},$$

$\gamma = 0.5572\ldots$ is the Euler constant.

If b is a negative number, then function Ψ may be presented with the formula

$$\Psi(a,b;x) = x^{1-b}\Psi(a-b+1,\ 2-b;\ x)$$

which holds for any value of x.

For $b \neq 0,\ -1,\ -2,\ -3,\ \ldots,$ the general solution of the degenerate hypergeometric equation may be written in the form

$$y = C_1\Phi(a,b;x) + C_2\Psi(a,b;x),$$

while for $b = 0,\ -1,\ -2,\ -3,\ \ldots,$ it may be written as

$$y = x^{1-b}\bigl[C_1\Phi(a-b+1,\ 2-b;\ x) + C_2\Psi(a-b+1,\ 2-b;\ x)\bigr].$$

The functions Φ and Ψ are described in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, vol. 1) in more detail (see also Suplement 2).

66. $\boldsymbol{xy''_{xx} + (ax+b)y'_x + c[(a-c)x+b]y = 0.}$

Particular solution: $y_0 = e^{-cx}$.

67. $\boldsymbol{xy''_{xx} + (2ax+b)y'_x + a(ax+b)y = 0.}$

Solution: $y = e^{-ax}(C_1 + C_2x^{1-b})$.

68. $\boldsymbol{xy''_{xx} + [(a+b)x+n+m]y'_x + (abx+an+bm)y = 0,}$
where n and m are positive integers; $a \neq b$ or $n \neq m$.

Solution:

$$y = C_1e^{-ax}\frac{d^{m-1}}{dx^{m-1}}x^{-n}e^{(a-b)x} + C_2e^{-bx}\frac{d^{n-1}}{dx^{n-1}}x^{-m}e^{(b-a)x}.$$

69. $\boldsymbol{xy''_{xx} + (ax+b)y'_x + (cx+d)y = 0.}$

This is a special case of equation 2.1.2.103.

70. $\boldsymbol{xy''_{xx} - (ax+1)y'_x - bx^2(bx+a)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{bx^2}{2}\Bigr)$.

71. $\boldsymbol{xy''_{xx} - (2ax+1)y'_x + (bx^3 + a^2x + a)y = 0}$.

Solution: $y = e^{ax}\Bigl[C_1 \sin\Bigl(\dfrac{x^2}{2}\sqrt{b}\Bigr) + C_2 \cos\Bigl(\dfrac{x^2}{2}\sqrt{b}\Bigr)\Bigr]$.

72. $\boldsymbol{xy''_{xx} + (ax+b)y'_x + cx(-cx^2 + ax + b + 1) = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{cx^2}{2}\Bigr)$.

73. $\boldsymbol{xy''_{xx} - (2ax^2+1)y'_x + bx^3y = 0}$.

Solution: $y = C_1 \exp\bigl[\frac{1}{2}\bigl(a + \sqrt{a^2-b}\,\bigr)x^2\bigr] + C_2 \exp\bigl[\frac{1}{2}\bigl(a - \sqrt{a^2-b}\,\bigr)x^2\bigr]$.

74. $\boldsymbol{xy''_{xx} + (abx^2 + b - 5)y'_x + 2a^2(b-2)x^3y = 0}$.

Particular solution: $y_0 = (ax^2+1)\exp(-ax^2)$.

75. $\boldsymbol{xy''_{xx} + (ax^2+bx)y'_x - [acx^2 + (a + bc + c^2)x + b + 2c]y = 0}$.

Particular solution: $y_0 = xe^{cx}$.

76. $\boldsymbol{xy''_{xx} + (ax^2+bx+2)y'_x + by = 0}$.

Particular solution: $y_0 = a + \dfrac{b}{x}$.

77. $\boldsymbol{xy''_{xx} + (ax^2+bx+c)y'_x + (2ax+b)y = 0}$.

This equation can be integrated to obtain the first order linear equation: $xy'_x + (ax^2 + bx + c - 1)y = C$.

78. $\boldsymbol{xy''_{xx} + (ax^2+bx+c)y'_x + (c-1)(ax+b)y = 0}$.

Particular solution: $y_0 = x^{1-c}$.

79. $\boldsymbol{xy''_{xx} + (ax^2+bx+c)y'_x + (Ax^2+Bx+C)y = 0}$.

1°. Let $A = ak$, $B = k(b-k)$, $C = ck$, where k is an arbitrary number.
Particular solution: $y_0 = e^{-kx}$.

2°. Let $A = a(b+k)$, $B = a(c+1) - k(b+k)$, $C = -ck$.
Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^2}{2} + kx\Bigr)$.

3°. Let $A = a(b+k)$, $B = 2a - bk - k^2$, $C = b(c-1) + k(c-2)$.
Particular solution: $y_0 = x^{1-c}\exp\Bigl(-\dfrac{ax^2}{2} + kx\Bigr)$.

4°. Let $A = -ak$, $B = a(c-1) - k(b+k)$, $C = b(c-1) + k(c-2)$.
Particular solution: $y_0 = x^{1-c}e^{kx}$.

80. $xy''_{xx} + (ax^2 + bx + 2)y'_x + (cx^2 + dx + b)y = 0.$

The substitution $u = xy$ leads to an equation of the form 2.1.2.103: $u''_{xx} + (ax + b)u'_x + (cx + d - a)u = 0$.

81. $xy''_{xx} + (ax^3 + b)y'_x + a(b - 1)x^2 y = 0.$

Particular solution: $y_0 = x^{1-b}$.

82. $xy''_{xx} + x(ax^2 + b)y'_x + (3ax^2 + b)y = 0.$

Particular solution: $y_0 = x \exp\Bigl(-\dfrac{ax^3}{3} - bx\Bigr)$.

83. $xy''_{xx} + (ax^3 + bx^2 + 2)y'_x + bxy = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

84. $xy''_{xx} + (abx^3 + bx^2 + ax - 1)y'_x + a^2 bx^3 y = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}$.

85. $xy''_{xx} + (ax^3 + bx^2 + cx + d)y'_x + (d - 1)(ax^2 + bx + c)y = 0.$

Particular solution: $y_0 = x^{1-d}$.

86. $xy''_{xx} + ax^n y'_x + (abx^n - ax^{n-1} - b^2 x + 2b)y = 0.$

Particular solution: $y_0 = xe^{-bx}$.

87. $xy''_{xx} + (ax^n + 2)y'_x + ax^{n-1} y = 0.$

Particular solution: $y_0 = x^{-1}$.

88. $xy''_{xx} + (x^n + 1 - n)y'_x + bx^{2n-1} y = 0.$

For $b \neq \frac{1}{4}$, the general solution has the form

$$y = C_1 \exp\Bigl(\frac{\beta_1}{n} x^n\Bigr) + C_2 \exp\Bigl(\frac{\beta_2}{n} x^n\Bigr),$$

where β_1 and β_2 are the roots of the quadratic equation $\beta^2 + \beta + b = 0$.

For $b = \frac{1}{4}$, the solution is

$$y = (C_1 + C_2 x^n) \exp\Bigl(-\frac{x^n}{2n}\Bigr).$$

89. $xy''_{xx} + (ax^n + b)y'_x + anx^{n-1} y = 0.$

Particular solution: $y_0 = x^{1-b} \exp\Bigl(-\dfrac{ax^n}{n}\Bigr)$.

90. $xy''_{xx} + (ax^n + b)y'_x + a(b - 1)x^{n-1} y = 0.$

Particular solution: $y_0 = x^{1-b}$.

91. $xy''_{xx} + (ax^n + b)y'_x + a(b + n - 1)x^{n-1}y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{ax^n}{n}\Bigr).$

92. $xy''_{xx} + (ax^n + b)y'_x + c(ax^n - cx + b)y = 0.$

Particular solution: $y_0 = e^{-cx}.$

93. $xy''_{xx} + (abx^n + b - 3n + 1)y'_x + a^2n(b - n)x^{2n-1}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n).$

94. $xy''_{xx} + (ax^n + b)y'_x + (cx^{2n-1} + dx^{n-1})y = 0.$

This is a special case of equation 2.1.2.141 with $\gamma = 0$.

95. $xy''_{xx} + (ax^n + bx^{n-1} + 2)y'_x + bx^{n-2}y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

96. $xy''_{xx} + (ax^n + bx)y'_x + (abx^n + anx^{n-1} - b)y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{ax^n}{n}\Bigr).$

97. $xy''_{xx} + (abx^n + bx^{n-1} + ax - 1)y'_x + a^2bx^ny = 0.$

Particular solution: $y_0 = (ax + 1)e^{-ax}.$

98. $xy''_{xx} + (ax^n + bx^m + c)y'_x + (c - 1)(ax^{n-1} + bx^{m-1})y = 0.$

Particular solution: $y_0 = x^{1-c}.$

99. $xy''_{xx} + (abx^{n+m} + anx^n + bx^m + 1 - 2n)y'_x + a^2bnx^{2n+m-1}y = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n).$

100. $(x + a)y''_{xx} + (bx + c)y'_x + by = 0.$

Particular solution: $y_0 = \exp\Bigl(-\displaystyle\int \frac{bx + c - 1}{x + a}\,dx\Bigr).$

101. $(a_1x + a_0)y''_{xx} + (b_1x + b_0)y'_x - mb_1y = 0.$

If $m = 1, 2, 3, \ldots$, a polynomial of order m in x is a particular solution of the equation, which may be presented as

$$y_0 = \sum_{k=0}^{m}\Bigl(-\frac{1}{b_1}\Bigr)^k \bigl\{x^m I x^{-m-1}[(a_1x + a_0)D^2 + b_0D]\bigr\}^k x^m,$$

where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu + 1}$ with $\nu \neq -1$.

TABLE 2.2
Solutions of equation 2.1.2.103 for different values of the determining parameters

Solution: $y=e^{hx}z(\xi)$, where $\xi=\dfrac{x-\mu}{\lambda}$					
Constraints	h	λ	μ	z	Parameters
$a_2\neq 0$, $D\neq 0$	$\dfrac{D-a_1}{2a_2}$	$-\dfrac{a_2}{A(h)}$	$-\dfrac{b_2}{a_2}$	$\mathcal{J}(a,b;\xi)$	$a=B(h)/A(h)$, $b=(a_2b_1-a_1b_2)a_2^{-2}$
$a_2=0$, $a_1\neq 0$	$-\dfrac{a_0}{a_1}$	1	$-\dfrac{2b_2h+b_1}{a_1}$	$\mathcal{J}\bigl(a,\frac{1}{2};k\xi^2\bigr)$	$a=B(h)/(2a_1)$, $k=-a_1/(2b_2)$
$a_2\neq 0$, $a_1^2=4a_0a_2$	$-\dfrac{a_1}{2a_2}$	a_2	$-\dfrac{b_2}{a_2}$	$\xi^\alpha Z_{2\alpha}\bigl(\beta\sqrt{\xi}\,\bigr)$	$\alpha=\dfrac{1}{2}-\dfrac{2b_2h+b_1}{2a_2}$, $\beta=2\sqrt{B(h)}$
$a_2=a_1=0$, $a_0\neq 0$	$-\dfrac{b_1}{2b_2}$	1	$\dfrac{4b_0b_2-b_1^2}{4a_0b_2}$	$\xi^{1/2}Z_{1/3}\bigl(k\xi^{3/2}\bigr)$	$k=\dfrac{2}{3}\Bigl(\dfrac{a_0}{b_2}\Bigr)^{1/2}$
Notation: $D^2=a_1^2-4a_0a_2$, $A(h)=2a_2h+a_1$, $B(h)=b_2h^2+b_1h+b_0$					

102. $(ax+b)y''_{xx}+s(cx+d)y'_x-s^2[(a+c)x+b+d]y=0$.

Particular solution: $y_0=e^{sx}$.

103. $(a_2x+b_2)y''_{xx}+(a_1x+b_1)y'_x+(a_0x+b_0)y=0$.

Let $\mathcal{J}(a,b;x)$ be an arbitrary solution of the degenerate hyperheometric equation $xy''_{xx}+(b-x)y'_x-ay=0$ (see 2.1.2.65), and $Z_\nu(x)$ be an arbitrary solution of the Bessel equation (see 2.1.2.121). The results of solving the original equation are presented in Table 2.2.

104. $(x+\lambda)y''_{xx}+(ax^n+bx^m+c)y'_x+(anx^{n-1}+bmx^{m-1})y=0$.

Particular solution: $y_0=\exp\Bigl(-\displaystyle\int\frac{ax^n+bx^m+c-1}{x+\lambda}\,dx\Bigr)$.

105. $x^2y''_{xx}+ay=0$.

This is a special case of equation 2.1.2.118. The substitution $x=e^t$ leads to a constant coefficient equation: $y''_{tt}-y'_t+ay=0$.

106. $x^2y''_{xx}+(ax+b)y=0$.

This is a special case of equation 2.1.2.127.

107. $x^2y''_{xx}+[a^2x^2-n(n+1)]y=0,\qquad n=0,\ 1,\ 2,\ \ldots$

Solution: $yx^{n+1}=(x^3D)^n\Bigl(\dfrac{C_1e^{ax}+C_2e^{-ax}}{x^{2n-1}}\Bigr)$, where $D=\dfrac{d}{dx}$.

108. $x^2 y''_{xx} - [a^2 x^2 + n(n+1)]y = 0, \qquad n = 0,\ 1,\ 2,\ \dots$

Solution: $\quad yx^{n+1} = (x^3 D)^n \left(\dfrac{C_1 \cos ax + C_2 \sin ax}{x^{2n-1}} \right)$, where $D = \dfrac{d}{dx}$.

109. $x^2 y''_{xx} - (a^2 x^2 + 2abx + b^2 - b)y = 0.$

Particular solution: $\quad y_0 = x^b e^{ax}$.

110. $x^2 y''_{xx} + (ax^2 + bx + c)y = 0.$

The substitution $y = x^\lambda u$, where λ is a root of the quadratic equation $\lambda^2 - \lambda + c = 0$, leads to an equation of the form 2.1.2.103: $xu''_{xx} + 2\lambda u'_x + (ax + b)u = 0$.

For $a = -\frac{1}{4}$, $b = k$, $c = \frac{1}{4} - m^2$, the original equation is reffered to as Whittaker's equation.

111. $x^2 y''_{xx} - (ax^3 + \frac{5}{16})y = 0.$

Particular solution: $\quad y_0 = x^{-1/4} \exp\left(\dfrac{2}{3} \sqrt{a}\, x^{3/2} \right)$.

112. $x^2 y''_{xx} - [a^2 x^4 + a(2b-1)x^2 + b(b+1)]y = 0.$

Particular solution: $\quad y_0 = x^{-b} \exp\left(-\dfrac{ax^2}{2} \right)$.

113. $x^2 y''_{xx} + (ax^n + b)y = 0.$

This is a special case of equation 2.1.2.127.

114. $x^2 y''_{xx} - [a^2 x^{2n} + a(2b + n - 1)x^n + b(b-1)]y = 0.$

Particular solution: $\quad y_0 = x^b \exp\left(\dfrac{a}{n} x^n \right)$.

115. $x^2 y''_{xx} + (ax^{2n} + bx^n + c)y = 0.$

This is a special case of equation 2.1.2.141.

116. $x^2 y''_{xx} + \left(ax^{3n} + bx^{2n} + \dfrac{1 - n^2}{4} \right) y = 0.$

The transformation $\xi = ax^n + b$, $w = yx^{\frac{n-1}{2}}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + (an)^{-2} \xi w = 0$.

117. $x^2 y''_{xx} + \left[ax^{2n}(bx^n + c)^m + \dfrac{1 - n^2}{4} \right] y = 0.$

The transformation $\xi = bx^n + c$, $w = yx^{\frac{n-1}{2}}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + (an)^{-2} \xi^m w = 0$.

118. $\boldsymbol{x^2y''_{xx} + axy'_x + by = 0}$.

The Euler equation.

Solution:

$$y = \begin{cases} |x|^{\frac{1-a}{2}} \left(C_1|x|^{\mu} + C_2|x|^{-\mu}\right) & \text{if } (1-a)^2 > 4b, \\ |x|^{\frac{1-a}{2}} \left(C_1 + C_2 \ln|x|\right) & \text{if } (1-a)^2 = 4b, \\ |x|^{\frac{1-a}{2}} \left[C_1 \sin(\mu \ln|x|) + C_2 \cos(\mu \ln|x|)\right] & \text{if } (1-a)^2 < 4b, \end{cases}$$

where $\mu = \frac{1}{2}|(1-a)^2 - 4b|^{1/2}$.

119. $\boldsymbol{x^2y''_{xx} + xy'_x + [x^2 - (n+\frac{1}{2})^2]y = 0, \qquad n = 0,\ 1,\ 2,\ \ldots}$

This is a special case of equation 2.1.2.121.

Solution: $\quad y = x^{n+1/2}\left[\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^n \left(C_1\dfrac{\sin x}{x} + C_2\dfrac{\cos x}{x}\right)\right]$.

120. $\boldsymbol{x^2y''_{xx} + xy'_x - [x^2 + (n+\frac{1}{2})^2]y = 0, \qquad n = 0,\ 1,\ 2,\ \ldots}$

This is a special case of equation 2.1.2.122.

Solution: $\quad y = x^{n+1/2}\left[\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^n \left(C_1\dfrac{e^x}{x} + C_2\dfrac{e^{-x}}{x}\right)\right]$.

121. $\boldsymbol{x^2y''_{xx} + xy'_x + (x^2 - \nu^2)y = 0}$.

The Bessel equation.

1°. Let ν be an arbitrary noninteger. Then the general solution is

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x), \tag{1}$$

where J_ν and Y_ν are Bessel functions of the first and second kind:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{2}$$

Solution (1) is denoted by $y = Z_\nu(x)$ which is reffered to as the cylindric function.

The cylindric functions possess the following properties:

$$\begin{aligned} 2\nu Z_\nu(x) &= x[Z_{\nu-1}(x) + Z_{\nu+1}(x)], \\ \frac{d}{dx}[x^\nu Z_\nu(x)] &= x^\nu Z_{\nu-1}(x), \\ \frac{d}{dx}[x^{-\nu} Z_\nu(x)] &= -x^{-\nu} Z_{\nu+1}(x). \end{aligned}$$

Functions J_ν and Y_ν may be presented in terms of definite integrals (with $x > 0$):

$$\begin{aligned} \pi J_\nu(x) &= \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu \int_0^\infty \exp(-x\sinh t - \nu t)\,dt, \\ \pi Y_\nu(x) &= \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu)e^{-x\sinh t}\,dt. \end{aligned}$$

2°. In the case $\nu = n + \frac{1}{2}$, where $n = 0, 1, 2, \ldots$, the Bessel functions are expressed in terms of elementary functions:

$$
\begin{aligned}
J_{n+\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x},\\
J_{-n-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x},\\
Y_{n+\frac{1}{2}}(x) &= (-1)^{n+1} J_{-n-\frac{1}{2}}(x).
\end{aligned}
$$

3°. Let $\nu = n$ be an arbitrary integer. The the following relations hold:

$$
J_{-n}(x) = (-1)^n J_n(x), \quad Y_{-n}(x) = (-1)^n Y_n(x).
$$

The solution is given by formula (1), wherein function $J_n(x)$ is obtained by substituting $\nu = n$ into formula (2), while function $Y_n(x)$ is found by taking the limit as $\nu \to n$ and for positive n becomes

$$
\begin{aligned}
Y_n(x) = \frac{2}{\pi} J_n(x) \ln\frac{x}{2} &- \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{2}{x}\right)^{n-2k}\\
&- \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{n+2k} \frac{\psi(k+1)+\psi(n+k+1)}{k!\,(n+k)!},
\end{aligned}
$$

where $\psi(1) = -\gamma$, $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$, $\gamma = 0.5572\ldots$ is the Euler constant, $\psi(x) = [\ln\Gamma(x)]'_x$ is the logarithmic derivative of the gamma-function.

For nonnegative integer n and large x, we may write

$$
\begin{aligned}
\sqrt{\pi x}\, J_{2n}(x) &= (-1)^n(\cos x + \sin x) + O(x^{-2}),\\
\sqrt{\pi x}\, J_{2n+1}(x) &= (-1)^{n+1}(\cos x - \sin x) + O(x^{-2}).
\end{aligned}
$$

Function J_n may be presented in terms of the definite integral

$$
J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin t - nt)\,dt; \qquad n = 0, 1, 2, \ldots
$$

The Bessel functions are described in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, vol. 2) in more detail (see also Suplement 2).

122. $\boldsymbol{x^2 y''_{xx} + x y'_x - (x^2 + \nu^2) y = 0}$.

The modified Bessel equation.

It can be reduced to the equation 2.1.2.121 by means of the substitution $x = i\bar{x}$. Solution:

$$
y = C_1 I_\nu(x) + C_2 K_\nu(x),
$$

where I_ν and K_ν are modified Bessel functions:

$$
I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}, \qquad K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu} - I_\nu}{\sin \pi\nu}.
$$

$I_\nu(x)$ may be expressed in terms of Bessel function:

$$I_\nu(x) = e^{-\pi\nu i/2} J_\nu(xe^{\pi i/2}), \quad i^2 = -1.$$

The case $\nu = n + \frac{1}{2}$, where $n = 0,\ 1,\ 2,\ \ldots$, is given in 2.1.2.120.

If $\nu = n$ is a nonnegative integer, we have

$$K_n(x) = (-1)^{n+1} I_n(x) \ln\frac{x}{2} + \frac{1}{2}\sum_{m=0}^{n-1}(-1)^m\Big(\frac{x}{2}\Big)^{2m-n}\frac{(n-m-1)!}{m!}$$
$$+\frac{1}{2}(-1)^n\sum_{m=0}^{\infty}\Big(\frac{x}{2}\Big)^{n+2m}\frac{\psi(n+m+1)+\psi(m+1)}{m!\,(n+m)!}; \quad n = 0, 1, 2, \ldots,$$

where $\psi(z)$ is the logarithmic derivative of the gamma-function (see 2.1.2.121); for $n = 0$, the first sum is omitted.

As $x \to +\infty$, the leading terms of the asymptotic expansion are

$$I_\nu(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \quad K_\nu(x) \simeq \frac{\sqrt{\pi}}{\sqrt{2x}} e^{-x}.$$

The modified Bessel functions are described in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, vol. 2) in more detail (see also Suplement 2).

123. $\boldsymbol{x^2 y''_{xx} + 2xy'_x - (a^2x^2 + 2)y = 0}$.

Solution: $x^2 y = C_1(ax - 1)e^{ax} + C_2(ax + 1)e^{-ax}$.

124. $\boldsymbol{x^2 y''_{xx} - 2axy'_x + [b^2x^2 + a(a+1)]y = 0}$.

Solution: $y = x^a(C_1 \sin bx + C_2 \cos bx)$.

125. $\boldsymbol{x^2 y''_{xx} - 2axy'_x + [-b^2x^2 + a(a+1)]y = 0}$.

Solution: $y = x^a(C_1 e^{bx} + C_2 e^{-bx})$.

126. $\boldsymbol{x^2 y''_{xx} + \lambda xy'_x + (ax^2 + bx + c)y = 0}$.

The substitution $y = x^k u$, where k is a root of the quadratic equation $k^2 + (\lambda - 1)k + c = 0$, leads to an equation of the form 2.1.2.103:

$$xu''_{xx} + (\lambda + 2k)u'_x + (ax + b)u = 0.$$

127. $\boldsymbol{x^2 y''_{xx} + axy'_x + (bx^n + c)y = 0}, \quad n \neq 0$.

The case $b = 0$ corresponds to the Euler equation 2.1.2.118.

For $b \neq 0$, the solution is

$$y = x^{\frac{1-a}{2}}\Big[C_1 J_\nu\Big(\frac{2}{n}\sqrt{b}\,x^{\frac{n}{2}}\Big) + C_2 Y_\nu\Big(\frac{2}{n}\sqrt{b}\,x^{\frac{n}{2}}\Big)\Big],$$

where $\nu = \frac{1}{n}\sqrt{(1-a)^2 - 4c}$, J_ν and Y_ν are Bessel functions of the first and second kind.

128. $x^2y''_{xx} + axy'_x + x^n(bx^n + c)y = 0.$

The substitution $\xi = x^n$ leads to an equation of the form 2.1.2.103:

$$n^2\xi y''_{\xi\xi} + n(n-1+a)y'_\xi + (b\xi + c)y = 0.$$

129. $x^2y''_{xx} + (ax+b)y'_x + cy = 0.$

The transformation $x = \xi^{-1}$, $y = \xi^k e^\xi w$, where k is a root of the quadratic equation $k^2 + (1-a)k + c = 0$, leads to the equation of the form 2.1.2.103:

$$\xi w''_{\xi\xi} + [(2-b)\xi + 2k + 2 - a]w'_\xi + [(1-b)\xi + 2k + 2 - a - bk]w = 0.$$

130. $x^2y''_{xx} + ax^2y'_x + (bx^2 + cx + d)y = 0.$

The substitution $y = u\exp\bigl(-\frac{1}{2}ax\bigr)$ leads to an equation of the form 2.1.2.110:

$$x^2y''_{xx} + [(\tfrac{1}{4}a^2 + b)x^2 + cx + d]u = 0.$$

131. $x^2y''_{xx} + (ax^2 + b)y'_x + c[(a-c)x^2 + b]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

132. $x^2y''_{xx} + (ax^2 + bx)y'_x - by = 0.$

Particular solution: $y_0 = x^{-b}e^{-ax}$.

133. $x^2y''_{xx} + (ax^2 + bx)y'_x + [k(a-k)x^2 + (an + bk - 2kn)x + n(b-n-1)]y = 0.$

Particular solution: $y_0 = x^{-n}e^{-kx}$.

134. $a_2x^2y''_{xx} + (a_1x^2 + b_1x)y'_x + (a_0x^2 + b_0x + c_0)y = 0.$

The substitution $y = x^k w$, where k is a root of the quadratic equation $a_2k^2 + (b_1 - a_2)k + c_0 = 0$, leads to the equation of the form 2.1.2.103:

$$a_2xw''_{xx} + (a_1x + 2a_2k + b_1)w'_x + (a_0x + a_1k + b_0)w = 0.$$

135. $x^2y''_{xx} + [ax^2 + (ab-1)x + b]y'_x + a^2bxy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}$.

136. $x^2y''_{xx} - 2x(x^2 - a)y'_x + \{2nx^2 + [(-1)^n - 1]a\}y = 0.$

For $n = 0, 1, 2, \ldots$, particular solutions are polynomials: $y_0 = P_n(x)$, where $P_0 = 1$, $P_1 = x$, $P_2 = 2x^2 - 1 - 2a$, $P_3 = 2x^3 - (3+2a)x$, ... The polynomials contain only even powers of x for even n and only odd powers of x for odd n.

137. $x^2y''_{xx} + x(ax^2 + bx + c)y'_x + (Ax^3 + Bx^2 + Cx + D)y = 0.$

1°. The substitution $y = x^k w$, where k is a root of the quadratic equation $k^2 + (c-1)k + D = 0$, leads to an equation of the form 2.1.2.79 (see also 2.1.2.75–2.1.2.78):

$$xw''_{xx} + (ax^2 + bx + c + 2k)w'_x + [Ax^2 + (B + ak)x + C + bk]y = 0.$$

2°. Let s and r be arbitrary parameters.

For $A = ar$, $B = as + br - r^2$, $C = bs + cr - 2rs$, $D = s(c-s-1)$, a particular solution is $y_0 = x^{-s}e^{-rx}$.

For $A = a(b-r)$, $B = a(c-s+1) + r(b-r)$, $C = bs + cr - 2rs$, $D = s(c-s-1)$, a particular solution is $y_0 = x^{-s}\exp(-\frac{1}{2}ax^2 - rx)$.

138. $x^2y''_{xx}+ax^ny'_x-(abx^n+acx^{n-1}+b^2x^2+2bcx+c^2-c)y=0.$

Particular solution: $y_0=x^ce^{bx}$.

139. $x^2y''_{xx}+ax^ny'_x+(abx^{n+2m}-b^2x^{4m+2}+amx^{n-1}-m^2-m)y=0.$

Particular solution: $y_0=x^{-m}\exp\Bigl(-\dfrac{b}{2m+1}x^{2m+1}\Bigr)$.

140. $x^2y''_{xx}+x(ax^n+b)y'_x+b(ax^n-1)y=0.$

Particular solution: $y_0=x^{-b}$.

141. $x^2y''_{xx}+x(ax^n+b)y'_x+(\alpha x^{2n}+\beta x^n+\gamma)y=0.$

The transformation $\xi=x^n$, $w=y\xi^{-k}$, where k is a root of the quadratic equation $n^2k^2+n(b-1)k+\gamma=0$, leads to an equation of the form 2.1.2.103:

$$n^2\xi w''_{xx}+[na\xi+2kn^2+n(n-1+b)]w'_x+(\alpha\xi+kna+\beta)w=0.$$

142. $x^2y''_{xx}+x(2ax^n+b)y'_x+[a^2x^{2n}+a(b+n-1)x^n+\alpha x^{2m}+\beta x^m+\gamma]y=0.$

The substitution $w=y\exp\Bigl(\dfrac{a}{n}x^n\Bigr)$ leads to the equation of the form 2.1.2.141:

$$x^2w''_{xx}+bxw'_x+(\alpha x^{2m}+\beta x^m+\gamma)w=0.$$

143. $x^2y''_{xx}+(ax^{n+2}+bx^2+c)y'_x+(anx^{n+1}+acx^n+bc)y=0.$

Particular solution: $y_0=\exp\Bigl(-\dfrac{a}{n+1}x^{n+1}-bx\Bigr)$.

144. $(1-x^2)y''_{xx}+n(n-1)y=0, \qquad n=0,\ 1,\ 2,\ \ldots$

This equation is encountered in Hydrodynamics when decribing axially symmetric Stokes flows.

For $n\ge 2$, the solution is

$$y=C_1\mathcal{J}_n(x)+C_2\mathcal{H}_n(x),$$

where $\mathcal{J}_n$ and $\mathcal{H}_n$ are the Gegenbauer functions which may be presented in terms of the Legendre functions of the first and second kind (see 2.1.2.147) as follows:

$$\mathcal{J}_n(x)=\frac{P_{n-2}(x)-P_n(x)}{2n-1},\quad \mathcal{H}_n(x)=\frac{Q_{n-2}(x)-Q_n(x)}{2n-1}.$$

For $n=0$ and $n=1$, the solution is $y=C_1+C_2x$.

145. $(x^2-a^2)y''_{xx}+by'_x-6y=0.$

Particular solution: $y_0=(4x-b)|x+a|^{\frac{2a+b}{2a}}|x-a|^{\frac{2a-b}{2a}}$.

146. $(x^2-1)y''_{xx}+xy'_x+ay=0$.

1°. For $a=k^2>0$, the solution is

$$y=\begin{cases} C_1\cos(k\operatorname{Arcosh}|x|)+C_2\sin(k\operatorname{Arcosh}|x|) & \text{if } |x|>1,\\ C_1\exp(k\arccos x)+C_2\exp(-k\arccos x) & \text{if } |x|<1,\end{cases}$$

where $\operatorname{Arcosh} x=\ln\bigl(x+\sqrt{x^2-1}\,\bigr)$.

2°. For $a=-k^2<0$, the solution is

$$y=\begin{cases} C_1\exp(k\operatorname{Arcosh}|x|)+C_2\exp(-k\operatorname{Arcosh}|x|) & \text{if } |x|>1,\\ C_1\cos(k\arccos x)+C_2\sin(k\arccos x) & \text{if } |x|<1.\end{cases}$$

3°. For $a=-n^2$, where n is a nonnegative integer, partial solutions are the Tchebycheff polynomials $T_n(x)=2^{1-n}\cos(n\arccos x)$.

147. $(1-x^2)y''_{xx}-2xy'_x+n(n+1)y=0, \qquad n=0,\ 1,\ 2,\ \dots$

The Legendre equation.

The solution is

$$y=C_1P_n(x)+C_2Q_n(x),$$

where the Legendre polynomials $P_n(x)$ and the Legendre functions of the second kind $Q_n(x)$ are given by the formulae

$$P_n(x)=\frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n,$$

$$Q_n(x)=\frac{1}{2}P_n(x)\ln\frac{1+x}{1-x}-\sum_{m=1}^{n}\frac{1}{m}P_{m-1}(x)P_{n-m}(x).$$

Functions $P_n=P_n(x)$ can be conveniently calculated by the recurrence relations

$$P_0=1,\quad P_1=x,\quad P_2=\frac{1}{2}(3x^2-1),\quad \dots,\quad P_{n+1}=\frac{2n+1}{n+1}xP_n-\frac{n}{n+1}P_{n-1}.$$

Three leading functions $Q_n=Q_n(x)$ are

$$Q_0=\frac{1}{2}\ln\frac{1+x}{1-x},\quad Q_1=\frac{x}{2}\ln\frac{1+x}{1-x}-1,\quad Q_2=\frac{3x^2-1}{4}\ln\frac{1+x}{1-x}-\frac{3}{2}x.$$

All n zeros of the polynomial $P_n(x)$ are real and lie on the interval $-1<x<+1$; functions $P_n(x)$ form an orthogonal system on the closed interval $-1\le x\le +1$, with the relations taking place

$$\int_{-1}^{+1}P_n(x)P_m(x)\,dx=\begin{cases}0 & \text{if } n\ne m,\\ \dfrac{2}{2n+1} & \text{if } n=m.\end{cases}$$

148. $(1-x^2)y''_{xx}-2xy'_x+\nu(\nu+1)y=0$.

The Legendre equation; ν is an arbitrary number.

The case $\nu=n$ where n is a nonnegative integer is considered in 2.1.2.147.

The substitution $z=x^2$ leads to the hypergeometric equation. Therefore, with $|x|<1$ the solution can be written as

$$y=C_1F\Bigl(-\frac{\nu}{2},\ \frac{1+\nu}{2},\ \frac{1}{2};\ x\Bigr)+C_2xF\Bigl(\frac{1-\nu}{2},\ 1+\frac{\nu}{2},\ \frac{3}{2};\ x\Bigr),$$

where F is the hypergeometric series (see 2.1.2.158).

The Legendre equation is discussed in the books by Abramowitz & Stegun (1964), Bateman & Erdélyi (1953, vol. 1), and Kamke (1976) in more detail (see also Suplement 2).

149. $(x^2-1)y''_{xx}+2(n+1)xy'_x-(\nu+n+1)(\nu-n)y=0, \qquad n=1,\ 2,\ 3,\ \ldots$

Solution: $y=\dfrac{d^n}{dx^n}y_\nu(x)$, where y_ν is the general solution of the Legendre equation 2.1.2.148.

150. $(x^2-1)y''_{xx}-2(n-1)xy'_x-(\nu-n+1)(\nu+n)y=0, \qquad n=1,\ 2,\ 3,\ \ldots$

Solution: $y=|x^2-1|^n\dfrac{d^n}{dx^n}y_\nu(x)$, where y_ν is the general solution of the Legendre equation 2.1.2.148.

151. $(ax^2+b)y''_{xx}+axy'_x+cy=0.$

The substitution $z=\displaystyle\int\frac{dx}{\sqrt{ax^2+b}}$ leads to a constant coefficient equation: $y''_{zz}+cy=0$.

152. $(x^2+a)y''_{xx}+2bxy'_x+2(b-1)y=0.$

Particular solution: $y_0=(x^2+a)^{1-b}$.

153. $(ax^2+b)y''_{xx}+(2n+1)axy'_x+cy=0, \qquad n=1,\ 2,\ 3,\ \ldots$

This equation can be obtained by n-fold differentiation of the equation of the form 2.1.2.151:

$$(ax^2+b)u''_{xx}+axu'_x+(c-an^2)u=0.$$

Solution: $y=\dfrac{d^n u}{dx^n}$.

154. $(1-x^2)y''_{xx}-xy'_x+(2ax^2+b)y=0.$

This is an algebraic form of the Mathieu equation.

The substitution $x=\cos z$ leads to the Mathieu equation 2.1.6.4: $y''_{zz}+(a+b+a\cos 2z)y=0$.

155. $(1-x^2)y''_{xx}+(ax+b)y'_x+cy=0.$

The substitution $2z=1+x$ leads to the hypergeometric equation 2.1.2.158:

$$z(1-z)y''_{zz}+[az+\tfrac{1}{2}(b-a)]y'_z+cy=0.$$

156. $(ax^2+b)y''_{xx}+(cx^2+d)y'_x+\lambda[(c-a\lambda)x^2+d-b\lambda]y=0.$

Particular solution: $y_0=e^{-\lambda x}$.

157. $(ax^2+b)y''_{xx}+[\lambda(c+a)x^2+(c-a)x+2b\lambda]y'_x+\lambda^2(cx^2+b)y=0.$

Particular solution: $y_0=(\lambda x+1)e^{-\lambda x}$.

158. $\boldsymbol{x(x-1)y''_{xx} + [(\alpha+\beta+1)x-\gamma]y'_x + \alpha\beta y = 0}$.

The Gauss hypergeometric equation.

For $\gamma \neq 0,\ -1,\ -2,\ -3,\ \dots$, a solution can be expressed in terms of the hypergeometric series:

$$F(\alpha,\beta,\gamma;x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}, \quad (\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1),$$

which is a fortiori convergent for $|x| < 1$.

For $\gamma > \beta > 0$, this solution can be expressed in terms of the definite integral

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma-function.

If γ is not an integer, the general solution of the hypergeometric equation has the form

$$y = C_1F(\alpha,\beta,\gamma;x) + C_2x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x).$$

In degenerate cases $\gamma = 0,\ -1,\ -2,\ -3,\ \dots$, a particular solution of the hypergeometric equation corresponds to $C_1 = 0$, $C_2 = 1$. If γ is a positive integer, another particular solution corresponds to $C_1 = 1$, $C_2 = 0$. In both these cases, the general solution can be constructed by means of the formula given in 2.1.1.

Table 2.3 represents some special cases where F is expressed in terms of elementary functions.

In Table 2.4 are given the general solutions of the hypergeometric equation for some values of the determining parameters.

Function F possesses the following properties:

$$\begin{aligned}
F(\alpha,\beta,\gamma;x) &= F(\beta,\alpha,\gamma;x),\\
F(\alpha,\beta,\gamma;x) &= (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),\\
F(\alpha,\beta,\gamma;x) &= (1-x)^{-\alpha}F(\alpha,\ \gamma-\beta,\ \gamma;\ \frac{x}{x-1}),\\
\frac{d^n}{dx^n}F(\alpha,\beta,\gamma;x) &= \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\ \beta+n,\ \gamma+n;\ x).
\end{aligned}$$

The hypergeometric functions are discussed in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1953, vol. 1) in more detail.

159. $\boldsymbol{x(x+a)y''_{xx} + (bx+c)y'_x + dy = 0}$.

The substitution $x = -az$ leads to the hypergeometric equation 2.1.2.158:

$$z(1-z)y''_{zz} + [(c/a) - bz]y'_z - dy = 0.$$

160. $\boldsymbol{2x(x-1)y''_{xx} + (2x-1)y'_x + (ax+b)y = 0}$.

The substitution $x = \cos^2\xi$ leads to the Mathieu equation 2.1.6.4:

$$y''_{\xi\xi} - (a + 2b + a\cos 2\xi)y = 0.$$

TABLE 2.3

Some special cases in which the hypergeometric function $F(\alpha,\beta,\gamma;z)$ is expressed in terms of elementary functions

α	β	γ	z	F
$-n$	β	γ	x	$\displaystyle\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(\gamma)_k}\,\frac{x^k}{k!},\quad \text{where } n=1,2,3,\dots$
$-n$	β	$-n-m$	x	$\displaystyle\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(-n-m)_k}\,\frac{x^k}{k!},\quad \text{where } n=1,2,3,\dots$
α	β	β	x	$(1-x)^{-\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{1}{2}$	x^2	$\frac{1}{2}\left[(1+x)^{-2\alpha}+(1-x)^{-2\alpha}\right]$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	x^2	$\dfrac{(1+x)^{1-2\alpha}-(1-x)^{1-2\alpha}}{2x(1-2\alpha)}$
α	$-\alpha$	$\frac{1}{2}$	$-x^2$	$\frac{1}{2}\left\{\left[\sqrt{1+x^2}+x\right]^{2\alpha}+\left[\sqrt{1+x^2}-x\right]^{2\alpha}\right\}$
α	$1-\alpha$	$\frac{1}{2}$	$-x^2$	$\dfrac{\left[\sqrt{1+x^2}+x\right]^{2\alpha-1}+\left[\sqrt{1+x^2}-x\right]^{2\alpha-1}}{2\sqrt{1+x^2}}$
α	$\alpha-\frac{1}{2}$	$2\alpha-1$	x	$2^{2\alpha-2}\left[1+\sqrt{1-x}\,\right]^{2-2\alpha}$
α	$1-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\dfrac{\sin[(2\alpha-1)x]}{(\alpha-1)\sin(2x)}$
α	$2-\alpha$	$\frac{3}{2}$	$\sin^2 x$	$\dfrac{\sin[(2\alpha-2)x]}{(\alpha-1)\sin(2x)}$
α	$1-\alpha$	$\frac{1}{2}$	$\sin^2 x$	$\dfrac{\cos[(2\alpha-1)x]}{\cos x}$
α	$\alpha+1$	$\frac{1}{2}\alpha$	x	$(1+x)(1-x)^{-\alpha-1}$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	x	$\left(\dfrac{1+\sqrt{1-x}}{2}\right)^{-2\alpha}$
α	$\alpha+\frac{1}{2}$	2α	x	$\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1+\sqrt{1-x}}{2}\right)^{1-2\alpha}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	x^2	$\dfrac{1}{x}\arcsin x$
$\frac{1}{2}$	1	$\frac{3}{2}$	$-x^2$	$\dfrac{1}{x}\arctan x$
1	1	2	$-x$	$\dfrac{1}{x}\ln(x+1)$
$\frac{1}{2}$	1	$\frac{3}{2}$	x^2	$\dfrac{1}{2x}\ln\dfrac{1+x}{1-x}$
$n+1$	$n+m+1$	$n+m+l+2$	x	$\dfrac{(-1)^m(n+m+l+1)!}{n!\,l!\,(n+m)!\,(m+l)!}\,\dfrac{d^{n+m}}{dx^{n+m}}\left\{(1-x)^{m+l}\dfrac{d^lF}{dx^l}\right\},$ $F=-\dfrac{\ln(1-x)}{x},\quad n,m,l=0,1,2,3,\dots$

TABLE 2.4
General solutions of the hypergeometric equation for some values of the determining parameters

α	β	γ	Solution: $y=y(x)$
0	β	γ	$C_1+C_2\int \lvert x\rvert^{-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\,dx$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	$C_1\bigl(1+\sqrt{1-x}\bigr)^{-2\alpha}+C_2x^{-2\alpha}\bigl(1+\sqrt{1-x}\bigr)^{2\alpha}$
α	$\alpha-\frac{1}{2}$	$\frac{1}{2}$	$C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	$\dfrac{1}{\sqrt{x}}\Bigl[C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}\Bigr]$
1	β	γ	$\lvert x\rvert^{1-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\Bigl(C_1+C_2\int \lvert x\rvert^{\gamma-2}\lvert x-1\rvert^{\beta-\gamma}\,dx\Bigr)$
α	β	α	$\lvert x-1\rvert^{-\beta}\Bigl(C_1+C_2\int \lvert x\rvert^{-\alpha}\lvert x-1\rvert^{\beta-1}\,dx\Bigr)$
α	β	$\alpha+1$	$\lvert x\rvert^{-\alpha}\Bigl(C_1+C_2\int \lvert x\rvert^{\alpha-1}\lvert x-1\rvert^{-\beta}\,dx\Bigr)$

161. $\boldsymbol{(x^2+2ax+b)y''_{xx}+(x+a)y'_x-m^2y=0}$.

Solution: $y=C_1\bigl(x+a+\sqrt{x^2+2ax+b}\bigr)^m+C_2\bigl(x+a+\sqrt{x^2+2ax+b}\bigr)^{-m}$.

162. $\boldsymbol{(ax^2+bx+c)y''_{xx}+(dx+k)y'_x+(d-2a)y=0}$.

Integrating yields a first order linear equation:

$$(ax^2+bx+c)y'_x+[(d-2a)x+k-b]y=C.$$

163. $\boldsymbol{(ax^2+bx+c)y''_{xx}+(kx+d)y'_x-ky=0}$.

Particular solution: $y_0=kx+d$.

164. $\boldsymbol{(ax^2+2bx+c)y''_{xx}+(ax+b)y'_x+dy=0}$.

The substitution $\xi=\displaystyle\int\frac{dx}{\sqrt{ax^2+2bx+c}}$ leads to a constant coefficient equation: $y''_{\xi\xi}+dy=0$.

165. $\boldsymbol{(ax^2+2bx+c)y''_{xx}+3(ax+b)y'_x+dy=0}$.

The substitution $u=y\sqrt{|ax^2+2bx+c|}$ leads to an equation of the form 2.1.2.164:

$$(ax^2+2bx+c)u''_{xx}+(ax+b)u'_x+(d-a)u=0.$$

166. $(a_2x^2+b_2x+c_2)y''_{xx}+(b_1x+c_1)y'_x+c_0y=0.$

Let λ_1 and λ_2 be the roots of the quadratic equation $a_2\lambda^2+b_2\lambda+c_2=0$.

1°. For $\lambda_1\neq\lambda_2$, the substitution $z=\dfrac{x-\lambda_1}{\lambda_2-\lambda_1}$ leads to the hypergeometric equation 2.1.2.158:

$$z(1-z)y''_{zz}-(Ax+B)y'_z-Cy=0,$$

where $A=\dfrac{b_1}{a_2}$, $B=\dfrac{b_1\lambda_1+c_1}{a_2(\lambda_2-\lambda_1)}$, $C=\dfrac{c_0}{a_2}$.

2°. For $\lambda_1=\lambda_2=\lambda$, the transformation $x=\lambda+\xi^{-1}$, $y=\xi^k u$, where k is a root of the quadratic equation $a_2k^2+(a_2-b_1)k+c_0=0$, leads to an equation of the form 2.1.2.103:

$$a_2\xi u''_{\xi\xi}-[(c_1+\lambda b_1)\xi+b_1-2a_2(k+1)]u'_\xi-k(c_1+\lambda b_1)u=0.$$

3°. Let $c_0=-a_2n(n-1)-b_1n$, where n is a positive integer. Then, amongst solutions there exists a polynomial of the degree $\le n$.

167. $(ax^2+bx+c)y''_{xx}-(x^2-k^2)y'_x+(x+k)y=0.$

Particular solution: $y_0=x-k$.

168. $(ax^2+bx+c)y''_{xx}+(x^3+k^3)y'_x-(x^2-kx+k^2)y=0.$

Particular solution: $y_0=x+k$.

169. $x^3y''_{xx}+(ax+b)y=0.$

This is a special case of equation 2.1.2.127 with $n=-1$.

170. $x^3y''_{xx}+(ax^2+b)y'_x+cxy=0.$

The substitution $x=1/z$ leads to an equation of the form 2.1.2.141:

$$z^2y''_{zz}+z(2-a-bz)y'_z+cy=0.$$

171. $x^3y''_{xx}+(ax^2+bx)y'_x+by=0.$

Particular solution: $y_0=a-2+\dfrac{b}{x}$.

172. $x^3y''_{xx}+(ax^2+bx)y'_x+cy=0.$

The substitution $x=1/z$ leads to an equation of the form 2.1.2.103:

$$zy''_{zz}+(2-a-bz)y'_z+cy=0.$$

173. $x^3y''_{xx}+(ax^2+bx)y'_x+(cx+d)y=0.$

The substitution $y=x^ku$, where $k=-d/b$,leads to the equation of the form 2.1.2.129:

$$x^2u''_{xx}+[(a+2k)x+b]u'_x+[k(a+k-1)+c]u=0.$$

If $c=0$ and $d=b(a-2)$, a particular solution is $y_0=e^{b/x}$.

174. $x^3 y''_{xx} + (ax^3 - x^2 + abx + b)y'_x + a^2 bxy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}.$

175. $x^3 y''_{xx} + x(ax^n + b)y'_x - (ax^n - abx^{n-1} + b)y = 0.$

Particular solution: $y_0 = x\exp\Big(\dfrac{b}{x}\Big).$

176. $x(ax^2 + b)y''_{xx} + 2(ax^2 + b)y'_x - 2axy = 0.$

Particular solution: $y_0 = ax + \dfrac{b}{x}.$

177. $x(x^2 + a)y''_{xx} + (bx^2 + c)y'_x + sxy = 0.$

The substitution $az = -x^2$ leads to the hypergeometric equation 2.1.2.158:

$$z(1-z)y''_{zz} + \frac{1}{2}\Big[1 + \frac{c}{a} - (1+b)z\Big]y'_z - \frac{1}{4}sy = 0.$$

178. $x^2(ax + b)y''_{xx} + [cx^2 + (2b + a\lambda)x + b\lambda]y'_x + \lambda(c - 2a)y = 0.$

Particular solution: $y_0 = \exp\Big(\dfrac{\lambda}{x}\Big).$

179. $x^2(ax + b)y''_{xx} - 2x(ax + 2b)y'_x + 2(ax + 3b)y = 0.$

Solution: $y = \dfrac{C_1 x^2 + C_2 x^3}{ax + b}.$

180. $x^2(ax + b)y''_{xx} + [a(2 - n - m)x^2 - b(n + m)x]y'_x + [am(n - 1)x + bn(m + 1)]y = 0.$

Solution: $y = \dfrac{C_1 x^n + C_2 x^{m+1}}{ax + b}.$

181. $x^2(x + a_2)y''_{xx} + x(b_1 x + a_1)y'_x + (b_0 x + a_0)y = 0.$

The substitution $y = x^k u$, where k is a root of the quadratic equation $a_2 k^2 + k(a_1 - a_2) + a_0 = 0$ leads to an equation of the form 2.1.2.159:

$$x(x + a_2)u''_{xx} + [(2k + b_1)x + 2ka_2 + a_1]u'_x + [k^2 + k(b_1 - 1) + b_0]u = 0.$$

182. $(ax^3 + bx^2 + cx)y''_{xx} + (\alpha x^2 + \beta x + 2c)y'_x + (\beta - 2b)y = 0.$

Particular solution: $y_0 = 2a - \alpha + \dfrac{2b - \beta}{x}.$

183. $(ax^3 + bx^2 + cx)y''_{xx} + (\alpha x^2 + \beta x + 2c)y'_x - (\alpha x + 2b - \beta)y = 0.$

Particular solution:

$$y_0 = \alpha x + 2(\beta - b) + \frac{\lambda}{x}, \qquad \text{where} \quad \lambda = \frac{c\alpha + (b - \beta)(2b - \beta)}{\alpha - a}.$$

184. $(ax^3+bx^2+cx)y''_{xx}+[-2ax^2-(b+1)x+k]y'_x+2(ax+1)y=0.$

Particular solution: $y_0=(ak+b-1)x^2+(c+k)(2x-k)$.

185. $(ax^3+bx^2+cx)y''_{xx}+(nx^2+mx+k)y'_x+(k-1)[(n-ak)x+m-bk]y=0.$

Particular solution: $y_0=x^{1-k}$.

186. $(ax^3+bx^2+cx)y''_{xx}+[(m-a)x^2+(2cm-1)x-c]y'_x+(-2mx+1)y=0.$

Particular solution: $y_0=(a+m)x^2+(2b+4cm-1)(x+c)$.

187. $(ax^3+bx^2+cx)y''_{xx}+(nx^2+mx+k)y'_x+[-2(a+n)x+1]y=0.$

With the constraint

$$2(2a+n)(c+k)+(2b+2m+1)[m+1+2k(a+n)]=0,$$

a particular solution has the form $y_0=(2a+n)x^2+(2b+2m+1)(x-k)$.

188. $(ax^3+x^2+b)y''_{xx}+a^2x(x^2-b)y'_x-a^3bxy=0.$

Particular solution: $y_0=(ax+2)e^{-ax}$.

189. $2x(ax^2+bx+c)y''_{xx}+(ax^2-c)y'_x+\lambda x^2y=0.$

The substitution $\xi=\displaystyle\int\Big(\frac{x}{ax^2+bx+c}\Big)^{1/2}dx$ leads to a constant coefficient equation: $2y''_{\xi\xi}+\lambda y=0$.

190. $x(ax^2+bx+1)y''_{xx}+(\alpha x^2+\beta x+\gamma)y'_x+(nx+m)y=0.$

The substitution $y=x^{1-\gamma}w$ leads to an equation of the similar form:

$$x(ax^2+bx+1)w''_{xx}+[(\alpha+2a-2a\gamma)x^2+(\beta+2b-2b\gamma)x+2-\gamma]w'_x+\\ \big\{[n+(1-\gamma)(\alpha-a\gamma)]x+m+(1-\gamma)(\beta-b\gamma)\big\}w=0.$$

191. $x(x-1)(x-a)y''_{xx}+$
$\big\{(\alpha+\beta+1)x^2-[\alpha+\beta+1+a(\gamma+\delta)-\delta]x+a\gamma\big\}y'_x+(\alpha\beta x-q)y=0.$

Heun's equation.

For $|a|\ge 1$ and $\gamma\neq 0,\ -1,\ -2,\ -3,\ \dots$, a solution can be represented as the power series

$$F(a,q;\alpha,\beta,\gamma,\delta,x)=\sum_{n=0}^{\infty}c_nx^n,$$

wherein the coefficients are determined by the recurrence formulae

$$c_0=1,\quad a\gamma c_1=q,$$
$$a(n+1)(\gamma+n)c_{n+1}=\Big[a(\gamma+\delta+n-1)+\alpha+\beta-\delta+n+\frac{q}{n}\Big]nc_n\\ -\big[(n-1)(n-2)+(n-1)(\alpha+\beta+1)+\alpha\beta\big]c_{n-1}.$$

The series is a fortiori convergent for $|x|\le 1$.

TABLE 2.5

Some transformations retaining the form of Heun's equation

No	New variables	Parameters of transformed equation for $w=w(\xi)$					
1*	$\xi=x,\ w=y$	a	q	α	β	γ	δ
2	$\xi=1-x,\ w=y$	$1-a$	$\alpha\beta-q$	α	β	δ	γ
3	$\xi=x,\ w=\|x\|^{\gamma-1}y$	a	q_1	$\alpha-\gamma+1$	$\beta-\gamma+1$	$2-\gamma$	δ
4	$\xi=\dfrac{x}{a},\ w=\|x\|^{\gamma-1}y$	$\dfrac{1}{a}$	$\dfrac{q_2}{a}$	$\alpha-\gamma+1$	$\beta-\gamma+1$	$2-\gamma$	$\alpha+\beta-\gamma-\delta+1$
5	$\xi=\dfrac{1}{x},\ w=\|x\|^{\alpha}y$	$\dfrac{1}{a}$	q_3	α	$\alpha-\gamma+1$	$\alpha-\beta+1$	δ
6	$\xi=\dfrac{x}{a},\ w=y$	$\dfrac{1}{a}$	q	α	β	γ	$\alpha+\beta-\gamma-\delta+1$
7	$\xi=1-\dfrac{x}{a},\ w=y$	$1-\dfrac{1}{a}$	q	α	β	$\alpha+\beta-\gamma-\delta+1$	γ
8	$\xi=\dfrac{x}{a},\ w=\|x\|^{\alpha}y$	a	q	$\alpha-\gamma+1$	$\alpha+\gamma-1$	$\alpha-\beta+1$	$\alpha+\beta-\gamma-\delta+1$
9	$\xi=\dfrac{x-1}{x},\ w=\|x\|^{\alpha}y$	$1-\dfrac{1}{a}$	q	α	$\alpha-\gamma+1$	δ	$\alpha-\beta+1$
10	$\xi=\dfrac{a(x-1)}{x(a-1)},\ w=\|x\|^{\alpha}y$	$\dfrac{a}{a-1}$	q	α	$\alpha-\gamma+1$	δ	$\alpha+\beta-\gamma-\delta+1$
11	$\xi=\dfrac{x}{x-1},\ w=\|x-1\|^{\alpha}y$	$\dfrac{a}{a-1}$	q	α	$\alpha-\delta+1$	γ	$\alpha-\beta+1$
12	$\xi=\dfrac{x(a-1)}{a(x-1)},\ w=\|x-1\|^{\alpha}y$	$1-\dfrac{1}{a}$	q	α	$\alpha-\delta+1$	γ	$\alpha+\beta-\gamma-\delta+1$
Notation: $q_1=q+(\alpha-\gamma+1)(\beta-\gamma+1)-\alpha\beta+\delta(\gamma-1)$, $q_2=q_1+a\delta(1-\gamma)$, $q_3=qa^{-1}+\alpha(\alpha-\gamma+1)+\alpha a^{-1}(\delta-\beta)-a\delta$.							

* This row corresponds to the original equation, while the others refer to the transformed equation for $w=w(\xi)$

If γ is not an integer, the general solution of Heun's equation can be presented as follows:

$$y = C_1 F(a,q;\alpha,\beta,\gamma,\delta,x) + C_2|x|^{1-\gamma}F(a,\ q_1;\ \alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ \delta,\ x),$$

where $q_1 = q + (\alpha-\gamma+1)(\beta-\gamma+1) - \alpha\beta + \delta(\gamma-1)$.

In Table 2.5 are listed some transformations retaining the form of Heun's equation. (Whenever at least one of the indicated equations is integrable by quadrature with some values of parameters, all the other equations are also integrable for those values of the parameters.)

For Heun's equation, see also the book by Bateman & Erdélyi (1955, vol. 3).

192. $(ax^3+bx^2+cx+d)y''_{xx} - (x^2-\lambda^2)y'_x + (x+\lambda)y = 0.$

Particular solution: $y_0 = x - \lambda$.

193. $\boldsymbol{2(ax^3 + bx^2 + cx + d)y''_{xx} + (3ax^2 + 2bx + c)y'_x + \lambda y = 0.}$

The substitution $\xi = \int \frac{dx}{\sqrt{ax^3 + bx^2 + cx + d}}$ leads to a constant coefficient equation: $2y''_{\xi\xi} + \lambda y = 0$.

194. $\boldsymbol{2(ax^3 + bx^2 + cx + d)y''_{xx} + 3(3ax^2 + 2bx + c)y'_x + (6ax + 2b + \lambda)y = 0.}$

This equation is obtained by differentiating the equation 2.1.2.193.

195. $\boldsymbol{(ax^3 + bx^2 + cx + d)y''_{xx} + [\alpha x^2 + (\alpha\gamma + \beta)x + \beta\gamma]y'_x - (\alpha x + \beta)y = 0.}$

Particular solution: $y_0 = x + \gamma$.

196. $\boldsymbol{(ax^3 + bx^2 + cx + d)y''_{xx} + (x^3 + \lambda^3)y'_x - (x^2 - \lambda x + \lambda^2)y = 0.}$

Particular solution: $y_0 = x + \lambda$.

197. $\boldsymbol{2x(ax^2 + bx + c)y''_{xx} + [a(2-k)x^2 + b(1-k)x - ck]y'_x + \lambda x^{k+1}y = 0.}$

The substitution $\xi = \int \frac{x^{k/2}\,dx}{\sqrt{ax^2 + bx + c}}$ leads to a constant coefficient equation: $2y''_{\xi\xi} + \lambda y = 0$.

198. $\boldsymbol{x^4 y''_{xx} + ay = 0.}$

The transformation $z = 1/x$, $u = y/x$ leads to a constant coefficient equation: $u''_{zz} + au = 0$.

199. $\boldsymbol{x^4 y''_{xx} + (ax^2 + bx + c)y = 0.}$

The transformation $z = 1/x$, $u = y/x$ leads to an equation of the form 2.1.2.110: $z^2 u''_{zz} + (cz^2 + bz + a)u = 0$.

200. $\boldsymbol{x^4 y''_{xx} - (a + b)x^2 y'_x + [(a + b)x + ab]y = 0.}$

Solution:

$$y = \begin{cases} C_1 x e^{-a/x} + C_2 x e^{-b/x} & \text{if } a \neq b, \\ (C_1 x + C_2)e^{-a/x} & \text{if } a = b. \end{cases}$$

201. $\boldsymbol{x^4 y''_{xx} + 2x^2(x + a)y'_x + by = 0.}$

The substitution $z = 1/x$ leads to a constant coefficient equation: $y''_{zz} - 2ay'_z + by = 0$.

202. $\boldsymbol{x^4 y''_{xx} + {+}ax^n y'_x - (ax^{n-1} + abx^{n-2} + b^2)y = 0.}$

Particular solution: $y_0 = xe^{-b/x}$.

203. $\boldsymbol{x^2(x - a)^2 y''_{xx} + by = 0.}$

Solution: $C_1|x|^m|x-a|^{1-m} + C_2|x|^{1-m}|x-a|^m$, where m is a root of the quadratic equation $m(m-1)a^2 = -b$.

204. $x^2(x-a)^2y''_{xx}+by=cx^2(x-a)^2$.

Solution:

$$y=|x|^m|x-a|^{1-m}\Big(C_1+\frac{c}{a(2m-1)}\int|x|^{1-m}|x-a|^m\,dx\Big)$$
$$+|x|^{1-m}|x-a|^m\Big(C_2-\frac{c}{a(2m-1)}\int|x|^m|x-a|^{1-m}\,dx\Big),$$

where m is a root of the quadratic equation $m(m-1)a^2=-b$.

205. $ax^2(x-1)^2y''_{xx}+(bx^2+cx+d)y=0$.

Find roots p and q of the quadratic equations

$$ap(p-1)+d=0,\qquad aq(q-1)+b+c+d=0.$$

Then, the substitution $y=x^p(x-1)^qw$ leads to the hypergeometric equation 2.1.2.158:

$$ax(x-1)w''_{xx}+2a[(p+q)x-p]w'_x+(2apq-c-2d)w=0.$$

206. $x^2(x^2+a)y''_{xx}+(bx^2+c)xy'_x+dy=0$.

The substitution $\xi=x^2$ leads to an equation of the form 2.1.2.181:

$$4\xi^2(\xi+a)y''_{\xi\xi}+2\xi[(b+1)\xi+a+c]y'_\xi+dy=0.$$

207. $(x^2+1)^2y''_{xx}+ay=0$.

Solution:

$$\frac{y}{\sqrt{x^2+1}}=\begin{cases}C_1\cos(\beta\arctan x)+C_2\sin(\beta\arctan x) & \text{if } a+1=\beta^2>0,\\ C_1\cosh(\beta\arctan x)+C_2\sinh(\beta\arctan x) & \text{if } a+1=-\beta^2<0,\\ C_1+C_2\arctan x & \text{if } a=-1.\end{cases}$$

208. $(x^2-1)^2y''_{xx}+ay=0$.

Solution:

$$y=\begin{cases}\sqrt{|x^2-1|}\left[C_1\cos\Big(\beta\ln\Big|\frac{x+1}{x-1}\Big|\Big)+C_2\sin\Big(\beta\ln\Big|\frac{x+1}{x-1}\Big|\Big)\right] & \\ & \text{if } a-1=4\beta^2>0,\\ (x+1)\left[C_1\Big|\frac{x+1}{x-1}\Big|^{(2\beta-1)/2}+C_2\Big|\frac{x+1}{x-1}\Big|^{-(2\beta-1)/2}\right] & \text{if } a-1=-4\beta^2<0,\\ \sqrt{|x^2-1|}\Big(C_1+C_2\ln\Big|\frac{x+1}{x-1}\Big|\Big) & \text{if } a=1.\end{cases}$$

209. $(x^2\pm a^2)^2y''_{xx}+b^2y=0$.

This is the bending equation of a double-walled compressed bar with a parabolic cross-section.

For the upper sign (constricted bar), the solution is as follows:

$$y=\sqrt{x^2+a^2}\,(C_1\cos u+C_2\sin u),\qquad u=\frac{\sqrt{a^2+b^2}}{a}\arctan\Big(\frac{x}{a}\Big).$$

For the lower sign (bar with salients), the solution is

$$y=\sqrt{a^2-x^2}\,(C_1\cos u+C_2\sin u),\qquad u=\frac{\sqrt{b^2-a^2}}{2a}\ln\frac{a+x}{a-x};\qquad |x|<a.$$

210. $(ax^2+b)^2y''_{xx}+2ax(ax^2+b)y'_x+cy=0$.

The substitution $\xi=\displaystyle\int\frac{dx}{ax^2+b}$ leads to a constant coefficient equation: $y''_{\xi\xi}+cy=0$.

211. $(x^2-1)^2y''_{xx}+2x(x^2-1)y'_x-[\nu(\nu+1)(x^2-1)+n^2]y=0$,
where ν is an arbitray number, n is a nonnegative integer.

1°. With $n=0$, this equation coinsides with the Legendre equation 2.1.2.148. Denote its general solution by $y_\nu(x)$.

2°. With $n=1,\ 2,\ 3,\ \dots$, the general solution of the original equation is given by the formula

$$y=|x^2-1|^{n/2}\frac{d^n}{dx^n}y_\nu(x).$$

212. $(1-x^2)^2y''_{xx}-2x(1-x^2)y'_x+[\nu(\nu+1)(1-x^2)-\mu^2]y=0$,
where ν and μ are arbitrary numbers.

The Legendre equation.

The transformation $x=1-2\xi$, $y=(x^2-1)^{\mu/2}w$ leads to the hypergeometric equation 2.1.2.158:

$$\xi(\xi-1)w''_{\xi\xi}+(\mu+1)(1-2\xi)w'_\xi+(\nu-\mu)(\nu+\mu+1)w=0$$

with parameters $\alpha=\mu-\nu$, $\beta=\mu+\nu+1$, $\gamma=\mu+1$.

In particular, the original equation is integrable by quadrature with $\nu=\mu$ or $\nu=-\mu-1$.

See Supplement 2 for more detail on the Legendre equation.

213. $a(x^2-1)^2y''_{xx}+bx(x^2-1)y'_x+(cx^2+dx+e)y=0$.

The transformation

$$\xi=\tfrac{1}{2}(x+1),\qquad w=(x+1)^{-p}(x-1)^{-q}y,$$

where p and q are parameters which are determined by solving the second order algebraic system

$$4aq(q-1)+2bq+c+d+e=0,\qquad (p-q)[2a(p+q-1)+b]=d$$

leads to the hypergeometric equation 2.1.2.158 in $w=w(\xi)$.

214. $(ax^2+b)^2y''_{xx}+(2ax+c)(ax^2+b)y'_x+ky=0$.

The substitution $\xi=\displaystyle\int\frac{dx}{ax^2+b}$ leads to a constant coefficient equation: $y''_{\xi\xi}+cy'_\xi+ky=0$.

215. $(ax^2+b)^2y''_{xx}+(ax^2+b)(cx^2+d)y'_x+2(bc-ad)xy=0$.

Particular solution: $\quad y_0=\exp\Bigl(-\displaystyle\int\frac{cx^2+d}{ax^2+b}\,dx\Bigr)$.

216. $\boldsymbol{(x^2+a)^2 y''_{xx} + bx^n (x^2+a) y'_x - (bx^{n+1}+a) y = 0}$.

Particular solution: $y_0 = \sqrt{x^2+a}$.

217. $\boldsymbol{(x^2+a)^2 y''_{xx} + bx^n (x^2+a) y'_x - m[bx^{n+1} + (m-1)x^2 + a] y = 0}$.

Particular solution: $y_0 = (x^2+a)^{m/2}$.

218. $\boldsymbol{(x-a)^2 (x-b)^2 y''_{xx} - cy = 0, \quad a \neq b}$.

The transformation $\xi = \ln \dfrac{x-a}{x-b}$, $y = (x-b)\eta$ leads to a constant coefficient equation: $(a-b)^2(\eta''_{\xi\xi} - \eta'_\xi) - c\eta = 0$. Therefore, the solution is as follows:

$$y = C_1 |x-a|^{(1+\lambda)/2} |x-b|^{(1-\lambda)/2} + C_2 |x-a|^{(1-\lambda)/2} |x-b|^{(1+\lambda)/2},$$

where $\lambda^2 = 4c(a-b)^{-2} + 1 \neq 0$.

219. $\boldsymbol{(x-a)^2 (x-b)^2 y''_{xx} + (x-a)(x-b)(2x+\lambda) y'_x + \mu y = 0}$.

Let k_1 and k_2 be the roots of the quadratic equation

$$(a-b)^2 k^2 + (a-b)(a+b+\lambda)k + \mu = 0.$$

1°. With $k_1 \neq k_2$, the solution is

$$y = C_1 \left| \frac{x-a}{x-b} \right|^{k_1} + C_2 \left| \frac{x-a}{x-b} \right|^{k_2}.$$

2°. With $k_1 = k_2 = k$, the solution is

$$y = \left| \frac{x-a}{x-b} \right|^{k} \left(C_1 + C_2 \ln \left| \frac{x-a}{x-b} \right| \right).$$

220. $\boldsymbol{(ax^2+bx+c)^2 y''_{xx} + Ay = 0}$.

The transformation

$$\xi = \int \frac{dx}{ax^2+bx+c}, \qquad w = \frac{y}{\sqrt{ax^2+bx+c}}$$

leads to a constant coefficient equation: $w''_{\xi\xi} + (A + ac - \frac{1}{4}b^2) w = 0$.

221. $\boldsymbol{(ax^2+bx+c)^2 y''_{xx} + (2ax+k)(ax^2+bx+c) y'_x + my = 0}$.

The substitution $\xi = \displaystyle\int \frac{dx}{ax^2+bx+c}$ leads to a constant coefficient equation: $y''_{\xi\xi} + (k-b) y'_\xi + my = 0$.

222. $\boldsymbol{x^6 y''_{xx} - x^5 y'_x + ay = 0}$.

The transformation $\xi = x^{-2}$, $w = yx^{-2}$ leads to a constant coefficient equation: $4w''_{\xi\xi} + aw = 0$.

223. $x^6 y''_{xx} + (3x^2 + a)x^3 y'_x + by = 0.$

The substitution $\xi = x^{-2}$ leads to a constant coefficient equation: $4y''_{\xi\xi} - 2ay'_\xi + by = 0$.

224. $$y''_{xx} + y'_x \sum_{n=1}^{3} (1 - \alpha_n - \beta_n) \frac{b_n}{b_n x - a_n} - \frac{y}{(b_1 x - a_1)(b_2 x - a_2)(b_3 x - a_3)} \sum_{n=1}^{3} \alpha_n \beta_n \frac{\Delta_n \Delta_{n-1}}{b_n x - a_n} = 0,$$

where $\sum_{n=1}^{3} (\alpha_n + \beta_n) = 1$, $|a_n| + |b_n| > 0$, $\Delta_n = a_n b_{n+1} - a_{n+1} b_n \neq 0$, $a_{n+3} = a_n$, and $b_{n+3} = b_n$.

Denote this equation by

$$\left\{ \begin{array}{ccc|ccc|c} a_1 & a_2 & a_3 & \alpha_1 & \alpha_2 & \alpha_3 & x \\ b_1 & b_2 & b_3 & \beta_1 & \beta_2 & \beta_3 & y \end{array} \right\} = 0. \tag{1}$$

With $a_1 = b_2 = 0$, $a_3 = b_3 = 1$, $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = \alpha$, $\beta_1 = 1 - \gamma$, $\beta_2 = \beta$, and $\beta_3 = \gamma - \alpha - \beta$, equation (1) transforms into the hypergeometric equation 2.1.2.158.

The transformation

$$\xi = \frac{Ax + B}{Cx + D}, \quad w = \frac{|b_1 x - a_1|^r |b_3 x - a_3|^s}{|b_2 x - a_2|^{r+s}} y, \tag{2}$$

where $AD - BC \neq 0$, brings the original equation into an equation of the similar form:

$$\left\{ \begin{array}{ccc|ccc|c} A_1 & A_2 & A_3 & \alpha_1 + r & \alpha_2 - r - s & \alpha_3 + s & \xi \\ B_1 & B_2 & B_3 & \beta_1 + r & \beta_2 - r - s & \beta_3 + s & w \end{array} \right\} = 0, \tag{3}$$

where $A_n = Aa_n + Bb_n$, $B_n = Ca_n + Db_n$.

In (2), assume $r = -\alpha_1$, $s = -\alpha_3$, $A = b_1/\Delta_3$, $B = -a_1/\Delta_3$, $C = -b_2/\Delta_2$, and $D = a_2/\Delta_2$ to obtain the hyperheometric equation (3).

225. $x^n y''_{xx} + c(ax + b)^{n-4} y = 0.$

The transformation $\xi = \dfrac{x}{ax + b}$, $w = \dfrac{y}{ax + b}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + cb^{-2} \xi^{-n} w = 0$.

226. $x^n y''_{xx} + axy'_x - (b^2 x^n + 2bx^{n-1} + abx + a)y = 0.$

Particular solution: $y_0 = xe^{bx}$.

227. $x^n y''_{xx} + (ax + b)y'_x - ay = 0.$

Particular solution: $y_0 = ax + b$.

228. $x^n y''_{xx} + (ax^{n-1} + bx)y'_x + (a - 1)by = 0.$

Particular solution: $y_0 = x^{1-a}$.

229. $x^n y''_{xx} + (2x^{n-1} + ax^2 + bx)y'_x + by = 0.$

Particular solution: $\quad y_0 = a + \dfrac{b}{x}.$

230. $x^n y''_{xx} + (ax^n + b)y'_x + c[(a-c)x^n + b]y = 0.$

Particular solution: $\quad y_0 = e^{-cx}.$

231. $x^n y''_{xx} + (ax^n - x^{n-1} + abx + b)y'_x + a^2bxy = 0.$

Particular solution: $\quad y_0 = (ax+1)e^{-ax}.$

232. $x^n y''_{xx} + (ax^{n+m} + 1)y'_x + ax^m(1 + mx^{n-1})y = 0.$

Particular solution: $\quad y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr).$

233. $(ax^n + b)y''_{xx} + (cx^n + d)y'_x + \lambda[(c - a\lambda)x^n + d - b\lambda]y = 0.$

Particular solution: $\quad y_0 = e^{-\lambda x}.$

234. $(ax^n + bx + c)y''_{xx} = an(n-1)x^{n-2}y.$

Particular solution: $\quad y_0 = ax^n + bx + c.$

235. $x(x^n + 1)y''_{xx} + [(a-b)x^n + a - n]y'_x + b(1-a)x^{n-1}y = 0.$

Particular solution: $\quad y_0 = (x^n + 1)^{b/n}.$

236. $x(x^{2n} + a)y''_{xx} + (x^{2n} + a - an)y'_x - b^2x^{2n-1}y = 0.$

Solution: $\quad y = C_1\bigl(x^n + \sqrt{x^{2n}+a}\,\bigr)^{b/n} + C_2\bigl(x^n + \sqrt{x^{2n}+a}\,\bigr)^{-b/n}.$

237. $x^2(a^2x^{2n} - 1)y''_{xx} + x[a^2(n+1)x^{2n} + n - 1]y'_x - \nu(\nu+1)a^2n^2x^{2n}y = 0.$

Solution: $\quad y = y_\nu(ax^n),\quad$ where $y_\nu(x)$ is the general solution of the Legendre equation 2.1.2.148.

238. $x^2(ax^n - 1)y''_{xx} + x(apx^n + q)y'_x + (arx^n + s)y = 0.$

Find the roots A_1, A_2 and B_1, B_2 of the quadratic equations

$$A^2 - (q+1)A - s = 0, \quad B^2 - (p-1)B + r = 0$$

and define parameters c, α, β, and γ by the relations

$$c = A_1, \quad \alpha = \frac{A_1 + B_1}{n}, \quad \beta = \frac{A_1 + B_2}{n}, \quad \gamma = \frac{A_1 - A_2}{n} + 1.$$

Then the solution of the original equation has the form

$$y = x^c w(ax^n),$$

where $w(\xi)$ is the general solution of the hypergeometric equation 2.1.2.158:

$$\xi(\xi - 1)w''_{\xi\xi} + [(\alpha + \beta + 1)\xi - \gamma]w'_\xi + \alpha\beta w = 0.$$

239. $(x^n+a)^2 y''_{xx} - bx^{n-2}[(b-1)x^n + a(n-1)]y = 0.$

Particular solution: $y_0 = (x^n+a)^{b/n}$.

240. $(ax^n+b)^2 y''_{xx} + (ax^n+b)(cx^n+d)y'_x + n(bc-ad)x^{n-1}y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int \frac{cx^n+d}{ax^n+b}\,dx\Bigr)$.

241. $(x^n+a)^2 y''_{xx} + bx^m(x^n+a)y'_x - x^{n-2}(bx^{m+1}+an-a)y = 0.$

Particular solution: $y_0 = (x^n+a)^{1/n}$.

242. $(ax^n+b)^2 y''_{xx} + cx^m(ax^n+b)y'_x + (cx^m - anx^{n-1} - 1)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\int \frac{dx}{ax^n+b}\Bigr)$.

243. $x^2(ax^n+b)^2 y''_{xx} + (n+1)x(a^2x^{2n} - b^2)y'_x + cy = 0.$

The substitution $\xi = \frac{1}{nb}\ln\Bigl(\frac{ax^n}{ax^n+b}\Bigr)$ leads to a constant coefficient equation: $y''_{\xi\xi} - b(n+2)y'_\xi + cy = 0$.

244. $(ax^{n+1}+bx^n+c)y''_{xx} + (\alpha x^n + \beta x^{n-1} + \gamma)y'_x$
$+ [n(\alpha - a - an)x^{n-1} + (n-1)(\beta - bn)x^{n-2}]y = 0.$

Particular solution: $y_0 = \exp\Bigl[\int \frac{(an+a-\alpha)x^n + (bn-\beta)x^{n-1} - \gamma}{ax^{n+1}+bx^n+c}\,dx\Bigr]$.

245. $(ax^n+bx^m+c)y''_{xx} + (\lambda - x)y'_x + y = 0.$

Particular solution: $y_0 = x - \lambda$.

246. $(ax^n+bx^m+c)y''_{xx} + (\lambda^2 - x^2)y'_x + (x+\lambda)y = 0.$

Particular solution: $y_0 = x - \lambda$.

247. $2(ax^n+bx^m+c)y''_{xx} + (anx^{n-1} + bmx^{m-1})y'_x + dy = 0.$

The substitution $\xi = \int \frac{dx}{\sqrt{ax^n+bx^m+c}}$ leads to a constant coefficient equation: $2y''_{\xi\xi} + dy = 0$.

248. $(ax^n+b)^{m+1} y''_{xx} + (ax^n+b)y'_x - anmx^{n-1}y = 0.$

Particular solution: $y_0 = \exp\Bigl[-\int \frac{dx}{(ax^n+b)^m}\Bigr]$.

249. $xP_n y''_{xx} + [2P_n + (ax^2+bx)Q_{n-2}]y'_x + bQ_{n-2}y = 0,$
where $P_n = P_n(x)$ and $Q_{n-2} = Q_{n-2}(x)$ are arbitrary polynomials of the degrees n and $n-2$, respectively.

Particular solution: $y_0 = a + \frac{b}{x}$.

2.1.3. Equations Containing Exponential Functions

1. $\boldsymbol{y''_{xx} + ae^{\lambda x}y = 0, \qquad \lambda \neq 0.}$

Solution:

$$y = C_1 J_0\biggl(\frac{2\sqrt{a}}{\lambda}e^{\lambda x/2}\biggr) + C_2 Y_0\biggl(\frac{2\sqrt{a}}{\lambda}e^{\lambda x/2}\biggr),$$

where J_0 and Y_0 are Bessel functions.

2. $\boldsymbol{y''_{xx} + (ae^{x} - b)y = 0.}$

Solution:

$$y = C_1 J_{2\sqrt{b}}\bigl(2\sqrt{a}\,e^{x/2}\bigr) + C_2 Y_{2\sqrt{b}}\bigl(2\sqrt{a}\,e^{x/2}\bigr),$$

where J_ν and Y_ν are Bessel functions.

3. $\boldsymbol{y''_{xx} + a(\lambda e^{\lambda x} - ae^{2\lambda x})y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

4. $\boldsymbol{y''_{xx} - [a^2e^{2x} + a(2b+1)e^{x} + b^2]y = 0.}$

Particular solution: $y_0 = \exp(ae^{x} + bx).$

5. $\boldsymbol{y''_{xx} - (ae^{2\lambda x} + be^{\lambda x} + c)y = 0.}$

The transformation $z = e^{\lambda x}$, $w = z^{-k}y$, where $k = \sqrt{c}/\lambda$, leads to an equation of the form 2.1.2.103:

$$\lambda^2 z w''_{zz} + \lambda^2(2k+1)w'_z - (az+b)w = 0.$$

6. $\boldsymbol{y''_{xx} + (ae^{4\lambda x} + be^{3\lambda x} + ce^{2\lambda x} - \tfrac{1}{4}\lambda^2)y = 0.}$

The transformation $\xi = e^{\lambda x}$, $w = ye^{\lambda x/2}$ leads to an equation of the form 2.1.2.6:

$$w''_{\xi\xi} + \lambda^{-2}(a\xi^2 + b\xi + c)w = 0.$$

7. $\boldsymbol{y''_{xx} + [ae^{2\lambda x}(be^{\lambda x} + c)^n - \tfrac{1}{4}\lambda^2]y = 0.}$

The transformation $\xi = be^{\lambda x} + c$, $w = ye^{\lambda x/2}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + a(b\lambda)^{-2}\xi^n w = 0.$

8. $\boldsymbol{y''_{xx} + ay'_x + be^{2ax}y = 0.}$

The transformation $\xi = e^{ax}$, $u = ye^{ax}$ leads to a constant coefficient linear equation: $u''_{\xi\xi} + ba^{-2}u = 0.$

9. $\boldsymbol{y''_{xx} - ay'_x + be^{2ax}y = 0.}$

The substitution $\xi = e^{ax}$ leads to a constant coefficient equation: $y''_{\xi\xi} + ba^{-2}y = 0.$

10. $y''_{xx} + ay'_x + (be^{\lambda x} + c)y = 0.$

Solution:

$$y = e^{-ax/2}\left[C_1 J_\nu\left(\frac{2\sqrt{b}}{\lambda}e^{\lambda x/2}\right) + C_2 Y_\nu\left(\frac{2\sqrt{b}}{\lambda}e^{\lambda x/2}\right)\right], \qquad \nu = \frac{\sqrt{a^2 - 4c}}{\lambda},$$

where J_ν and Y_ν are Bessel functions.

11. $y''_{xx} - y'_x + \left(ae^{3\lambda x} + be^{2\lambda x} + \dfrac{1-\lambda^2}{4}\right)y = 0.$

The substitution $z = e^x$ leads to an equation of the form 2.1.2.116:

$$z^2 y''_{zz} + \left(az^{3\lambda} + bz^{2\lambda} + \frac{1-\lambda^2}{4}\right)y = 0.$$

12. $y''_{xx} - y'_x + \left[ae^{2\lambda x}(be^{\lambda x} + c)^n + \dfrac{1-\lambda^2}{4}\right]y = 0.$

The substitution $z = e^x$ leads to an equation of the form 2.1.2.117:

$$z^2 y''_{zz} + \left[az^{2\lambda}(bz^\lambda + c)^n + \frac{1-\lambda^2}{4}\right]y = 0.$$

13. $y''_{xx} + 2ae^{\lambda x}y'_x + ae^{\lambda x}(ae^{\lambda x} + \lambda)y = 0.$

Solution: $y = \exp\left(-\dfrac{a}{\lambda}e^{\lambda x}\right)(C_1 + C_2 x).$

14. $y''_{xx} + (a+b)e^{\lambda x}y'_x + ae^{\lambda x}(be^{\lambda x} + \lambda)y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{a}{\lambda}e^{\lambda x}\right).$

15. $y''_{xx} + ae^{\lambda x}y'_x - be^{\mu x}(e^{\lambda x} + be^{\mu x} + \mu)y = 0.$

Particular solution: $y_0 = \exp\left(\dfrac{b}{\mu}e^{\mu x}\right).$

16. $y''_{xx} + 2ke^{\mu x}y'_x + (ae^{2\lambda x} + be^{\lambda x} + k^2e^{2\mu x} + k\mu e^{\mu x} + c)y = 0.$

The substitution $w = y\exp\left(\dfrac{k}{\mu}e^{\mu x}\right)$ leads to an equation of the form 2.1.3.5: $w''_{xx} + (ae^{2\lambda x} + be^{\lambda x} + c)w = 0.$

17. $y''_{xx} + ae^{\lambda x}y'_x + b(ax^n e^{\lambda x} - bx^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{b}{n+1}x^{n+1}\right).$

18. $y''_{xx} + 2ae^{\lambda x}y'_x + (a^2e^{2\lambda x} + a\lambda e^{\lambda x} + bx^{2n} + cx^{n-1})y = 0.$

The substitution $w = y\exp\left(\dfrac{a}{\lambda}e^{\lambda x}\right)$ leads to an equation of the form 2.1.2.10: $w''_{xx} + (bx^{2n} + cx^{n-1})w = 0.$

19. $y''_{xx} - (a + 2be^{ax})y'_x + b^2e^{2ax}y = 0.$

Particular solution: $y_0 = \exp\left(\dfrac{b}{a}e^{ax}\right)$.

20. $y''_{xx} + (ae^{2\lambda x} + \lambda)y'_x - a\lambda e^{2\lambda x}y = 0.$

Particular solution: $y_0 = ae^{\lambda x} + \lambda e^{-\lambda x}$.

21. $y''_{xx} + (ae^{\lambda x} - \lambda)y'_x + be^{2\lambda x}y = 0.$

The substitution $\xi = e^{\lambda x}$ leads to a constant coefficient linear equation: $\lambda^2 y''_{\xi\xi} + a\lambda y'_\xi + by = 0$.

22. $y''_{xx} + (ae^{\lambda x} + b)y'_x + c(ae^{\lambda x} + b - c)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

23. $y''_{xx} + (a + be^{2\lambda x})y'_x + \lambda(a - \lambda - be^{2\lambda x})y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

24. $y''_{xx} + (abe^{\lambda x} + b - 3\lambda)y'_x + a^2\lambda(b - \lambda)e^{2\lambda x}y = 0.$

Particular solution: $y_0 = (ae^{\lambda x} + 1)\exp(-ae^{\lambda x})$.

25. $y''_{xx} + (2ae^{\lambda x} - \lambda)y'_x + (a^2e^{2\lambda x} + be^{\mu x})y = 0.$

This is a special case of equation 2.1.3.30.

26. $y''_{xx} + (2ae^{\lambda x} + b)y'_x + [a^2e^{2\lambda x} + a(b + \lambda)e^{\lambda x} + c]y = 0.$

The substitution $w = y\exp\left(\dfrac{a}{\lambda}e^{\lambda x}\right)$ leads to a constant coefficient linear equation: $w''_{xx} + bw'_x + cw = 0$.

27. $y''_{xx} + (ae^{\lambda x} + 2b - \lambda)y'_x + (ce^{2\lambda x} + abe^{\lambda x} + b^2 - b\lambda)y = 0.$

The transformtion $\xi = \frac{1}{\lambda}e^{\lambda x}$, $w = e^{bx}y$ leads to a constant coefficient linear equation: $w''_{\xi\xi} + aw'_\xi + cw = 0$.

28. $y''_{xx} + (ae^x + b)y'_x + [c(a - c)e^{2x} + (ak + bc + c - 2ck)e^x + k(b - k)]y = 0.$

Particular solution: $y_0 = \exp(-ce^x - kx)$.

29. $y''_{xx} + (ae^{\lambda x} + b)y'_x + (\alpha e^{2\lambda x} + \beta e^{\lambda x} + \gamma)y = 0.$

The substitution $\xi = e^x$ leads to an equation of the form 2.1.2.141:

$$\xi^2 y''_{\xi\xi} + (a\xi^\lambda + b + 1)\xi y'_\xi + (\alpha\xi^{2\lambda} + \beta\xi^\lambda + \gamma)y = 0.$$

30. $y''_{xx} + (2ae^{\lambda x} - \lambda)y'_x + (a^2e^{2\lambda x} + be^{2\mu x} + ce^{\mu x} + k)y = 0.$

The substitution $w = y\exp\left(\dfrac{a}{\lambda}e^{\lambda x} - \dfrac{\lambda x}{2}\right)$ leads to an equation of the form 2.1.3.5:

$$w''_{xx} + \left(be^{2\mu x} + ce^{\mu x} + k - \tfrac{1}{4}\lambda^2\right)w = 0.$$

31. $\boldsymbol{y''_{xx} + (2ae^{\lambda x} + b - \lambda)y'_x + (a^2e^{2\lambda x} + abe^{\lambda x} + ce^{2\mu x} + de^{\mu x} + k)y = 0}$.

The substitution $w = y\exp\Bigl(\dfrac{a}{\lambda}e^{\lambda x} + \dfrac{b-\lambda}{2}x\Bigr)$ leads to an equation of the form 2.1.3.5:

$$w''_{xx} + \bigl[ce^{2\mu x} + de^{\mu x} + k - \tfrac{1}{4}(b-\lambda)^2\bigr]w = 0.$$

32. $\boldsymbol{y''_{xx} + (ae^{\lambda x} + be^{\mu x})y'_x + ae^{\lambda x}(be^{\mu x} + \lambda)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

33. $\boldsymbol{y''_{xx} + e^{\lambda x}(ae^{2\mu x} + b)y'_x + \mu[e^{\lambda x}(b - ae^{2\mu x}) - \mu]y = 0}$.

Particular solution: $y_0 = ae^{\mu x} + be^{-\mu x}$.

34. $\boldsymbol{y''_{xx} + (ae^{\lambda x} + be^{\mu x} + c)y'_x + (a\lambda e^{\lambda x} + b\mu e^{\mu x})y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x} - \dfrac{b}{\mu}e^{\mu x} - cx\Bigr)$.

35. $\boldsymbol{y''_{xx} + (ae^{\lambda x} + be^{\mu x} + c)y'_x + [abe^{(\lambda+\mu)x} + ace^{\lambda x} + b\mu e^{\mu x}]y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\mu}e^{\mu x} - cx\Bigr)$.

36. $\boldsymbol{y''_{xx} + (ae^{\lambda x} + 2be^{\mu x} - \lambda)y'_x + [abe^{(\lambda+\mu)x} + ce^{2\lambda x} + b^2e^{2\mu x} + b(\mu - \lambda)e^{\mu x}]y = 0}$.

1°. With $\lambda = 0$, the equation transforms into 2.1.3.26, and with $\mu = 0$ it transfroms into 2.1.3.27.

2°. With $\lambda\mu \neq 0$, the transformation $\xi = \frac{1}{\lambda}e^{\lambda x}$, $w = y\exp(\frac{b}{\mu}e^{\mu x})$ leads to a constant coefficient equation: $w''_{\xi\xi} + aw'_\xi + cw = 0$.

37. $\boldsymbol{y''_{xx} + [abe^{(\lambda+\mu)x} + a\lambda e^{\lambda x} + be^{\mu x} - 2\lambda]y'_x + a^2b\lambda e^{(2\lambda+\mu)x}y = 0}$.

Particular solution: $y_0 = (ae^{\lambda x} + 1)\exp(-ae^{\lambda x})$.

38. $\boldsymbol{y''_{xx} + (ax + b)e^{\lambda x}y'_x - ae^{\lambda x}y = 0}$.

Particular solution: $y_0 = ax + b$.

39. $\boldsymbol{y''_{xx} + (axe^{\lambda x} + 2b)y'_x + (abxe^{\lambda x} - ae^{\lambda x} + b^2)y = 0}$.

Particular solution: $y_0 = xe^{-bx}$.

40. $\boldsymbol{y''_{xx} + x(ae^{\lambda x} + be^{\mu x})y'_x - (ae^{\lambda x} + be^{\mu x})y = 0}$.

Particular solution: $y_0 = x$.

41. $\boldsymbol{y''_{xx} + (ax^n + be^{\lambda x})y'_x + (abx^ne^{\lambda x} + anx^{n-1})y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

42. $y''_{xx} + a\exp(bx^n)y'_x + c[a\exp(bx^n) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

43. $y''_{xx} + (ax+b)\exp(\lambda x^n)y'_x - a\exp(\lambda x^n)y = 0.$

Particular solution: $y_0 = ax + b$.

44. $y''_{xx} + ax^n\exp(bx^m)y'_x - ax^{n-1}\exp(bx^m)y = 0.$

Particular solution: $y_0 = x$.

45. $xy''_{xx} - (2ax^2+1)y'_x + 4bx^3\exp(2\lambda x^2)y = 0.$

Solution:

$$y = \exp\Bigl(\frac{ax^2}{2}\Bigr)\biggl[C_1 J_{\frac{a}{2\lambda}}\biggl(\frac{\sqrt{b}}{\lambda}e^{\lambda x^2}\biggr) + C_2 Y_{\frac{a}{2\lambda}}\biggl(\frac{\sqrt{b}}{\lambda}e^{\lambda x^2}\biggr)\biggr],$$

where J_ν and Y_ν are Bessel functions.

46. $xy''_{xx} + axe^{\lambda x}y'_x + ae^{\lambda x}(1+\lambda x)y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr)$.

47. $xy''_{xx} + axe^{\lambda x}y'_x - [a(bx+1)e^{\lambda x} + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

48. $xy''_{xx} + (axe^{\lambda x} + b)y'_x + a(b-1)e^{\lambda x}y = 0.$

Particular solution: $y_0 = x^{1-b}$.

49. $xy''_{xx} + [a(bx+1)e^{\lambda x} + bx - 1]y'_x + ab^2xe^{\lambda x}y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}$.

50. $xy''_{xx} + [(ax^2+bx)e^{\lambda x} + 2]y'_x + be^{\lambda x}y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

51. $xy''_{xx} + (ax^n + be^{\lambda x})y'_x + ax^{n-1}(be^{\lambda x} + n - 1)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n}x^n\Bigr)$.

52. $xy''_{xx} + (axe^{\lambda x} + bx^n)y'_x + [a(bx^n - 1)e^{\lambda x} + bnx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{b}{n}x^n\Bigr)$.

53. $xy''_{xx} + [(ax^n+1)e^{\lambda x} + anx^n + 1 - 2n]y'_x + a^2nx^{2n-1}e^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax^n+1)\exp(-ax^n)$.

54. $xy''_{xx} + (ae^{\lambda x} + be^{\mu x})y'_x + (a\lambda e^{\lambda x} + b\mu e^{\mu x})y = 0.$

Integrate to obtain a first order linear equation: $xy'_x + (ae^{\lambda x} + be^{\mu x} - 1)y = C$.

55. $xy''_{xx} + [ax^n \exp(bx^m) + c]y'_x + a(c-1)x^{n-1}\exp(bx^m)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

56. $(x+a)y''_{xx} + (be^{\lambda x} + c)y'_x + b\lambda e^{\lambda x}y = 0.$

Particular solution: $y_0 = \exp\left(\int \frac{1-c-be^{\lambda x}}{x+a}\,dx\right)$.

57. $4x^2y''_{xx} + [ax^{2n}\exp(bx^n) + 1 - n^2]y = 0.$

The transformation $\xi = bx^n$, $w = yx^{\frac{n-1}{2}}$ leads to an equation of the form 2.1.3.1: $4w''_{\xi\xi} + a(bn)^{-2}e^{\xi}w = 0$.

58. $x^2y''_{xx} + 2axy'_x + [(b^2e^{2cx} - \nu^2)c^2x^2 + a(a-1)]y = 0.$

Solution: $y = x^{-a}[C_1J_\nu(be^{cx}) + C_2Y_\nu(be^{cx})]$, where J_ν and Y_ν are Bessel functions.

59. $x^2y''_{xx} + axe^{\lambda x}y'_x + b(ae^{\lambda x} - b - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

60. $x^2y''_{xx} + x(ae^{\lambda x} + 2b)y'_x + [a(cx+b)e^{\lambda x} - c^2x^2 + b(b-1)]y = 0.$

Particular solution: $y_0 = x^{-b}e^{-cx}$.

61. $x^4y''_{xx} + (e^{2/x} - \nu^2)y = 0.$

Solution: $y = x[C_1J_\nu(e^{1/x}) + C_2Y_\nu(e^{1/x})]$, where J_ν and Y_ν are Bessel functions.

62. $x^4y''_{xx} + \left[a\exp\left(\frac{2\lambda}{x}\right) + b\exp\left(\frac{\lambda}{x}\right) + c\right]y = 0.$

The transformation $\xi = 1/x$, $w = y/x$ leads to an equation of the form 2.1.3.5: $w''_{\xi\xi} + (ae^{2\lambda\xi} + be^{\lambda\xi} + c)w = 0$.

63. $x^4y''_{xx} + ax^2e^{\lambda x}y'_x + [a(b-x)e^{\lambda x} - b^2]y = 0.$

Particular solution: $y_0 = x\exp(b/x)$.

64. $(x^2+a)^2y''_{xx} + be^{\lambda x}(x^2+a)y'_x - (bxe^{\lambda x} + a)y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

65. $(x^n+a)^2y''_{xx} + b(x^n+a)e^{\lambda x}y'_x - x^{n-2}(bxe^{\lambda x} + an - a)y = 0.$

Particular solution: $y_0 = (x^n+a)^{1/n}$.

66. $(ax^n+b)^2 y''_{xx} + c(ax^n+b)e^{\lambda x} y'_x + (ce^{\lambda x} - anx^{n-1} - 1)y = 0.$

Particular solution: $\quad y_0 = \exp\Bigl(-\int \dfrac{dx}{ax^n+b}\Bigr).$

67. $(a^2e^{2\lambda x}+b)y''_{xx} - b\lambda y'_x - a^2\lambda^2\nu^2 e^{2\lambda x} y = 0.$

Solution: $\quad y = C_1\bigl(ae^{\lambda x} + \sqrt{a^2e^{2\lambda x}+b}\,\bigr)^{\nu} + C_2\bigl(ae^{\lambda x} + \sqrt{a^2e^{2\lambda x}+b}\,\bigr)^{-\nu}.$

68. $2(ae^{\lambda x}+b)y''_{xx} + a\lambda e^{\lambda x} y'_x + cy = 0.$

The substitution $\xi = \displaystyle\int \frac{dx}{\sqrt{ae^{\lambda x}+b}}$ leads to a constant coefficient linear equation: $2y''_{\xi\xi} + cy = 0.$

69. $(ae^{\lambda x}+b)y''_{xx} + (ce^{\lambda x}+d)y'_x + k[(c-ak)e^{\lambda x} + d - bk]y = 0.$

Particular solution: $\quad y_0 = e^{-kx}.$

70. $(ae^{\lambda x}+b)y''_{xx} + (ce^{\lambda x}+d)y'_x + (ne^{\lambda x}+m)y = 0.$

For $a=0$, this is an equation of the form 2.1.3.29. For $a \neq 0$, the transformation $\xi = ae^{\lambda x}$, $w = y\xi^{-k}$, where k is a root of the quadratic equation $b\lambda^2k^2 + d\lambda k + m = 0$, leads to an equation of the form 2.1.2.159:

$$a\lambda^2\xi(\xi+b)w''_{\xi\xi} + \lambda[(2ak\lambda + a\lambda + c)\xi + a(2bk\lambda + b\lambda + d)]w'_\xi + (ak^2\lambda^2 + ck\lambda + n)w = 0.$$

71. $(e^x+k)y''_{xx} + (ae^{\lambda x} + be^{\mu x} + c)y'_x + (a\lambda e^{\lambda x} + b\mu e^{\mu x} - e^x)y = 0.$

Integrating yields a first order linear equation:

$$(e^x+k)y'_x + (ae^{\lambda x} + be^{\mu x} - e^x + c)y = C.$$

72. $(ae^{\lambda x}+b)^2 y''_{xx} + ce^{\lambda x}(\lambda b - ce^{\lambda x})y = 0.$

Particular solution: $\quad y = (ae^{\lambda x}+b)^k, \;$ where $k = -\dfrac{c}{a\lambda}.$

73. $(ae^{\lambda x}+b)^2 y''_{xx} + \sigma(ae^{\lambda x}+b)y'_x + ce^{\lambda x}(\sigma + \lambda - ce^{\lambda x})y = 0.$

Particular solution: $\quad y = (ae^{\lambda x}+b)^k, \;$ where $k = -\dfrac{c}{a\lambda}.$

74. $(ae^{\lambda x}+b)^2 y''_{xx} + (a\lambda e^{\lambda x}+c)(ae^{\lambda x}+b)y'_x + my = 0.$

The substitution $\xi = \displaystyle\int \frac{dx}{ae^{\lambda x}+b}$ leads to the constant coefficient linear equation: $y''_{\xi\xi} + cy'_\xi + my = 0.$

75. $(ae^{\lambda x}+b)^2 y''_{xx} + ke^{\mu x}(ae^{\lambda x}+b)y'_x + ce^{\lambda x}(ke^{\mu x} - ce^{\lambda x} + \lambda b)y = 0.$

Particular solution: $\quad y = (ae^{\lambda x}+b)^k, \;$ where $k = -\dfrac{c}{a\lambda}.$

76. $\boldsymbol{4(ae^{\lambda x}+b)^n y''_{xx}+[ke^{2\lambda x}(ce^{\lambda x}+d)^{n-4}-\lambda^2(ae^{\lambda x}+b)^n]y=0}$.

The transformation

$$\xi=\frac{ae^{\lambda x}+b}{ce^{\lambda x}+d},\qquad w=\frac{ye^{\lambda x/2}}{ce^{\lambda x}+d}$$

leads to an equation of the form 2.1.2.7: $4w''_{\xi\xi}+k(\Delta\lambda)^{-2}\xi^{-n}w=0$, where $\Delta=ad-bc$.

77. $\boldsymbol{(ae^{\lambda x}+bx+c)y''_{xx}-a\lambda^2e^{\lambda x}y=0}$.

Particular solution: $y_0=ae^{\lambda x}+bx+c$.

78. $\boldsymbol{[(ax+b)e^{\lambda x}+c]y''_{xx}-c\lambda^2y=0}$.

Particular solution: $y_0=ce^{-\lambda x}+ax+b$.

2.1.4. Equations Containing Hyperbolic Functions

1. $\boldsymbol{y''_{xx}-(a-2q\cosh 2x)y=0}$.

The modified Mathieu equation.

The substitution $x=i\xi$ leads to the Mathieu equation 2.1.6.4:

$$y''_{\xi\xi}+(a-2q\cos 2\xi)y=0.$$

For eigenvalues $a=a_n(q)$ and $a=b_n(q)$, the corresponding solutions of the modified Mathieu equation are

$$\mathrm{Ce}_{2n+p}(x,q)=\mathrm{ce}_{2n+p}(ix,q)=\sum_{k=0}^{\infty}A_{2k+p}^{2n+p}\cosh[(2k+p)x],$$

$$\mathrm{Se}_{2n+p}(x,q)=-i\,\mathrm{se}_{2n+p}(ix,q)=\sum_{k=0}^{\infty}B_{2k+p}^{2n+p}\sinh[(2k+p)x],$$

where p may be equal to 0 and 1, and coefficients A_{2k+p}^{2n+p} and B_{2k+p}^{2n+p} are indicated in 2.1.6.4.

The modified Mathieu equation is discussed in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1955, vol. 3) in more detail.

2. $\boldsymbol{y''_{xx}+(a\cosh^2 x+b)y=0}$.

Utilize the formula $\cosh 2x=2\cosh^2 x-1$ to obtain an equation of the form 2.1.4.1:

$$y''_{xx}+\Big(\frac{a}{2}+b+\frac{a}{2}\cosh 2x\Big)y=0.$$

3. $\boldsymbol{y''_{xx}-a[a\cosh^2(bx)+b\sinh(bx)]y=0}$.

Particular solution: $y_0=\exp\Big[\frac{a}{b}\sinh(bx)\Big]$.

4. $\boldsymbol{y''_{xx}-a[a\sinh^2(bx)+b\cosh(bx)]y=0}$.

Particular solution: $y_0=\exp\Big[\frac{a}{b}\cosh(bx)\Big]$.

5. $\boldsymbol{y''_{xx} + (a\cosh^2 x + b\sinh^2 x + c)y = 0.}$

Utilize the formulae $2\sinh^2 x = \cosh(2x) - 1$ and $2\cosh^2 x = \cosh(2x) + 1$ to yield an equation of the form 2.1.4.1:

$$y''_{xx} + \left[\frac{a-b}{2} + c + \frac{a+b}{2}\cosh(2x)\right]y = 0.$$

6. $\boldsymbol{y''_{xx} + [a\tanh(\lambda x) + b]y = 0.}$

The transformation

$$\xi = \frac{1-\tanh(\lambda x)}{1+\tanh(\lambda x)}, \quad w = y\xi^{-k/\lambda},$$

where k is a root of the quadratic equation $4k^2 + b - a = 0$, leads to an equation of the form 2.1.2.159:

$$4\lambda^2\xi(\xi+1)w''_{\xi\xi} + 4\lambda(2k+\lambda)(\xi+1)w'_\xi + (4k^2+a+b)w = 0.$$

7. $\boldsymbol{y''_{xx} - 4a^2\tanh^2(3ax)y = 0.}$

Particular solution: $y_0 = \sinh(3ax)[\cosh(3ax)]^{-1/3}$.

8. $\boldsymbol{y''_{xx} + [a\lambda - a(a+\lambda)\tanh^2(\lambda x)]y = 0.}$

Particular solution: $y_0 = [\cosh(\lambda x)]^{-a/\lambda}$.

9. $\boldsymbol{y''_{xx} + [3a\lambda - \lambda^2 - a(a+\lambda)\tanh^2(\lambda x)]y = 0.}$

Particular solution: $y_0 = \sinh(\lambda x)[\cosh(\lambda x)]^{-a/\lambda}$.

10. $\boldsymbol{y''_{xx} + [a\coth(\lambda x) + b]y = 0.}$

The transformation

$$\xi = \frac{1-\tanh(\lambda x)}{1+\tanh(\lambda x)}, \quad w = y\xi^{-k/\lambda},$$

where k is a root of the quadratic equation $4k^2 + b - a = 0$, leads to an equation of the form 2.1.2.159:

$$4\lambda^2\xi(\xi-1)w''_{\xi\xi} + 4\lambda(2k+\lambda)(\xi-1)w'_\xi + (4k^2+a+b)w = 0.$$

11. $\boldsymbol{y''_{xx} - 4a^2\coth^2(3ax)y = 0.}$

Particular solution: $y_0 = \cosh(3ax)[\sinh(3ax)]^{-1/3}$.

12. $\boldsymbol{y''_{xx} + [a\lambda - a(a+\lambda)\coth^2(\lambda x)]y = 0.}$

Particular solution: $y_0 = [\sinh(\lambda x)]^{-a/\lambda}$.

13. $\boldsymbol{y''_{xx} + [3a\lambda - \lambda^2 - a(a+\lambda)\coth^2(\lambda x)]y = 0.}$

Particular solution: $y_0 = \cosh(\lambda x)[\sinh(\lambda x)]^{-a/\lambda}$.

14. $\boldsymbol{y''_{xx} + ay'_x - \lambda[\lambda + a\tanh(\lambda x)]y = 0}$.

Particular solution: $y_0 = \cosh(\lambda x)$.

15. $\boldsymbol{y''_{xx} + a\sinh(\lambda x)y'_x + b[a\sinh(\lambda x) - b]y = 0}$.

Particular solution: $y_0 = e^{-bx}$.

16. $\boldsymbol{y''_{xx} + a\sinh(\lambda x)y'_x - \lambda[\lambda + \cosh(\lambda x)]y = 0}$.

Particular solution: $y_0 = \sinh(\lambda x)$.

17. $\boldsymbol{y''_{xx} + a\cosh(\lambda x)y'_x - \lambda[\lambda + \sinh(\lambda x)]y = 0}$.

Particular solution: $y_0 = \cosh(\lambda x)$.

18. $\boldsymbol{y''_{xx} + 2\tanh x\, y'_x + ay = 0}$.

Solution:

$$y\cosh x = \begin{cases} C_1\cos(bx) + C_2\sin(bx) & \text{if } a - 1 = b^2 > 0, \\ C_1\cosh(bx) + C_2\sinh(bx) & \text{if } a - 1 = -b^2 < 0. \end{cases}$$

19. $\boldsymbol{y''_{xx} + a\tanh(\lambda x)y'_x + b[a\tanh(\lambda x) - b]y = 0}$.

Particular solution: $y_0 = e^{-bx}$.

20. $\boldsymbol{y''_{xx} - \lambda\tanh(\lambda x)y'_x - a^2\cosh^2(\lambda x)y = 0}$.

Solution: $y = C_1\exp\Big[\dfrac{a}{\lambda}\sinh(\lambda x)\Big] + C_2\exp\Big[-\dfrac{a}{\lambda}\sinh(\lambda x)\Big]$.

21. $\boldsymbol{y''_{xx} - \tanh x\, y'_x + a^2\coth^2 x\,(\sinh x)^{2m-2}y = 0}$.

Solution:

$$y = \sqrt{\sinh x}\,\left[C_1 J_{\frac{1}{2m}}\Big(\frac{a}{m}\sinh^m x\Big) + C_2 Y_{\frac{1}{2m}}\Big(\frac{a}{m}\sinh^m x\Big)\right],$$

where J_ν and Y_ν are Bessel functions.

22. $\boldsymbol{y''_{xx} + 2\tanh x\, y'_x + (ax^2 + bx + c)y = 0}$.

The substitution $u = y\cosh x$ leads to an equation of the form 2.1.2.6:

$$u''_{xx} + (ax^2 + bx + c - 1)u = 0.$$

23. $\boldsymbol{y''_{xx} + 2\tanh x\, y'_x + (ax^n + 1)y = 0}$.

The substitution $u = y\cosh x$ leads to an equation of the form 2.1.2.7: $u''_{xx} + ax^n u = 0$.

24. $\boldsymbol{y''_{xx} + 2\tanh x\, y'_x + (ax^{2n} + bx^{n-1} + 1)y = 0}$.

The substitution $u = y\cosh x$ leads to an equation of the form 2.1.2.10:

$$u''_{xx} + (ax^{2n} + bx^{n-1})u = 0.$$

25. $y''_{xx} + 2n \coth x \, y'_x + (n^2 - a^2)y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $y = \left(\dfrac{1}{\sinh x}\dfrac{d}{dx}\right)^n (C_1 e^{ax} + C_2 e^{-ax}).$

26. $y''_{xx} + (2\tanh x + a)y'_x + (a\tanh x + b)y = 0.$

The substitution $u = y\cosh x$ leads to a constant coefficient linear equation: $u''_{xx} + au'_x + (b-1)u = 0.$

27. $y''_{xx} + a\tanh^n(\lambda x)y'_x - \lambda[\lambda + a\tanh^{n+1}(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(\lambda x).$

28. $y''_{xx} + (ax+b)\sinh^n(\lambda x)y'_x - a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b.$

29. $y''_{xx} + (ax+b)\tanh^n(\lambda x)y'_x - a\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b.$

30. $y''_{xx} + ax^n\cosh^m(\lambda x)y'_x - ax^{n-1}\cosh^m(\lambda x)y = 0.$

Particular solution: $y_0 = x.$

31. $y''_{xx} + ax^n\tanh^m(\lambda x)y'_x - ax^{n-1}\tanh^m(\lambda x)y = 0.$

Particular solution: $y_0 = x.$

32. $xy''_{xx} + ax\cosh^n(\lambda x)y'_x - [a(bx+1)\cosh^n(\lambda x) + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}.$

33. $xy''_{xx} + ax\tanh^n(\lambda x)y'_x - [a(bx+1)\tanh^n(\lambda x) + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}.$

34. $xy''_{xx} + [ax\cosh^n(\lambda x) + b]y'_x + a(b-1)\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = x^{1-b}.$

35. $xy''_{xx} + [ax\tanh^n(\lambda x) + b]y'_x + a(b-1)\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = x^{1-b}.$

36. $xy''_{xx} + [(ax^2+bx)\cosh^n(\lambda x) + 2]y'_x + b\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

37. $xy''_{xx} + [(ax^2+bx)\tanh^n(\lambda x) + 2]y'_x + b\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

38. $xy''_{xx} + (a\sinh^n x + bx^{m+1})y'_x + bx^m(a\sinh^n x + m)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{m+1}x^{m+1}\Bigr).$

39. $xy''_{xx} + (a\tanh^n x + bx^{m+1})y'_x + bx^m(a\tanh^n x + m)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{m+1}x^{m+1}\Bigr).$

40. $x^2y''_{xx} + ax\cosh^n(\lambda x)y'_x + b[a\cosh^n(\lambda x) - b - 1]y = 0.$

Particular solution: $y_0 = x^{-b}.$

41. $x^2y''_{xx} + ax\tanh^n(\lambda x)y'_x + b[a\tanh^n(\lambda x) - b - 1]y = 0.$

Particular solution: $y_0 = x^{-b}.$

42. $(a\sinh x + b)y''_{xx} + (c\sinh x + d)y'_x + \lambda[(c - a\lambda)\sinh x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}.$

43. $(a\tanh x + b)y''_{xx} + (c\tanh x + d)y'_x + \lambda[(c - a\lambda)\tanh x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}.$

44. $[a\tanh(\lambda x) + b]y''_{xx} + [c\tanh(\lambda x) + d]y'_x + [n\tanh(\lambda x) + m]y = 0.$

The transformation

$$\xi = \frac{1 + \tanh(\lambda x)}{1 - \tanh(\lambda x)}, \quad w = y\xi^{-k/\lambda},$$

where k is a root of the quadratic equation $4(a-b)k^2 + 2(c-d)k + n - m = 0$, leads to an equation of the form 2.1.2.159:

$$4\lambda^2\xi[(a+b)\xi + b - a]w''_{\xi\xi} + 2\lambda\{[2(2k+\lambda)(a+b) + c + d]\xi + 2(2k+\lambda)(b-a) + d - c\}w'_\xi + [4(a+b)k^2 + 2(c+d)k + n + m]w = 0.$$

45. $[a + b\coth(\lambda x)]y''_{xx} + [c + d\coth(\lambda x)]y'_x + [n + m\coth(\lambda x)]y = 0.$

Multiply this equation by $\tanh(\lambda x)$ to obtain the equation 2.1.4.44.

46. $\cosh^2(ax)y''_{xx} - by = 0.$

The substitution $ax = \ln\sqrt{\dfrac{\xi}{1-\xi}}$ $(0 < \xi < 1)$ leads to the hypergeometric equation 2.1.2.158:

$$\xi(\xi - 1)y''_{\xi\xi} + (2\xi - 1)y'_\xi + a^{-2}by = 0.$$

47. $\sinh^2(ax)y''_{xx} - by = 0.$

The substitution $ax = \pm\ln\dfrac{\xi}{\sqrt{\xi^2+1}}$ $(\xi > 0)$ leads to the equation 2.1.2.177:

$$\xi(\xi^2 + 1)y''_{\xi\xi} + (3\xi^2 + 1)y'_\xi - 4a^{-2}b\xi y = 0.$$

48. $y''_{xx}\sinh^2 x - [a^2\sinh^2 x + n(n-1)]y = 0, \qquad a \neq 0; \quad n = 1,\ 2,\ 3,\ \ldots$

Solution: $\quad y = \sinh^n x\Big(\dfrac{1}{\sinh x}\dfrac{d}{dx}\Big)^n (C_1 e^{ax} + C_2 e^{-ax}).$

49. $\sinh^n(\lambda x)y''_{xx} + [a\cosh^{n-4}(\lambda x) - \lambda^2\sinh^n(\lambda x)]y = 0.$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0$.

50. $\cosh^n(\lambda x)y''_{xx} + [a\sinh^{n-4}(\lambda x) - \lambda^2\cosh^n(\lambda x)]y = 0.$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0$.

51. $[a\sinh(\lambda x) + bx + c]y''_{xx} - a\lambda^2\sinh(\lambda x)y = 0.$

Particular solution: $\quad y_0 = a\sinh(\lambda x) + bx + c.$

52. $[a\cosh(\lambda x) + bx + c]y''_{xx} - a\lambda^2\cosh(\lambda x)y = 0.$

Particular solution: $\quad y_0 = a\cosh(\lambda x) + bx + c.$

2.1.5. Equations Containing Logarithmic Functions

1. $y''_{xx} - (a^2x^2\ln^2 x + a\ln x + a)y = 0.$

Particular solution: $\quad y_0 = e^{-ax^2/4}x^{ax^2/2}.$

2. $y''_{xx} - (a^2x^{2n}\ln^2 x + anx^{n-1}\ln x + ax^{n-1})y = 0.$

Particular solution: $\quad y_0 = e^{-F}x^{(n+1)F}$, where $F = \dfrac{ax^{n+1}}{(n+1)^2}$.

3. $y''_{xx} + a\ln^n(bx)y'_x + c[a\ln^n(bx) - c]y = 0.$

Particular solution: $\quad y_0 = e^{-cx}.$

4. $y''_{xx} + [a\ln^n(bx) + c]y'_x + ac\ln^n(bx)y = 0.$

Particular solution: $\quad y_0 = e^{-cx}.$

5. $y''_{xx} + (ax + b)\ln^n(cx)y'_x - a\ln^n(cx)y = 0.$

Particular solution: $\quad y_0 = ax + b.$

6. $y''_{xx} + ax^n\ln^m(bx)y'_x - ax^{n-1}\ln^m(bx)y = 0.$

Particular solution: $\quad y_0 = x.$

7. $xy''_{xx} - (a^2x\ln^2 x + a)y = 0.$

Particular solution: $\quad y_0 = e^{-ax}x^{ax}.$

8. $xy''_{xx} - [a^2x\ln^{2n}(bx) + an\ln^{n-1}(bx)]y = 0.$

Particular solution: $y_0 = \exp\left[a\int \ln^n(bx)\,dx\right].$

9. $xy''_{xx} + ax\ln x\, y'_x + a(\ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{1-ax}.$

10. $xy''_{xx} + (ax\ln x + b)y'_x + (ab\ln x + a)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}.$

11. $xy''_{xx} + (2ax\ln x + 1)y'_x + (a^2x\ln^2 x + a\ln x + a)y = 0.$

Solution: $y = e^{ax}x^{-ax}(C_1 + C_2\ln x).$

12. $xy''_{xx} + \ln x(ax + b)y'_x + a(b\ln^2 x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}.$

13. $xy''_{xx} + ax\ln^n(bx)y'_x + an\ln^{n-1}(bx)y = 0.$

Particular solution: $y_0 = \exp\left[-a\int \ln^n(bx)\,dx\right].$

14. $xy''_{xx} + ax\ln^n xy'_x + (a\ln^n x + an\ln^{n-1} x)y = 0.$

Particular solution: $y_0 = x\exp\left(-a\int \ln^n x\,dx\right).$

15. $xy''_{xx} + (ax^n\ln x + 1)y'_x - ax^{n-1}y = 0.$

Particular solution: $y_0 = \ln x.$

16. $xy''_{xx} + (ax\ln^n x + 1)y'_x - a\ln^{n-1} x\, y = 0.$

Particular solution: $y_0 = \ln x.$

17. $xy''_{xx} + (ax\ln^n x + b)y'_x + a(b-1)\ln^n x\, y = 0.$

Particular solution: $y_0 = x^{1-b}.$

18. $xy''_{xx} + [(ax^2 + bx)\ln^n(cx) + 2]y'_x + b\ln^n(cx)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

19. $xy''_{xx} + (ax^n + b\ln^m x)y'_x + ax^{n-1}(b\ln^m x + n - 1)y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{a}{n}x^n\right).$

20. $xy''_{xx} + (ax^n + bx\ln^m x)y'_x + [b(ax^n - 1)\ln^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\left(-\dfrac{a}{n}x^n\right).$

21. $x^2y''_{xx} + (a\ln x + b)y = 0.$

The transformation $\xi = a\ln x + b - \frac{1}{4}$, $w = yx^{-1/2}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + a^{-2}\xi w = 0$.

22. $x^2y''_{xx} + (a\ln^2 x + b\ln x + c)y = 0.$

The transformation $\xi = \ln x$, $w = yx^{-1/2}$ leads to an equation of the form 2.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c - \frac{1}{4})w = 0$.

23. $x^2y''_{xx} + [a(b\ln x + c)^n + \frac{1}{4}]y = 0.$

The transformation $\xi = b\ln x + c$, $w = yx^{-1/2}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + ab^{-2}\xi^n w = 0$.

24. $x^2y''_{xx} + xy'_x + a\ln^n(bx)y = 0.$

Solution:

$$y = \sqrt{\ln(bx)}\left[C_1 J_{\frac{1}{2m}}\left(\frac{\sqrt{a}}{m}\ln^m(bx)\right) + C_2 Y_{\frac{1}{2m}}\left(\frac{\sqrt{a}}{m}\ln^m(bx)\right)\right],$$

where $m = \frac{1}{2}(n+2)$; J_ν and Y_ν are Bessel functions.

25. $x^2y''_{xx} + xy'_x + (a\ln^{2n} x + b\ln^{n-1} x)y = 0.$

The substitution $\xi = \ln x$ leads to an equation of the form 2.1.2.10: $y''_{\xi\xi} + (a\xi^{2n} + b\xi^{n-1})y = 0$.

26. $x^2y''_{xx} + x(2a\ln x + 1)y'_x + (x^2 + a^2\ln^2 x + b)y = 0.$

The substitution $y = w\exp(-\frac{1}{2}a\ln^2 x)$ leads to the Bessel equation 2.1.2.121: $x^2w''_{\xi\xi} + xw'_\xi + (x^2 + b - a)w = 0$.

27. $x^2y''_{xx} + x(2\ln x + a + 1)y'_x + (\ln^2 x + a\ln x + b)y = 0.$

The transformation $\xi = \ln x$, $w = y\exp(\frac{1}{2}\ln^2 x)$ leads to a constant coefficient equation: $w''_{\xi\xi} + aw'_\xi + (b-1)w = 0$.

28. $x^2y''_{xx} + x(2\ln x + a)y'_x + [\ln^2 x + (a-1)\ln x + bx^n + c]y = 0.$

The substitution $w = y\exp(\frac{1}{2}\ln^2 x)$ leads to an equation of the form 2.1.2.127:

$$x^2w''_{xx} + axw'_x + (bx^n + c - 1)w = 0.$$

29. $x^2y''_{xx} + ax\ln^n(bx)y'_x + c[a\ln^n(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

30. $x^2y''_{xx} + x(ax^n + b\ln x)y'_x + b(ax^n\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp\left(-\frac{b}{2}\ln^2 x\right)$.

31. $x(x+a)y''_{xx}+x(b\ln x+c)y'_x+by=0.$

Particular solution: $y_0=\exp\left(-\int\frac{b\ln x+c-1}{x+a}\,dx\right)$.

32. $x^4y''_{xx}+ax^2\ln^n(bx)y'_x+[a(c-x)\ln^n(bx)-c^2]y=0.$

Particular solution: $y_0=x\exp\left(\frac{c}{x}\right)$.

33. $(a\ln x+b)y''_{xx}+(c\ln x+d)y'_x+\lambda[(c-a\lambda)\ln x+d-b\lambda]y=0.$

Particular solution: $y_0=e^{-\lambda x}$.

34. $x\ln x\,y''_{xx}-ny'_x-a^2x(\ln x)^{2n+1}y=0.$

Solution: $y=C_1e^{aF}+C_2e^{-aF}$, where $F=\int\ln^n x\,dx$.

35. $x\ln(ax)y''_{xx}-[n\ln(ax)+m]y'_x-b^2x^{2n+1}\ln^{2m+1}(ax)y=0.$

Solution: $y=C_1e^{bF}+C_2e^{-bF}$, where $F=\int x^n\ln^m(ax)\,dx$.

36. $x^2\ln(ax)y''_{xx}+y=0.$

Solution: $y=C_1\ln(ax)+C_2\ln(ax)\int[\ln(ax)]^{-2}\,dx$.

37. $x\ln^2x\,y''_{xx}+(ax+1)\ln x\,y'_x+bxy=0.$

The substitution $\xi=\int\frac{dx}{\ln x}$ leads to a constant coefficient linear equation: $y''_{\xi\xi}+ay'_\xi+by=0$.

38. $x(ax\ln x+bx+c)y''_{xx}-ay=0.$

Particular solution: $y_0=ax\ln x+bx+c$.

39. $x^2(a\ln x+bx+c)y''_{xx}+ay=0.$

Particular solution: $y_0=a\ln x+bx+c$.

40. $\ln^n(ax)y''_{xx}+(b^2-x^2)y'_x+(x+b)y=0.$

Particular solution: $y_0=x-b$.

2.1.6. Equations Containing Trigonometric Functions

1. $y''_{xx}+a^2y=b\sin(\lambda x).$

Solution:

$$y=\begin{cases}C_1\sin(ax)+C_2\cos(ax)+\dfrac{b}{a^2-\lambda^2}\sin(\lambda x) & \text{if } a\neq\lambda,\\ C_1\sin(ax)+C_2\cos(ax)-\dfrac{b}{2a}x\cos(ax) & \text{if } a=\lambda.\end{cases}$$

TABLE 2.6
The general solution of the Mathieu equation expressed in terms of subsidiary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$.

Constraint	General solution $y = y(x)$	Period of φ_1 and φ_2	Index
$y_1(\pi) > 1$	$C_1 e^{2\mu x}\varphi_1(x) + C_2 e^{-2\mu x}\varphi_2(x)$	π	μ is a real number
$y_1(\pi) < -1$	$C_1 e^{2\rho x}\varphi_1(x) + C_2 e^{-2\rho x}\varphi_2(x)$	2π	$\mu = \rho + \frac{1}{2}i, \quad i^2 = -1$ ρ is imaginary part of μ
$\lvert y_1(\pi)\rvert < 1$	$(C_1 \cos\nu x + C_2 \sin\nu x)\varphi_1(x)$ $+(C_1 \cos\nu x - C_2 \sin\nu x)\varphi_2(x)$	π	$\mu = i\nu$ is a pure imaginary number, $\cos(2\pi\nu) = y_1(\pi)$
$y_1(\pi) = \pm 1$	$C_1\varphi_1(x) + C_2 x\varphi_2(x)$	π	$\mu = 0$

2. $\boldsymbol{y''_{xx} + a^2 y = b\cos(\lambda x)}$.

Solution:

$$y = \begin{cases} C_1 \sin(ax) + C_2 \cos(ax) + \dfrac{b}{a^2 - \lambda^2}\cos(\lambda x) & \text{if } a \neq \lambda, \\ C_1 \sin(ax) + C_2 \cos(ax) + \dfrac{b}{2a} x \sin(ax) & \text{if } a = \lambda. \end{cases}$$

3. $\boldsymbol{y''_{xx} + [a\sin(\lambda x) + b]y = 0}$.

The substitution $\lambda x = 2\xi + \frac{\pi}{2}$ leads to the Mathieu equation 2.1.6.4:

$$y''_{\xi\xi} + (4a\lambda^{-2}\cos 2\xi + b\lambda^{-2})y = 0.$$

4. $\boldsymbol{y''_{xx} + (a - 2q\cos 2x)y = 0}$.

The Mathieu equation.

1°. Given numbers a and q, there exists the general solution $y(x)$ and a characteristic index μ such that

$$y(x + \pi) = e^{2\pi\mu} y(x).$$

For small values of q, the approximate value of μ can be found from the equation

$$\cosh(\pi\mu) = 1 + 2\sin^2\left(\tfrac{\pi}{2}\sqrt{a}\right) + \frac{\pi q^2}{(1-a)\sqrt{a}}\sin\left(\pi\sqrt{a}\right) + O(q^4).$$

If $y_1(x)$ is the solution of the Mathieu equation, which satisfies the initial conditions $y_1(0) = 1$ and $y_1'(0) = 0$, the characteristic index can be determined from the relation

$$\cosh(2\pi\mu) = y_1(\pi).$$

The solution $y_1(x)$ and hence μ can be determined with any degree of accuracy by means of numerical and approximate methods.

The general solution differs depending on the value of $y_1(\pi)$ and can be expressed in terms of two subsidiary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$ (see Table 2.6).

TABLE 2.6a

Periodical solutions of the Mathieu equation $\mathrm{ce}_n = \mathrm{ce}_n(x,q)$ and $\mathrm{se}_n = \mathrm{se}_n(x,q)$ (for odd n, functions ce_n and se_n are 2π-periodical, and for even n, they are π-periodical); definite eigenvalues $a = a_n(q)$ and $b = b_n(q)$ correspond to each value of parameter q.

Mathieu functions	Recurrence relations for coefficients	Normalization conditions
$\mathrm{ce}_{2n} = \sum\limits_{m=0}^{\infty} A_{2m}^{2n} \cos 2mx$	$qA_2^{2n} = a_{2n}A_0^{2n}$; $qA_4^{2n} = (a_{2n}-4)A_2^{2n} - 2qA_0^{2n}$; $qA_{2m+2}^{2n} = (a_{2n}-4m^2)A_{2m}^{2n} - qA_{2m-2}^{2n}, \quad m \ge 2$	$(A_0^{2n})^2 + \sum\limits_{m=0}^{\infty}(A_{2m}^{2n})^2 = \begin{cases} 2 & \text{if } n=0 \\ 1 & \text{if } n \ge 1 \end{cases}$
$\mathrm{ce}_{2n+1} = \sum\limits_{m=0}^{\infty} A_{2m+1}^{2n+1} \cos(2m+1)x$	$qA_3^{2n+1} = (a_{2n+1}-1-q)A_1^{2n+1}$; $qA_{2m+3}^{2n+1} = [a_{2n+1}-(2m+1)^2]A_{2m+1}^{2n+1} - qA_{2m-1}^{2n+1}, \quad m \ge 1$	$\sum\limits_{m=0}^{\infty}(A_{2m+1}^{2n+1})^2 = 1$
$\mathrm{se}_{2n} = \sum\limits_{m=0}^{\infty} B_{2m}^{2n} \sin 2mx$, $\mathrm{se}_0 = 0$	$qB_4^{2n} = (b_{2n}-4)B_2^{2n}$; $qB_{2m+2}^{2n} = (b_{2n}-4m^2)B_{2m}^{2n} - qB_{2m-2}^{2n}, \quad m \ge 2$	$\sum\limits_{m=0}^{\infty}(B_{2m}^{2n})^2 = 1$
$\mathrm{se}_{2n+1} = \sum\limits_{m=0}^{\infty} B_{2m+1}^{2n+1} \sin(2m+1)x$	$qB_3^{2n+1} = (b_{2n+1}-1-q)B_1^{2n+1}$; $qB_{2m+3}^{2n+1} = [b_{2n+1}-(2m+1)^2]B_{2m+1}^{2n+1} - qB_{2m-1}^{2n+1}, \quad m \ge 1$	$\sum\limits_{m=0}^{\infty}(B_{2m+1}^{2n+1})^2 = 1$

2°. In applications, of major interest are periodical solutions of the Mathieu equation which exist for definite values of parameters a and q (those values of a are referred to as eigenvalues). The most important solutions are listed in Table 2.6a.

The Mathieu functions possess the following properties:

$$\mathrm{ce}_{2n}(x,\,-q) = (-1)^n\,\mathrm{ce}_{2n}\Big(\frac{\pi}{2}-x,\ q\Big), \qquad \mathrm{ce}_{2n+1}(x,\,-q) = (-1)^n\,\mathrm{se}_{2n+1}\Big(\frac{\pi}{2}-x,\ q\Big),$$
$$\mathrm{se}_{2n}(x,\,-q) = (-1)^{n-1}\,\mathrm{se}_{2n}\Big(\frac{\pi}{2}-x,\ q\Big), \qquad \mathrm{se}_{2n+1}(x,\,-q) = (-1)^n\,\mathrm{ce}_{2n+1}\Big(\frac{\pi}{2}-x,\ q\Big).$$

Selecting sufficiently large number m and omitting the term with the maximum number in the recurrence relations (indicated in Table 2.6a), we can obtain approximate relations for eigenvalues a_n (or b_n) with respect to parameter q. Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients A_m^n (or B_m^n) to zero, we obtain an algebraic equation for finding $a_n(q)$ (or $b_n(q)$).

For fixed real $q \neq 0$, eigenvalues a_n and b_n are all real and different, while

$$\begin{aligned} &\text{if} \quad q>0 \quad \text{then} \quad a_0 < b_1 < a_1 < b_2 < a_2 < \cdots \\ &\text{if} \quad q<0 \quad \text{then} \quad a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 < \cdots \end{aligned}$$

The eigenvalues possess the properties

$$a_{2n}(-q) = a_{2n}(q), \quad b_{2n}(-q) = b_{2n}(q), \quad a_{2n+1}(-q) = b_{2n+1}(q).$$

The solution of the Mathieu equation corresponding to eigenvalue a_n (or b_n) has n zeros on the interval $0 \le x < \pi$ (q is a real number).

Listed below are two leading terms of asymptotic expansions of the Mathieu functions $\mathrm{ce}_n(x,q)$ and $\mathrm{se}_n(x,q)$, as well as of the corresponding eigenvalues $a_n(q)$ and $b_n(q)$, as $q \to 0$:

$$\mathrm{ce}_0 = \frac{1}{\sqrt{2}}\Bigl(1 - \frac{q}{2}\cos 2x\Bigr), \quad a_0 = -\frac{q^2}{2} + \frac{7q^4}{128};$$

$$\mathrm{ce}_1 = \cos x - \frac{q}{8}\cos 3x, \quad a_1 = 1 + q;$$

$$\mathrm{ce}_2 = \cos 2x + \frac{q}{4}\Bigl(1 - \frac{\cos 4x}{3}\Bigr), \quad a_2 = 4 + \frac{5q^2}{12};$$

$$\mathrm{ce}_n = \cos nx + \frac{q}{4}\left[\frac{\cos(n+2)x}{n+1} - \frac{\cos(n-2)x}{n-1}\right], \quad a_n = n^2 + \frac{q^2}{2(n^2-1)} \quad (n \ge 3);$$

$$\mathrm{se}_1 = \sin x - \frac{q}{8}\sin 3x, \quad b_1 = 1 - q;$$

$$\mathrm{se}_2 = \sin 2x - q\frac{\sin 4x}{12}, \quad b_2 = 4 - \frac{q^2}{12};$$

$$\mathrm{se}_n = \sin nx - \frac{q}{4}\left[\frac{\sin(n+2)x}{n+1} - \frac{\sin(n-2)x}{n-1}\right], \quad b_n = n^2 + \frac{q^2}{2(n^2-1)} \quad (n \ge 3).$$

The Mathieu functions are discussed in the books by Abramowitz & Stegun (1964) and Bateman & Erdélyi (1955, vol. 3) in more detail.

5. $\boldsymbol{y''_{xx} + (a\sin^2 x + b)y = 0}$.

Utilizing the formula $2\sin^2 x = 1 - \cos 2x$, we obtain the Mathieu equation 2.1.6.4: $y''_{xx} + (\frac{1}{2}a + b - \frac{1}{2}a\cos 2x)y = 0$.

6. $\boldsymbol{y''_{xx} + (a\cos^2 x + b)y = 0}$.

Utilizing the formula $2\cos^2 x = 1 + \cos 2x$, we obtain the Mathieu equation 2.1.6.4: $y''_{xx} + (\frac{1}{2}a + b - \frac{1}{2}a\cos 2x)y = 0$.

7. $\boldsymbol{y''_{xx} - a[a\sin^2(bx) + b\cos(bx)]y = 0}$.

Particular solution: $y_0 = \exp\Bigl[-\frac{a}{b}\cos(bx)\Bigr]$.

8. $\boldsymbol{y''_{xx} - a[a\cos^2(bx) + b\sin(bx)]y = 0}$.

Particular solution: $y_0 = \exp\Bigl[-\frac{a}{b}\sin(bx)\Bigr]$.

9. $\boldsymbol{y''_{xx} + a[\lambda + (\lambda - a)\tan^2(\lambda x)]y = 0}$.

Particular solution: $y_0 = [\cos(\lambda x)]^{a/\lambda}$.

10. $\boldsymbol{y''_{xx} + (a\tan^2 x + b)y = 0}$.

The transformation $\xi = \sin^2 x$, $w = y\cos^m x$, where m is a root of the quadratic equation $m^2 + m + a = 0$ leads to the hypergeometric equation 2.1.2.158:

$$\xi(\xi - 1)w''_{\xi\xi} + [(1-m)\xi - \tfrac{1}{2}]w'_\xi - \tfrac{1}{4}(m+b)w = 0.$$

11. $\boldsymbol{y''_{xx} + a[\lambda + (\lambda - a)\cot^2(\lambda x)]y = 0}$.

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

12. $\boldsymbol{y''_{xx} + (a\cot^2 x + b)y = 0}$.

The substitution $x = \xi + \frac{\pi}{2}$ leads to the equation 2.1.6.10: $y''_{\xi\xi} + (a\tan^2 x + b)y = 0$.

13. $\boldsymbol{y''_{xx} + a\sin(bx)y'_x + c[ax^n \sin(bx) - cx^{2n} + nx^{n-1}]y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{n+1}x^{n+1}\Bigr)$.

14. $\boldsymbol{y''_{xx} + (a\sin x + b)y'_x + a(b\sin x + \cos x)y = 0}$.

Particular solution: $y_0 = \exp(a\cos x)$.

15. $\boldsymbol{y''_{xx} + a\sin^n(bx)y'_x + c[a\sin^n(bx) - c]y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

16. $\boldsymbol{y''_{xx} + [a\sin^n(bx) + c]y'_x + ac\sin^n(bx)y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

17. $\boldsymbol{y''_{xx} + (a\sin^n x + b\sin x)y'_x + b(a\sin^{n+1} x + \cos x)y = 0}$.

Particular solution: $y_0 = \exp(b\cos x)$.

18. $\boldsymbol{y''_{xx} + (ax + b)\sin^n(cx)y'_x - a\sin^n(cx)y = 0}$.

Particular solution: $y_0 = ax + b$.

19. $\boldsymbol{y''_{xx} + ax^n \sin^m(bx)y'_x - ax^{n-1}\sin^m(bx)y = 0}$.

Particular solution: $y_0 = x$.

20. $\boldsymbol{y''_{xx} + ax^n \sin^m(bx)y'_x + c[ax^{n+k}\sin^m(bx) - cx^{2k} + kx^{k-1}]y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

21. $\boldsymbol{y''_{xx} + (a\cos x + b)y'_x + a(b\cos x - \sin x)y = 0}$.

Particular solution: $y_0 = \exp(-a\sin x)$.

22. $\boldsymbol{y''_{xx} + a\cos^n(bx)y'_x + c[a\cos^n(bx) - c]y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

23. $\boldsymbol{y''_{xx} + [a\cos^n(bx) + c]y'_x + ac\cos^n(bx)y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

24. $\boldsymbol{y''_{xx} + (a\cos^n x + b\cos x)y'_x + b(a\cos^{n+1} x - \sin x)y = 0}$.

Particular solution: $y_0 = \exp(-b\sin x)$.

25. $y''_{xx} + (ax + b)\cos^n(cx)y'_x - a\cos^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

26. $y''_{xx} + ax^n\cos^m(bx)y'_x - ax^{n-1}\cos^m(bx)y = 0.$

Particular solution: $y_0 = x$.

27. $y''_{xx} + ax^n\cos^m(bx)y'_x + c[ax^{n+k}\cos^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

28. $y''_{xx} + (a - \lambda)\tan(\lambda x)y'_x + a\lambda y = 0.$

Particular solution: $y_0 = [\cos(\lambda x)]^{a/\lambda}$.

29. $y''_{xx} + a\tan x\, y'_x + by = 0.$

1°. The substitution $\xi = \sin x$ leads to an equation of the form 2.1.2.155:

$$(\xi^2 - 1)y''_{\xi\xi} + (1 - a)\xi y'_\xi - by = 0.$$

2°. Solution with $a = -2$:

$$y\cos x = \begin{cases} C_1\sin(kx) + C_2\cos(kx) & \text{if } b + 1 = k^2 > 0, \\ C_1\sinh(kx) + C_2\cosh(kx) & \text{if } b + 1 = -k^2 < 0. \end{cases}$$

3°. Solution with $a = 2$, $b = 3$:

$$y = C_1\cos^3 x + C_2\sin x(1 + 2\cos^2 x).$$

30. $y''_{xx} + n\tan x\, y'_x + a^2(\cos x)^{2n}y = 0.$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \int \cos^n x\, dx$.

31. $y''_{xx} + \tan x\, y'_x + a^2\cos^2 x(\sin x)^{2n-2}y = 0.$

Solution:

$$y = \sqrt{\sin x}\left[C_1 J_{\frac{1}{2n}}\Bigl(\frac{a}{n}\sin^n x\Bigr) + C_2 Y_{\frac{1}{2n}}\Bigl(\frac{a}{n}\sin^n x\Bigr)\right],$$

where J_ν and Y_ν are Bessel functions.

32. $y''_{xx} + a\tan x\, y'_x + (b\tan^2 x + c)y = 0.$

This is a special case of equation 2.1.6.55.

33. $y''_{xx} + \tan x\, y'_x - a(a - 1)\cot^2 x\, y = 0.$

Solution:

$$y = \begin{cases} C_1|\sin x|^a + C_2|\sin x|^{1-a} & \text{if } a \neq \frac{1}{2}, \\ \sqrt{|\sin x|}\,(C_1 + C_2\ln|\sin x|) & \text{if } a = \frac{1}{2}. \end{cases}$$

34. $y''_{xx} - 2\lambda \tan(\lambda x) y'_x + (ax^2 + bx + c)y = 0.$

The substitution $u = y\cos(\lambda x)$ leads to an equation of the form 2.1.2.6:

$$u''_{xx} + (ax^2 + bx + c + \lambda^2)u = 0.$$

35. $y''_{xx} - 2\lambda \tan(\lambda x) y'_x + (ax^{2n} + bx^{n-1} - \lambda^2)y = 0.$

The substitution $u = y\cos(\lambda x)$ leads to an equation of the form 2.1.2.10:

$$u''_{xx} + (ax^{2n} + bx^{n-1})u = 0.$$

36. $y''_{xx} + a\tan^n(bx) y'_x + c[a\tan^n(bx) - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

37. $y''_{xx} + a\tan^n(\lambda x) y'_x + b[a\tan^{n+1}(\lambda x) + (\lambda - b)\tan^2(\lambda x) + \lambda]y = 0.$

Particular solution: $y_0 = [\cos(\lambda x)]^{b/\lambda}$.

38. $y''_{xx} + a\tan^n x\, y'_x + (a\tan^{n+1} x - a\tan^{n-1} x + 4)y = 0.$

Particular solution: $y_0 = \sin x \cos x$.

39. $y''_{xx} + [a\tan^n(bx) + c]y'_x + ac\tan^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

40. $y''_{xx} + \tan x(a\tan^n x + b - 1)y'_x + (ab\tan^{n+2} x - a\tan^n x + 2b + 2)y = 0.$

Particular solution: $y_0 = \sin x \cos^b x$.

41. $y''_{xx} + (ax + b)\tan^n(cx) y'_x - a\tan^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

42. $y''_{xx} + ax^n \tan^m(bx) y'_x - ax^{n-1}\tan^m(bx)y = 0.$

Particular solution: $y_0 = x$.

43. $y''_{xx} + ax^n \tan^m(bx) y'_x + c[ax^{n+k}\tan^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

44. $y''_{xx} + \cot x\, y'_x + \nu(\nu + 1)y = 0.$

The substitution $\xi = \cos x$ leads to the Legendre equation 2.1.2.148:

$$(\xi^2 - 1)y''_{\xi\xi} + 2\xi y'_\xi - \nu(\nu + 1)y = 0.$$

45. $y''_{xx} + 2a\cot(ax) y'_x + (b^2 - a^2)y = 0.$

Particular solution: $y_0 = \dfrac{\cos(bx)}{\sin(ax)}$.

46. $\boldsymbol{y''_{xx} + (\lambda - a)\cot(\lambda x) y'_x + a\lambda y = 0}$.

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

47. $\boldsymbol{y''_{xx} + a\cot(\lambda x) y'_x + by = 0}$.

The substitution $\xi = \lambda x + \frac{\pi}{2}$ leads to an equation of the form 2.1.6.29:

$$y''_{\xi\xi} - a\lambda^{-1}\tan\xi\, y'_\xi + b\lambda^{-2}y = 0.$$

48. $\boldsymbol{y''_{xx} - 2a\cot(2ax) y'_x - b^2\sin^2(2ax) y = 0}$.

Solution: $y = C_1 \exp\Bigl[\dfrac{b}{a}\sin^2(ax)\Bigr] + C_2 \exp\Bigl[-\dfrac{b}{a}\sin^2(ax)\Bigr]$.

49. $\boldsymbol{y''_{xx} - n\cot x\, y'_x + a^2(\sin x)^{2n} y = 0}$.

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \int \sin^n x\, dx$.

50. $\boldsymbol{y''_{xx} - 2\cot(2x) y'_x + a\tan^2 x\, y = 0}$.

The substitution $\xi = \cos x$ leads to the Euler equation 2.1.2.118: $\xi^2 y''_{\xi\xi} - \xi y'_\xi + ay = 0$.

51. $\boldsymbol{y''_{xx} + a\cot(\lambda x) y'_x + b[\lambda + (\lambda - a - b)\cot^2(\lambda x)] y = 0}$.

Particular solution: $y_0 = [\sin(\lambda x)]^{b/\lambda}$.

52. $\boldsymbol{y''_{xx} + a\cot x\, y'_x + (b\cot^2 x + c) y = 0}$.

This is a special case of equation 2.1.6.55.

53. $\boldsymbol{y''_{xx} + 2\lambda\cot(\lambda x) y'_x + (ax^2 + bx + c) y = 0}$.

The substitution $u = y\sin(\lambda x)$ leads to 2.1.2.6: $u''_{xx} + (ax^2 + bx + c + \lambda^2)u = 0$.

54. $\boldsymbol{y''_{xx} + 2\lambda\cot(\lambda x) y'_x + (ax^{2n} + bx^{n-1} - \lambda^2) y = 0}$.

The substitution $u = y\sin(\lambda x)$ leads to 2.1.2.10: $u''_{xx} + (ax^{2n} + bx^{n-1})u = 0$.

55. $\boldsymbol{y''_{xx} + (a\tan x + b\cot x) y'_x + (\alpha\tan^2 x + \beta\cot^2 x + \gamma) y = 0}$.

The transformation $\xi = \sin^2 x$, $y = w\sin^n x\cos^m x$, where n and m are roots of the quadratic equations

$$n^2 + (b-1)n + \beta = 0, \qquad m^2 - (a+1)m + \alpha = 0,$$

leads to the hypergeometric equation 2.1.2.158:

$$4\xi(\xi-1)w''_{\xi\xi} + 2[(2n+2m+2+b-a)\xi - 2n - b - 1]w'_\xi + (2nm+n+m+bm-an-\gamma)w = 0.$$

56. $\boldsymbol{y''_{xx} + a\cot^n(bx) y'_x + c[a\cot^n(bx) - c] y = 0}$.

Particular solution: $y_0 = e^{-cx}$.

57. $y''_{xx} + [a\cot^n(bx) + c]y'_x + ac\cot^n(bx)y = 0.$

Particular solution: $y_0 = e^{-cx}$.

58. $y''_{xx} + (ax + b)\cot^n(cx)y'_x - a\cot^n(cx)y = 0.$

Particular solution: $y_0 = ax + b$.

59. $y''_{xx} + ax^n\cot^m(bx)y'_x - ax^{n-1}\cot^m(bx)y = 0.$

Particular solution: $y_0 = x$.

60. $y''_{xx} + ax^n\cot^m(bx)y'_x + c[ax^{n+k}\cot^m(bx) - cx^{2k} + kx^{k-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{c}{k+1}x^{k+1}\Bigr)$.

61. $xy''_{xx} + [(ax^2 + bx)\sin^n(cx) + 2]y'_x + b\sin^n(cx)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

62. $xy''_{xx} + (ax^{n+1} + b\sin^m x)y'_x + ax^n(b\sin^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

63. $xy''_{xx} + (ax^n + bx\sin^m x)y'_x + [b(ax^n - 1)\sin^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr)$.

64. $xy''_{xx} + ax^n\sin^m(bx)y'_x - [a(cx + 1)x^{n-1}\sin^m(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

65. $xy''_{xx} + [ax^n\sin^m(bx) + c]y'_x + a(c - 1)x^{n-1}\sin^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

66. $xy''_{xx} + ax\cos^n(bx)y'_x - [a(cx + 1)\cos^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}$.

67. $xy''_{xx} + [(ax^2 + bx)\cos^n(cx) + 2]y'_x + b\cos^n(cx)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

68. $xy''_{xx} + (ax^{n+1} + b\cos^m x)y'_x + ax^n(b\cos^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

69. $xy''_{xx} + [ax^n\cos^m(bx) + c]y'_x + a(c - 1)x^{n-1}\cos^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}$.

70. $xy''_{xx} + (ax^n + bx\cos^m x)y'_x + [b(ax^n - 1)\cos^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

71. $xy''_{xx} - 2\lambda x\tan(\lambda x)y'_x + (ax + b)y = 0.$

The substitution $u = y\cos(\lambda x)$ leads to an equation of the form 2.1.2.59:

$$xu''_{xx} + [(a + \lambda^2)x + b]u = 0.$$

72. $xy''_{xx} + ax\tan^n(bx)y'_x - [a(cx + 1)\tan^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}.$

73. $xy''_{xx} + [(ax^2 + bx)\tan^n(cx) + 2]y'_x + b\tan^n(cx)y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

74. $xy''_{xx} + (ax^{n+1} + b\tan^m x)y'_x + ax^n(b\tan^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr).$

75. $xy''_{xx} + (ax^n + bx\tan^m x)y'_x + [b(ax^n - 1)\tan^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

76. $xy''_{xx} + [ax^n\tan^m(bx) + c]y'_x + a(c - 1)x^{n-1}\tan^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}.$

77. $xy''_{xx} + 2\lambda x\cot(\lambda x)y'_x + (ax + b)y = 0.$

The substitution $u = y\sin(\lambda x)$ leads to an equation of the form 2.1.2.59:

$$xu''_{xx} + [(a + \lambda^2)x + b]u = 0.$$

78. $xy''_{xx} + ax\cot^n(bx)y'_x - [a(cx + 1)\cot^n(bx) + c^2x + 2c]y = 0.$

Particular solution: $y_0 = xe^{cx}.$

79. $xy''_{xx} + (ax^{n+1} + b\cot^m x)y'_x + ax^n(b\cot^m x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr).$

80. $xy''_{xx} + (ax^n + bx\cot^m x)y'_x + [b(ax^n - 1)\cot^m x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

81. $xy''_{xx} + [ax^n\cot^m(bx) + c]y'_x + a(c - 1)x^{n-1}\cot^m(bx)y = 0.$

Particular solution: $y_0 = x^{1-c}.$

82. $x^2 y''_{xx} + x(a\sin^n x + 1)y'_x + b(a\sin^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

83. $x^2 y''_{xx} + x(a\sin^n x + b)y'_x + b(a\sin^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

84. $x^2 y''_{xx} + ax^n \sin^m(bx)y'_x + c[ax^{n-1}\sin^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

85. $x^2 y''_{xx} + x(a\cos^n x + 1)y'_x + b(a\cos^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

86. $x^2 y''_{xx} + x(a\cos^n x + b)y'_x + b(a\cos^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

87. $x^2 y''_{xx} + ax^n \cos^m(bx)y'_x + c[ax^{n-1}\cos^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

88. $x^2 y''_{xx} - 2\lambda x^2 \tan(\lambda x)y'_x + (ax^2 + bx + c)y = 0.$

The substitution $u = y\cos(\lambda x)$ leads to an equation of the form 2.1.2.110:

$$x^2 u''_{xx} + [(a+\lambda^2)x^2 + bx + c]u = 0.$$

89. $x^2 y''_{xx} + x(1 - 2x\tan x)y'_x - (x\tan x + \nu^2)y = 0.$

Solution: $y\cos x = C_1 J_\nu(x) + C_2 Y_\nu(x)$, where J_ν and Y_ν are Bessel functions.

90. $x^2 y''_{xx} - x(2x\tan x + k)y'_x + (ax^2 + bx + c + kx\tan x)y = 0.$

The substitution $u = y\cos x$ leads to an equation of the form 2.1.2.126:

$$x^2 u''_{xx} - kxu'_x + [(a+1)x^2 + bx + c]u = 0.$$

91. $x^2 y''_{xx} + x(a\tan^n x + 1)y'_x + b(a\tan^n x - b)y = 0.$

Particular solution: $y_0 = x^{-b}$.

92. $x^2 y''_{xx} + x(a\tan^n x + b)y'_x + b(a\tan^n x - 1)y = 0.$

Particular solution: $y_0 = x^{-b}$.

93. $x^2 y''_{xx} + ax^n \tan^m(bx)y'_x + c[ax^{n-1}\tan^m(bx) - c - 1]y = 0.$

Particular solution: $y_0 = x^{-c}$.

94. $x^2 y''_{xx} + 2\lambda x^2 \cot(\lambda x)y'_x + (ax^2 + bx + c)y = 0.$

The substitution $u = y\sin(\lambda x)$ leads to an equation of the form 2.1.2.110:

$$x^2 u''_{xx} + [(a+\lambda^2)x^2 + bx + c]u = 0.$$

95. $\boldsymbol{x^2 y''_{xx} + x(2x\cot x + k)y'_x + (ax^2 + bx + c + kx\cot x)y = 0.}$

The substitution $u = y\sin x$ leads to an equation of the form 2.1.2.126:

$$x^2 u''_{xx} + kxu'_x + [(a+1)x^2 + bx + c]u = 0.$$

96. $\boldsymbol{x^2 y''_{xx} + x(a\cot^n x + 1)y'_x + b(a\cot^n x - b)y = 0.}$

Particular solution: $y_0 = x^{-b}$.

97. $\boldsymbol{x^2 y''_{xx} + x(a\cot^n x + b)y'_x + b(a\cot^n x - 1)y = 0.}$

Particular solution: $y_0 = x^{-b}$.

98. $\boldsymbol{x^2 y''_{xx} + ax^n \cot^m(bx)y'_x + c[ax^{n-1}\cot^m(bx) - c - 1]y = 0.}$

Particular solution: $y_0 = x^{-c}$.

99. $\boldsymbol{x^4 y''_{xx} + \left[a\sin\left(\frac{\lambda}{x}\right) + b\right]y = 0.}$

The transformation $\xi = 1/x$, $w = y/x$ leads to an equation of the form 2.1.6.3:

$$w''_{\xi\xi} + [a\sin(\lambda\xi) + b]w = 0.$$

100. $\boldsymbol{x^4 y''_{xx} + ax^2\sin^n(bx)y'_x + [a(c - x)\sin^n(bx) - c^2]y = 0.}$

Particular solution: $y_0 = x\exp\left(\frac{c}{x}\right)$.

101. $\boldsymbol{x^4 y''_{xx} + ax^2\cos^n(bx)y'_x + [a(c - x)\cos^n(bx) - c^2]y = 0.}$

Particular solution: $y_0 = x\exp\left(\frac{c}{x}\right)$.

102. $\boldsymbol{x^4 y''_{xx} + ax^2\tan^n(bx)y'_x + [a(c - x)\tan^n(bx) - c^2]y = 0.}$

Particular solution: $y_0 = x\exp\left(\frac{c}{x}\right)$.

103. $\boldsymbol{x^4 y''_{xx} + ax^2\cot^n(bx)y'_x + [a(c - x)\cot^n(bx) - c^2]y = 0.}$

Particular solution: $y_0 = x\exp\left(\frac{c}{x}\right)$.

104. $\boldsymbol{\sin(2x)y''_{xx} - y'_x + 2a^2\sin^2 x\, y = 0.}$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \int\sqrt{\tan x}\,dx$.

105. $\boldsymbol{\sin(2x)y''_{xx} - 2ny'_x + 2a^2\sin^2 x(\tan x)^{2n-1}y = 0.}$

Solution: $y = C_1\sin(au) + C_2\cos(au)$, where $u = \int\tan^n x\,dx$.

106. $\boldsymbol{\sin x\, y''_{xx} + \cos x\, y'_x + \nu(\nu + 1)\sin x\, y = 0.}$

The substitution $\xi = \cos x$ leads to the Legendre equation 2.1.2.148:

$$(1 - \xi^2)y''_{\xi\xi} - 2\xi\, y'_\xi + \nu(\nu + 1)y = 0.$$

107. $\sin x\, y''_{xx} + (2n+1)\cos x\, y'_x + (\nu - n)(\nu + n + 1)\sin x\, y = 0,$
where ν is an arbitrary number, n is a positive integer.

The substitution $\xi = \cos x$ leads to an equation of the form 2.1.2.148:

$$(\xi^2 - 1)y''_{\xi\xi} + 2(n+1)\xi\, y'_\xi + (n-\nu)(\nu+n+1)y = 0.$$

108. $\sin^2 x\, y''_{xx} + ay = 0.$

This is a special case of equation 2.1.6.110.

109. $\sin^2 x\, y''_{xx} - [a\sin^2 x + n(n-1)]y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $\quad y = \sin^n x\left(\dfrac{1}{\sin x}\dfrac{d}{dx}\right)^n \left(C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}}\right).$

110. $\sin^2 x\, y''_{xx} + (a\sin^2 x + b)y = 0.$

Set $x = 2\xi$. Utilizing the trigonometric formulae $\sin 2\xi = 2\sin\xi\cos\xi$ and $b = b(\sin^2\xi + \cos^2\xi)^2$ and dividing both sides of the equation by $\sin^2 x$, we arrive at an equation of the form 2.1.6.55:

$$y''_{\xi\xi} + (b\tan^2\xi + b\cot^2\xi + 4a + 2b)y = 0.$$

111. $\sin^2 x\, y''_{xx} - \{[(a^2b^2 - (a+1)^2]\sin^2 x + a(a+1)b\sin 2x + a(a-1)\}y = 0.$

Particular solution: $\quad y_0 = e^{abx}\sin^a x\,(\cos x + b\sin x).$

112. $\sin^2 x\, y''_{xx} + \sin x\cos x\, y'_x + [\nu(\nu+1)\sin^2 x - n^2]y = 0,$
where ν is an arbitrary number, n is a nonnegative integer.

The transformation $\xi = \cos x$, $y = w\sin^n x$ leads to an equation of the form 2.1.2.149:

$$(\xi^2 - 1)w''_{\xi\xi} + 2(n+1)\xi w'_\xi + (n-\nu)(n+\nu+1)w = 0.$$

113. $\sin^2 x\, y''_{xx} + \sin x(a\cos x + b)y'_x + (\alpha\cos^2 x + \beta\cos x + \gamma)y = 0.$

Set $x = 2\xi$. Utilizing the trigonometric formulae

$$\sin(2\xi) = 2\sin\xi\cos\xi, \quad \cos(2\xi) = \cos^2\xi - \sin^2\xi, \quad b = b(\sin^2\xi + \cos^2\xi),$$
$$\beta = \beta(\sin^2\xi + \cos^2\xi), \quad \gamma = \gamma(\sin^2\xi + \cos^2\xi)^2,$$

and dividing all the terms by $\sin^2 x$, we arrive at an equation of the form 2.1.6.55:

$$y''_{\xi\xi} + [(b-a)\tan\xi + (b+a)\cot\xi]y'_\xi + [(\alpha-\beta+\gamma)\tan^2\xi + (\alpha+\beta+\gamma)\cot^2\xi + 2\gamma - 2\alpha]y = 0.$$

114. $\cos^2 x\, y''_{xx} - [a\cos^2 x + n(n-1)]y = 0, \qquad n = 1,\ 2,\ 3,\ \ldots$

Solution: $\quad y = \cos^n x\left(\dfrac{1}{\cos x}\dfrac{d}{dx}\right)^n \left(C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}}\right).$

115. $\cos^2 x\, y''_{xx} + (a\cos^2 x + b)y = 0.$

The substitution $x = \xi + \frac{\pi}{2}$ leads to 2.1.6.110: $\sin^2\xi\, y''_{\xi\xi} + (a\sin^2\xi + b)y = 0.$

116. $\cos^2 x\, y''_{xx} + a\sin(2x)y'_x + [b\cos(2x) + c]y = 0.$

Dividing the equation by $\cos^2 x$ and utilizing the formulae

$$\sin(2x) = 2\sin x\cos x, \quad \cos(2x) = \cos^2 x - \sin^2 x, \quad c = c(\sin^2 x + \cos^2 x),$$

we obtain an equation of the form 2.1.6.55:

$$y''_{xx} + 2a\tan x\, y'_x + [(c-b)\tan^2 x + b + c]y = 0.$$

117. $\cos^2(ax)y''_{xx} + (n-1)a\sin(2ax)y'_x + na^2[(n-1)\sin^2(ax) + \cos^2(ax)]y = 0.$

Particular solution: $y_0 = \cos^n(ax)$.

118. $\cos^2 x\, y''_{xx} + \cos x(a\sin x + b)y'_x + (\alpha\sin^2 x + \beta\sin x + \gamma)y = 0.$

The substitution $x = \xi + \frac{\pi}{2}$ leads to an equation of the form 2.1.6.113: $\sin^2\xi y''_{\xi\xi} - \sin\xi\,(a\cos\xi + b)y'_\xi + (\alpha\cos^2\xi + \beta\cos\xi + \gamma)y = 0$.

119. $\sin x\cos^2 x\, y''_{xx} + \cos x(a\sin^2 x + b)y'_x + c\sin x\, y = 0.$

1°. Dividing the equation by $\sin x\cos^2 x$ and assuming $b = b(\sin^2 x + \cos^2 x)$, $c = c(\sin^2 x + \cos^2 x)$, we obtain the equation 2.1.6.55:

$$y''_{xx} + [(a+b)\tan x + b\cot x]y'_x + c(\tan^2 x + 1)y = 0.$$

2°. Particular solutions:

$$\begin{aligned} &y_0 = \cos^a x && \text{for } c = a(b+1),\\ &y_0 = \tan^{1-b} x && \text{for } c = (a+2)(b-1),\\ &y_0 = \sin^{1-b} x\cos^{a+b-1} x && \text{for } c = 2(a+b-1). \end{aligned}$$

120. $\sin x\cos^2 x\, y''_{xx} + \cos x(a\sin^2 x - 1)y'_x + b\sin^3 x\, y = 0.$

Solution: $y = C_1(\cos x)^{k_1} + C_2(\cos x)^{k_2}$, where k_1 and k_2 are the roots of the quadratic equation $k^2 - ak + b = 0$.

121. $\sin^2 x\cos^2 x\, y''_{xx} + (a\sin^2 x + b\cos^2 x + c\sin^2 x\cos^2 x)y = 0.$

Dividing the equation by $\sin^2 x\cos^2 x$ and assuming $a = a(\sin^2 x + \cos^2 x)$, $b = b(\sin^2 x + \cos^2 x)$, we arrive at the equation 2.1.6.55:

$$y''_{xx} + (a\tan^2 x + b\cot^2 x + a + b + c)y = 0.$$

122. $[a\sin(\lambda x) + bx + c]y''_{xx} + a\lambda^2\sin(\lambda x)y = 0.$

Particular solution: $y_0 = a\sin(\lambda x) + bx + c$.

123. $[a\cos(\lambda x) + bx + c]y''_{xx} + a\lambda^2\cos(\lambda x)y = 0.$

Particular solution: $y_0 = a\cos(\lambda x) + bx + c$.

124. $\sin^n(ax)y''_{xx} + (x^2 - b^2)y'_x - (x+b)y = 0.$

Particular solution: $y_0 = x - b$.

125. $\sin^n(\lambda x)y''_{xx} + [\lambda^2 \sin^n(\lambda x) + a\cos^{n-4}(\lambda x)]y = 0.$

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + a\lambda^{-2}\xi^{-n}w = 0$.

126. $(a\sin^n x + b)y''_{xx} + (c\sin^n x + d)y'_x + \lambda[(c - a\lambda)\sin^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

127. $\cos^n(\lambda x)y''_{xx} + [\lambda^2 \cos^n(\lambda x) + a\sin^{n-4}(\lambda x)]y = 0.$

The substitution $\lambda x = \frac{\pi}{2} - \lambda\xi$ leads to an equation of the form 2.1.6.125.

128. $\cos^n(ax)y''_{xx} + (x^2 - b^2)y'_x - (x + b)y = 0.$

Particular solution: $y_0 = x - b$.

129. $(a\cos^n x + b)y''_{xx} + (c\cos^n x + d)y'_x + \lambda[(c - a\lambda)\cos^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

130. $(a\tan^n x + b)y''_{xx} + (cx + d)y'_x - cy = 0.$

Particular solution: $y_0 = cx + d$.

131. $(a\tan^n x + b)y''_{xx} + (c\tan^n x + d)y'_x + \lambda[(c - a\lambda)\tan^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

132. $(a\cot^n x + b)y''_{xx} + (c\cot^n x + d)y'_x + \lambda[(c - a\lambda)\cot^n x + d - b\lambda]y = 0.$

Particular solution: $y_0 = e^{-\lambda x}$.

2.1.7. Equations Containing Inverse Trigonometric Functions

1. $y''_{xx} + (ax + b + c\arcsin x)y'_x + [c(ax + b)\arcsin x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx)$.

2. $y''_{xx} + b(\arcsin x)^n y'_x + c[b(\arcsin x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}$.

3. $y''_{xx} + b(\arcsin x)^n y'_x + a[bx^m(\arcsin x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\left(-\dfrac{a}{m+1}x^{m+1}\right)$.

4. $y''_{xx} + (ax + b)(\arcsin x)^n y'_x - a(\arcsin x)^n y = 0.$

Particular solution: $y_0 = ax + b$.

5. $y''_{xx} + ax^n(\arcsin x)^m y'_x - ax^{n-1}(\arcsin x)^m y = 0.$

Particular solution: $y_0 = x.$

6. $y''_{xx} + (ax + b + c\arccos x)y'_x + [c(ax+b)\arccos x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx).$

7. $y''_{xx} + b(\arccos x)^n y'_x + c[b(\arccos x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}.$

8. $y''_{xx} + b(\arccos x)^n y'_x + a[bx^m(\arccos x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr).$

9. $y''_{xx} + (ax+b)(\arccos x)^n y'_x - a(\arccos x)^n y = 0.$

Particular solution: $y_0 = ax + b.$

10. $y''_{xx} + ax^n(\arccos x)^m y'_x - ax^{n-1}(\arccos x)^m y = 0.$

Particular solution: $y_0 = x.$

11. $y''_{xx} + (ax + b + c\arctan x)y'_x + [c(ax+b)\arctan x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx).$

12. $y''_{xx} + b(\arctan x)^n y'_x + c[b(\arctan x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}.$

13. $y''_{xx} + b(\arctan x)^n y'_x + a[bx^m(\arctan x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr).$

14. $y''_{xx} + (ax+b)(\arctan x)^n y'_x - a(\arctan x)^n y = 0.$

Particular solution: $y_0 = ax + b.$

15. $y''_{xx} + ax^n(\arctan x)^m y'_x - ax^{n-1}(\arctan x)^m y = 0.$

Particular solution: $y_0 = x.$

16. $y''_{xx} + (ax + b + c\,\mathrm{arccot}\, x)y'_x + [c(ax+b)\,\mathrm{arccot}\, x + a]y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2 - bx).$

17. $y''_{xx} + b(\mathrm{arccot}\, x)^n y'_x + c[b(\mathrm{arccot}\, x)^n - c]y = 0.$

Particular solution: $y_0 = e^{-cx}.$

18. $y''_{xx} + b(\mathrm{arccot}\, x)^n y'_x + a[bx^m(\mathrm{arccot}\, x)^n - ax^{2m} + mx^{m-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{m+1}x^{m+1}\Bigr).$

19. $y''_{xx} + (ax+b)(\operatorname{arccot} x)^n y'_x - a(\operatorname{arccot} x)^n y = 0.$

Particular solution: $y_0 = ax + b.$

20. $y''_{xx} + ax^n(\operatorname{arccot} x)^m y'_x - ax^{n-1}(\operatorname{arccot} x)^m y = 0.$

Particular solution: $y_0 = x.$

21. $xy''_{xx} + ax \arcsin x\, y'_x - [a(bx+1)\arcsin x + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}.$

22. $xy''_{xx} + [a(bx+1)\arcsin x + bx - 1]y'_x + ab^2 x \arcsin x\, y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}.$

23. $xy''_{xx} + [(ax^2+bx)\arcsin x + 2]y'_x + b\arcsin x\, y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

24. $xy''_{xx} + [ax(\arcsin x)^n + b]y'_x + a(b-1)(\arcsin x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}.$

25. $xy''_{xx} + (ax^{n+1} + b\arcsin x)y'_x + ax^n(b\arcsin x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr).$

26. $xy''_{xx} + (ax^n + bx\arcsin x)y'_x + [b(ax^n - 1)\arcsin x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

27. $x^2y''_{xx} + bx\arcsin x\, y'_x + a(b\arcsin x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}.$

28. $x^2y''_{xx} + x(b\arcsin x + 2)y'_x + [b(ax+1)\arcsin x - a^2x^2]y = 0.$

Particular solution: $y_0 = \dfrac{1}{x}e^{-ax}.$

29. $xy''_{xx} + ax\arccos x\, y'_x - [a(bx+1)\arccos x + b(b+1)]y = 0.$

Particular solution: $y_0 = xe^{bx}.$

30. $xy''_{xx} + [a(bx+1)\arccos x + bx - 1]y'_x + ab^2x\arccos x\, y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}.$

31. $xy''_{xx} + [(ax^2+bx)\arccos x + 2]y'_x + b\arccos x\, y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

32. $\boldsymbol{xy''_{xx} + [ax(\arccos x)^n + b]y'_x + a(b-1)(\arccos x)^n y = 0.}$

Particular solution: $y_0 = x^{1-b}$.

33. $\boldsymbol{xy''_{xx} + (ax^{n+1} + b\arccos x)y'_x + ax^n(b\arccos x + n)y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

34. $\boldsymbol{xy''_{xx} + (ax^n + bx\arccos x)y'_x + [b(ax^n - 1)\arccos x + anx^{n-1}]y = 0.}$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr)$.

35. $\boldsymbol{x^2y''_{xx} + bx\arccos x\, y'_x + a(b\arccos x - a - 1)y = 0.}$

Particular solution: $y_0 = x^{-a}$.

36. $\boldsymbol{x^2y''_{xx} + x(b\arccos x + 2)y'_x + [b(ax+1)\arccos x - a^2x^2]y = 0.}$

Particular solution: $y_0 = \dfrac{1}{x}e^{-ax}$.

37. $\boldsymbol{xy''_{xx} + ax\arctan x\, y'_x - [a(bx+1)\arctan x + b(b+1)]y = 0.}$

Particular solution: $y_0 = xe^{bx}$.

38. $\boldsymbol{xy''_{xx} + [a(bx+1)\arctan x + bx - 1]y'_x + ab^2x\arctan x\, y = 0.}$

Particular solution: $y_0 = (bx+1)e^{-bx}$.

39. $\boldsymbol{xy''_{xx} + [(ax^2 + bx)\arctan x + 2]y'_x + b\arctan x\, y = 0.}$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

40. $\boldsymbol{xy''_{xx} + [ax(\arctan x)^n + b]y'_x + a(b-1)(\arctan x)^n y = 0.}$

Particular solution: $y_0 = x^{1-b}$.

41. $\boldsymbol{xy''_{xx} + (ax^{n+1} + b\arctan x)y'_x + ax^n(b\arctan x + n)y = 0.}$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

42. $\boldsymbol{xy''_{xx} + (ax^n + bx\arctan x)y'_x + [b(ax^n - 1)\arctan x + anx^{n-1}]y = 0.}$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr)$.

43. $\boldsymbol{xy''_{xx} + ax(\arctan^n x + b)y'_x - a(\arctan^n x + b)y = 0.}$

Particular solution: $y_0 = x$.

44. $\boldsymbol{xy''_{xx} + b\arctan^n x\, y'_x + a(b\arctan^n x - ax)y = 0.}$

Particular solution: $y_0 = e^{-ax}$.

45. $xy''_{xx} + a(\arctan^n x + bx)y'_x + ab\arctan^n xy = 0.$

Particular solution: $y_0 = e^{-bx}$.

46. $xy''_{xx} + b\arctan^n xy'_x + ax(b\arctan^n x - ax + 1)y = 0.$

Particular solution: $y_0 = \exp(-\frac{1}{2}ax^2)$.

47. $x^2y''_{xx} + bx\arctan xy'_x + a(b\arctan x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}$.

48. $x^2y''_{xx} + x(b\arctan x + 2)y'_x + [b(ax+1)\arctan xa^2x^2]y = 0.$

Particular solution: $y_0 = \dfrac{1}{2}e^{-ax}$.

49. $x^2y''_{xx} + ax(\arctan^n x + b)y'_x - a(\arctan^n x + b)y = 0.$

Particular solution: $y_0 = x$.

50. $x^2y''_{xx} + b\arctan^n xy'_x + a(b\arctan^n x - ax^2)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

51. $x^2y''_{xx} + a(\arctan^n x + bx^2)y'_x + ab\arctan^n x\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

52. $x^2y''_{xx} + x[(ax+b)\arctan^n x + 2]y'_x + b\arctan^n x\, y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

53. $xy''_{xx} + ax\,\mathrm{arccot}\, x\, y'_x - [a(bx+1)\,\mathrm{arccot}\, x + b(bx+2)]y = 0.$

Particular solution: $y_0 = xe^{bx}$.

54. $xy''_{xx} + [a(bx+1)\,\mathrm{arccot}\, x + bx - 1]y'_x + ab^2x\,\mathrm{arccot}\, x\, y = 0.$

Particular solution: $y_0 = (bx+1)e^{-bx}$.

55. $xy''_{xx} + [(ax^2+bx)\,\mathrm{arccot}\, x + 2]y'_x + b\,\mathrm{arccot}\, x\, y = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}$.

56. $xy''_{xx} + [ax(\mathrm{arccot}\, x)^n + b]y'_x + a(b-1)(\mathrm{arccot}\, x)^n y = 0.$

Particular solution: $y_0 = x^{1-b}$.

57. $xy''_{xx} + (ax^{n+1} + b\,\mathrm{arccot}\, x)y'_x + ax^n(b\,\mathrm{arccot}\, x + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr)$.

58. $xy''_{xx} + (ax^n + bx \operatorname{arccot} x)y'_x + [b(ax^n - 1)\operatorname{arccot} x + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

59. $x^2y''_{xx} + bx \operatorname{arccot} x\, y'_x + a(b \operatorname{arccot} x - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}.$

60. $x^2y''_{xx} + x(b \operatorname{arccot} x + 2)y'_x + [b(ax + 1)\operatorname{arccot} x - a^2x^2]y = 0.$

Particular solution: $y_0 = \dfrac{1}{x}e^{-ax}.$

61. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\arcsin x)^n y'_x - 2a[cx(\arcsin x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b.$

62. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\arccos x)^n y'_x - 2a[cx(\arccos x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b.$

63. $(x^2 + 1)y''_{xx} - [a^2(x^2 + 1)(\arctan x)^2 + a]y = 0.$

Particular solution: $y_0 = (x^2 + 1)^{-a/2}\exp(ax \arctan x).$

64. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\arctan x)^n y'_x - 2a[cx(\arctan x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b.$

65. $(ax^2 + b)y''_{xx} + c(ax^2 + b)(\operatorname{arccot} x)^n y'_x - 2a[cx(\operatorname{arccot} x)^n + 1]y = 0.$

Particular solution: $y_0 = ax^2 + b.$

66. $x^4y''_{xx} + ax^2 \arcsin x\, y'_x + [a(b - x)\arcsin x - b^2]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(\dfrac{b}{x}\Bigr).$

67. $x^4y''_{xx} + ax^2 \arccos x\, y'_x + [a(b - x)\arccos x - b^2]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(\dfrac{b}{x}\Bigr).$

68. $x^4y''_{xx} + ax^2 \arctan x\, y'_x + [a(b - x)\arctan x - b^2]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(\dfrac{b}{x}\Bigr).$

69. $x^4y''_{xx} + ax^2 \operatorname{arccot} x\, y'_x + [a(b - x)\operatorname{arccot} x - b^2]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(\dfrac{b}{x}\Bigr).$

70. $(x^2 + 1)^2y''_{xx} + [a(\arctan x)^2 + b \arctan x + c]y = 0.$

The transformation $\xi = \arctan x$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to an equation of the form 2.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c + 1)w = 0.$

71. $(x^2+1)^2 y''_{xx} + [b(\arctan x)^n - 1]y = 0.$

The transformation $\xi = \arctan x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + b\xi^n w = 0$.

72. $(x^2+1)^2 y''_{xx} + [a(\operatorname{arccot} x)^2 + b \operatorname{arccot} x + c]y = 0.$

The transformation $\xi = \operatorname{arccot} x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 2.1.2.6: $w''_{\xi\xi} + (a\xi^2 + b\xi + c + 1)w = 0$.

73. $(x^2+1)^2 y''_{xx} + [b(\operatorname{arccot} x)^n - 1]y = 0.$

The transformation $\xi = \operatorname{arccot} x$, $w = \dfrac{y}{\sqrt{x^2+1}}$ leads to an equation of the form 2.1.2.7: $w''_{\xi\xi} + b\xi^n w = 0$.

74. $(ax^2+b)^2 y''_{xx} + (cx+d)(\arcsin x)^n y'_x - c(\arcsin x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

75. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\arcsin x)^n y'_x - [bx(\arcsin x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

76. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\arcsin x)^n y'_x + [c(\arcsin x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\left(-\int \dfrac{dx}{ax^2+b}\right)$.

77. $(ax^2+b)^2 y''_{xx} + (cx+d)(\arccos x)^n y'_x - c(\arccos x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

78. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\arccos x)^n y'_x - [bx(\arccos x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

79. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\arccos x)^n y'_x + [c(\arccos x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\left(-\int \dfrac{dx}{ax^2+b}\right)$.

80. $(ax^2+b)^2 y''_{xx} + (cx+d)(\arctan x)^n y'_x - c(\arctan x)^n y = 0.$

Particular solution: $y_0 = cx + d$.

81. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\arctan x)^n y'_x - [bx(\arctan x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}$.

82. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\arctan x)^n y'_x + [c(\arctan x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\left(-\int \dfrac{dx}{ax^2+b}\right)$.

83. $(ax^2+b)^2 y''_{xx} + (cx+d)(\operatorname{arccot} x)^n y'_x - c(\operatorname{arccot} x)^n y = 0.$

Particular solution: $y_0 = cx + d.$

84. $(x^2+a)^2 y''_{xx} + b(x^2+a)(\operatorname{arccot} x)^n y'_x - [bx(\operatorname{arccot} x)^n + a]y = 0.$

Particular solution: $y_0 = \sqrt{x^2+a}.$

85. $(ax^2+b)^2 y''_{xx} + c(ax^2+b)(\operatorname{arccot} x)^n y'_x + [c(\operatorname{arccot} x)^n - 2ax - 1]y = 0.$

Particular solution: $y_0 = \exp\Big(-\int \dfrac{dx}{ax^2+b}\Big).$

2.1.8. Equations Containing Combinations of Exponential, Logarithmic, Trigonometric, and Other Functions

1. $y''_{xx} + ae^{\lambda x} y'_x + b[b + ae^{\lambda x}\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

2. $y''_{xx} + ae^{\lambda x} y'_x + b[b - ae^{\lambda x}\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx).$

3. $y''_{xx} + a\cosh^n(\lambda x) y'_x + b[b + a\cosh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

4. $y''_{xx} + a\cosh^n(\lambda x) y'_x + b[b - a\cosh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx).$

5. $y''_{xx} + a\cosh^n(kx) y'_x + be^{\lambda x}[a\cosh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big).$

6. $y''_{xx} + a\sinh^n(\lambda x) y'_x + b[b + a\sinh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

7. $y''_{xx} + a\sinh^n(\lambda x) y'_x + b[b - a\sinh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx).$

8. $y''_{xx} + a\sinh^n(kx) y'_x + be^{\lambda x}[a\sinh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Big(-\dfrac{b}{\lambda}e^{\lambda x}\Big).$

9. $y''_{xx} + a\tanh^n(\lambda x) y'_x + b[b + a\tanh^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

10. $y''_{xx} + a\tanh^n(\lambda x)y'_x + b[b - a\tanh^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

11. $y''_{xx} + a\tanh^n(kx)y'_x + be^{\lambda x}[a\tanh^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

12. $y''_{xx} + a\coth^n(\lambda x)y'_x + b[b + a\coth^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

13. $y''_{xx} + a\coth^n(\lambda x)y'_x + b[b - a\coth^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

14. $y''_{xx} + a\coth^n(kx)y'_x + be^{\lambda x}[a\coth^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

15. $y''_{xx} + a\ln^n(\lambda x)y'_x + b[b + a\ln^n(\lambda x)\tan(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

16. $y''_{xx} + a\ln^n(\lambda x)y'_x + b[b - a\ln^n(\lambda x)\cot(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

17. $y''_{xx} + a\ln^n(kx)y'_x + be^{\lambda x}[a\ln^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

18. $y''_{xx} + a\cos^n(kx)y'_x + be^{\lambda x}[a\cos^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

19. $y''_{xx} + a\sin^n(kx)y'_x + be^{\lambda x}[a\sin^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

20. $y''_{xx} + a\tan^n(kx)y'_x + be^{\lambda x}[a\tan^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

21. $y''_{xx} + a\cot^n(kx)y'_x + be^{\lambda x}[a\cot^n(kx) - be^{\lambda x} + \lambda]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{b}{\lambda}e^{\lambda x}\Bigr)$.

22. $y''_{xx} + (ae^{\lambda x} + b\ln^n x)y'_x + ae^{\lambda x}(b\ln^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

23. $y''_{xx} + (ae^{\lambda x} + b\cos x)y'_x + b(ae^{\lambda x}\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x).$

24. $y''_{xx} + (ae^{\lambda x} + b\cos^n x)y'_x + ae^{\lambda x}(b\cos^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

25. $y''_{xx} + (ae^{\lambda x} + b\cos^n x)y'_x + b\cos^{n-1} x\,(ae^{\lambda x}\cos x - n\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\displaystyle\int \cos^n x\,dx\Bigr).$

26. $y''_{xx} + (ae^{\lambda x} + b\sin x)y'_x + b(ae^{\lambda x}\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x).$

27. $y''_{xx} + (ae^{\lambda x} + b\sin^n x)y'_x + ae^{\lambda x}(b\sin^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

28. $y''_{xx} + (ae^{\lambda x} + b\sin^n x)y'_x + b\sin^{n-1} x\,(ae^{\lambda x}\sin x + n\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\displaystyle\int \sin^n x\,dx\Bigr).$

29. $y''_{xx} + (ae^{\lambda x} + b\tan x)y'_x + (b+1)(ae^{\lambda x}\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x.$

30. $y''_{xx} + (ae^{\lambda x} + b\tan^n x)y'_x + ae^{\lambda x}(b\tan^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

31. $y''_{xx} + (ae^{\lambda x} + b\cot x)y'_x + (b-1)(ae^{\lambda x}\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x.$

32. $y''_{xx} + (ae^{\lambda x} + b\cot^n x)y'_x + ae^{\lambda x}(b\cot^n x + \lambda)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

33. $y''_{xx} + (a\cosh^n x + b\cos x)y'_x + b(a\cosh^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x).$

34. $y''_{xx} + (a\cosh^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\cosh^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

35. $y''_{xx} + (a\cosh^n x + b\sin x)y'_x + b(a\cosh^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

36. $y''_{xx} + (a\cosh^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\cosh^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

37. $y''_{xx} + (a\cosh^n x + b\tan x)y'_x + (b+1)(a\cosh^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

38. $y''_{xx} + (a\cosh^n x + b\cot x)y'_x + (b-1)(a\cosh^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

39. $y''_{xx} + (a\sinh^n x + b\cos x)y'_x + b(a\sinh^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

40. $y''_{xx} + (a\sinh^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\sinh^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

41. $y''_{xx} + (a\sinh^n x + b\sin x)y'_x + b(a\sinh^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

42. $y''_{xx} + (a\sinh^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\sinh^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

43. $y''_{xx} + (a\sinh^n x + b\tan x)y'_x + (b+1)(a\sinh^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

44. $y''_{xx} + (a\sinh^n x + b\cot x)y'_x + (b-1)(a\sinh^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

45. $y''_{xx} + (a\tanh^n x + b\cos x)y'_x + b(a\tanh^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

46. $y''_{xx} + (a\tanh^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\tanh^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

47. $y''_{xx} + (a\tanh^n x + b\sin x)y'_x + b(a\tanh^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

48. $y''_{xx} + (a\tanh^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\tanh^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

49. $y''_{xx} + (a\tanh^n x + b\tan x)y'_x + (b+1)(a\tanh^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

50. $y''_{xx} + (a\tanh^n x + b\cot x)y'_x + (b-1)(a\tanh^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

51. $y''_{xx} + (a\coth^n x + b\cos x)y'_x + b(a\coth^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

52. $y''_{xx} + (a\coth^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\coth^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

53. $y''_{xx} + (a\coth^n x + b\sin x)y'_x + b(a\coth^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

54. $y''_{xx} + (a\coth^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\coth^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \sin^m x\,dx\Bigr)$.

55. $y''_{xx} + (a\coth^n x + b\tan x)y'_x + (b+1)(a\coth^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x$.

56. $y''_{xx} + (a\coth^n x + b\cot x)y'_x + (b-1)(a\coth^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x$.

57. $y''_{xx} + (a\ln^n x + b\cos x)y'_x + b(a\ln^n x\cos x - \sin x)y = 0.$

Particular solution: $y_0 = \exp(-b\sin x)$.

58. $y''_{xx} + (a\ln^n x + b\cos^m x)y'_x + b\cos^{m-1} x\,(a\ln^n x\cos x - m\sin x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\int \cos^m x\,dx\Bigr)$.

59. $y''_{xx} + (a\ln^n x + b\sin x)y'_x + b(a\ln^n x\sin x + \cos x)y = 0.$

Particular solution: $y_0 = \exp(b\cos x)$.

60. $y''_{xx} + (a\ln^n x + b\sin^m x)y'_x + b\sin^{m-1} x\,(a\ln^n x\sin x + m\cos x)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-b\displaystyle\int \sin^m x\,dx\Bigr).$

61. $y''_{xx} + (a\ln^n x + b\tan x)y'_x + (b+1)(a\ln^n x\tan x + 1)y = 0.$

Particular solution: $y_0 = \cos^{b+1} x.$

62. $y''_{xx} + (a\ln^n x + b\cot x)y'_x + (b-1)(a\ln^n x\cot x - 1)y = 0.$

Particular solution: $y_0 = \sin^{1-b} x.$

63. $y''_{xx} + ae^{\lambda x}\cos(bx)y'_x + b[b + ae^{\lambda x}\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

64. $y''_{xx} + ae^{\lambda x}\sin(bx)y'_x + b[b - ae^{\lambda x}\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx).$

65. $y''_{xx} + a\cosh(bx)\ln^n(\lambda x)y'_x - b[b + a\sinh(bx)\ln^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx).$

66. $y''_{xx} + a\cosh(bx)\cos^n(\lambda x)y'_x - b[b + a\sinh(bx)\cos^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx).$

67. $y''_{xx} + a\cosh^n(\lambda x)\cos(bx)y'_x + b[b + a\cosh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx).$

68. $y''_{xx} + a\cosh(bx)\sin^n(\lambda x)y'_x - b[b + a\sinh(bx)\sin^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx).$

69. $y''_{xx} + a\cosh^n(\lambda x)\sin(bx)y'_x + b[b - a\cosh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx).$

70. $y''_{xx} + a\cosh(bx)\tan^n(\lambda x)y'_x - b[b + a\sinh(bx)\tan^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx).$

71. $y''_{xx} + a\cosh(bx)\cot^n(\lambda x)y'_x - b[b + a\sinh(bx)\cot^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \cosh(bx).$

72. $y''_{xx} + a\sinh(bx)\ln^n(kx)y'_x - b[b + a\cosh(bx)\ln^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx).$

73. $y''_{xx} + a\sinh(bx)\cos^n(kx)y'_x - b[b + a\cosh(bx)\cos^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx).$

74. $y''_{xx} + a\sinh^n(\lambda x)\cos(bx)y'_x + b[b + a\sinh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

75. $y''_{xx} + a\sinh(bx)\sin^n(kx)y'_x - b[b + a\cosh(bx)\sin^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

76. $y''_{xx} + a\sinh^n(\lambda x)\sin(bx)y'_x + b[b - a\sinh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

77. $y''_{xx} + a\sinh(bx)\tan^n(kx)y'_x - b[b + a\cosh(bx)\tan^n(kx)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

78. $y''_{xx} + a\sinh(bx)\cot^n(\lambda x)y'_x - b[b + a\cosh(bx)\cot^n(\lambda x)]y = 0.$

Particular solution: $y_0 = \sinh(bx)$.

79. $y''_{xx} + a\tanh^n(\lambda x)\cos(bx)y'_x + b[b + a\tanh^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

80. $y''_{xx} + a\tanh^n(\lambda x)\sin(bx)y'_x + b[b - a\tanh^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

81. $y''_{xx} + a\coth^n(\lambda x)\cos(bx)y'_x + b[b + a\coth^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

82. $y''_{xx} + a\coth^n(\lambda x)\sin(bx)y'_x + b[b - a\coth^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

83. $y''_{xx} + a\ln^n(\lambda x)\cos(bx)y'_x + b[b + a\ln^n(\lambda x)\sin(bx)]y = 0.$

Particular solution: $y_0 = \cos(bx)$.

84. $y''_{xx} + a\ln^n(\lambda x)\sin(bx)y'_x + b[b - a\ln^n(\lambda x)\cos(bx)]y = 0.$

Particular solution: $y_0 = \sin(bx)$.

85. $y''_{xx} + (a + be^{2\lambda x})\ln^n(kx)y'_x + \lambda[(a - be^{2\lambda x})\ln^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

86. $y''_{xx} + (a + be^{2\lambda x})\cos^n(kx)y'_x + \lambda[(a - be^{2\lambda x})\cos^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

87. $y''_{xx} + (a + be^{2\lambda x})\sin^n(kx)y'_x + \lambda[(a - be^{2\lambda x})\sin^n(kx) - \lambda]y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

88. $y''_{xx} + (a + be^{2\lambda x}) \tan^n(kx) y'_x + \lambda[(a - be^{2\lambda x}) \tan^n(kx) - \lambda] y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

89. $y''_{xx} + (a + be^{2\lambda x}) \cot^n(kx) y'_x + \lambda[(a - be^{2\lambda x}) \cot^n(kx) - \lambda] y = 0.$

Particular solution: $y_0 = be^{\lambda x} + ae^{-\lambda x}$.

90. $y''_{xx} + (a \operatorname{sn}^2 x + b) y = 0.$

The Lamé equation in the form of Jacobi, $\operatorname{sn} x$ is the Jacobi elliptic function. See the books by Bateman & Erdélyi (1955, vol. 3) and Kamke (1976) for more information on this equation.

91. $y''_{xx} + [A\wp(x) + B] y = 0.$

The Lamé equation in the form of Weierstrass, $\wp$ is the Weierstrass function. See the books by Bateman & Erdélyi (1955, vol. 3) and Kamke (1976) for more information on this equation.

92. $xy''_{xx} + (ax \ln x + be^{\lambda x}) y'_x + a(be^{\lambda x} \ln x + 1) y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

93. $xy''_{xx} + (1 - axe^{\lambda x} \ln x) y'_x + ae^{\lambda x} y = 0.$

Particular solution: $y_0 = \ln x$.

94. $xy''_{xx} + (ax \ln x + b \cosh^n x) y'_x + a(b \cosh^n x \ln x + 1) y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

95. $xy''_{xx} + (1 - ax \cosh^n x \ln x) y'_x + a \cosh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

96. $xy''_{xx} + (ax \ln x + b \sinh^n x) y'_x + a(b \sinh^n x \ln x + 1) y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

97. $xy''_{xx} + (1 - ax \sinh^n x \ln x) y'_x + a \sinh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

98. $xy''_{xx} + (ax \ln x + b \tanh^n x) y'_x + a(b \tanh^n x \ln x + 1) y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

99. $xy''_{xx} + (1 - ax \tanh^n x \ln x) y'_x + a \tanh^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

100. $xy''_{xx} + (ax \ln x + b \coth^n x) y'_x + a(b \coth^n x \ln x + 1) y = 0.$

Particular solution: $y_0 = e^{ax} x^{-ax}$.

101. $xy''_{xx} + (1 - ax\coth^n x \ln x)y'_x + a\coth^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

102. $xy''_{xx} + (ax\ln x + b\cos^n x)y'_x + a(b\cos^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

103. $xy''_{xx} + (1 - ax\cos^n x \ln x)y'_x + a\cos^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

104. $xy''_{xx} + (ax\ln x + b\sin^n x)y'_x + a(b\sin^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

105. $xy''_{xx} + (1 - ax\sin^n x \ln x)y'_x + a\sin^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

106. $xy''_{xx} + (ax\ln x + b\tan^n x)y'_x + a(b\tan^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

107. $xy''_{xx} + (1 - ax\tan^n x \ln x)y'_x + a\tan^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

108. $xy''_{xx} + (ax\ln x + b\cot^n x)y'_x + a(b\cot^n x \ln x + 1)y = 0.$

Particular solution: $y_0 = e^{ax}x^{-ax}$.

109. $xy''_{xx} + (1 - ax\cot^n x \ln x)y'_x + a\cot^n x\, y = 0.$

Particular solution: $y_0 = \ln x$.

110. $x^2y''_{xx} + x(a\ln x + be^{\lambda x})y'_x + a(be^{\lambda x}\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

111. $x^2y''_{xx} + x(a\ln x + b\cosh^n x)y'_x + a(b\cosh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

112. $x^2y''_{xx} + x(a\ln x + b\sinh^n x)y'_x + a(b\sinh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

113. $x^2y''_{xx} + x(a\ln x + b\tanh^n x)y'_x + a(b\tanh^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

114. $x^2y''_{xx} + x(a\ln x + b\coth^n x)y'_x + a(b\coth^n x \ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

115. $x^2 y''_{xx} + x(a\ln x + b\cos^n x)y'_x + a(b\cos^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

116. $x^2 y''_{xx} + x(a\ln x + b\sin^n x)y'_x + a(b\sin^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

117. $x^2 y''_{xx} + x(a\ln x + b\tan^n x)y'_x + a(b\tan^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

118. $x^2 y''_{xx} + x(a\ln x + b\cot^n x)y'_x + a(b\cot^n x\ln x - \ln x + 1)y = 0.$

Particular solution: $y_0 = \exp(-\tfrac{1}{2}a\ln^2 x)$.

119. $\sin^2 x\, y''_{xx} + \sin x\,(a + be^{\lambda x})y'_x + a(be^{\lambda x} - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

120. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\cosh^n x)y'_x + a(b\cosh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

121. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\sinh^n x)y'_x + a(b\sinh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

122. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\tanh^n x)y'_x + a(b\tanh^n x - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

123. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\coth^n x)y'_x + a(b\coth^n x - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

124. $\sin^2 x\, y''_{xx} + \sin x\,(a + b\ln^n x)y'_x + a(b\ln^n x - \cos x)y = 0.$

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

125. $\cos^2 x\, y''_{xx} + \cos x\,(a + be^{\lambda x})y'_x + a(be^{\lambda x} + \sin x)y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a$.

126. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\cosh^n x)y'_x + a(b\cosh^n x + \sin x)y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a$.

127. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\sinh^n x) y'_x + a(b\sinh^n x + \sin x) y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a.$

128. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\tanh^n x) y'_x + a(b\tanh^n x + \sin x) y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a.$

129. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\coth^n x) y'_x + a(b\coth^n x + \sin x) y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a.$

130. $\cos^2 x\, y''_{xx} + \cos x\,(a + b\ln^n x) y'_x + a(b\ln^n x + \sin x) y = 0.$

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a.$

2.1.9. Equations Containing Arbitrary Functions

Notation: $f = f(x)$ and $g = g(x)$ are arbitrary functions; a, b, c, d, n, m, k, λ, α, β, and γ are arbitrary parameters.

1. $y''_{xx} + ay = f.$

Solution:

$$y = \begin{cases} C_1\cos(kx) + C_2\sin(kx) + \dfrac{1}{k}\displaystyle\int_{x_0}^{x} f(\xi)\sin[k(x-\xi)]\,d\xi & \text{if } a = k^2 > 0, \\ C_1\cosh(kx) + C_2\sinh(kx) + \dfrac{1}{k}\displaystyle\int_{x_0}^{x} f(\xi)\sinh[k(x-\xi)]\,d\xi & \text{if } a = -k^2 < 0, \end{cases}$$

where x_0 is an arbitrary number.

2. $y''_{xx} + ay'_x + by = f.$

The substitution $y = w\exp(-\frac{1}{2}ax)$ leads to an equation of the form 2.1.9.1:

$$w''_{xx} + (b - \tfrac{1}{4}a^2)w = f\exp(\tfrac{1}{2}ax).$$

3. $y''_{xx} + fy'_x = g.$

Solution: $y = C_1 + \displaystyle\int e^{-F}\Big(C_2 + \int e^F g\,dx\Big)\,dx$, where $F = \displaystyle\int f\,dx.$

4. $y''_{xx} + xfy'_x - fy = 0.$

Particular solution: $y_0 = x.$

5. $y''_{xx} + fy'_x + a(f - a)y = 0.$

Particular solution: $y_0 = e^{-ax}.$

6. $y''_{xx} + fy'_x + a(x^n f - ax^{2n} + nx^{n-1})y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr).$

7. $y''_{xx} + (f + ax^n + b)y'_x + [(ax^n + b)f + anx^{n-1}]y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1} - bx\Bigr).$

8. $xy''_{xx} + xfy'_x - [(ax+1)f + a(ax+2)]y = 0.$

Particular solution: $y_0 = xe^{ax}.$

9. $xy''_{xx} + (xf + a)y'_x + (a-1)fy = 0.$

Particular solution: $y_0 = x^{1-a}.$

10. $xy''_{xx} + [(ax+1)f + ax - 1]y'_x + a^2 xfy = 0.$

Particular solution: $y_0 = (ax+1)e^{-ax}.$

11. $xy''_{xx} + [(ax^2 + bx)f + 2]y'_x + bfy = 0.$

Particular solution: $y_0 = a + \dfrac{b}{x}.$

12. $xy''_{xx} + (f + ax^{n+1})y'_x + ax^n(f + n)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{n+1}x^{n+1}\Bigr).$

13. $xy''_{xx} + (xf + ax^n)y'_x + [(ax^n - 1)f + anx^{n-1}]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(-\dfrac{a}{n}x^n\Bigr).$

14. $xy''_{xx} + [(ax^n + 1)f + anx^n + 1 - 2n]y'_x + a^2 nx^{2n-1}fy = 0.$

Particular solution: $y_0 = (ax^n + 1)\exp(-ax^n).$

15. $x^2 y''_{xx} + \alpha xy'_x + \beta y = f.$

The nonhomogeneous Euler equation.

The substitution $x = e^t$ leads to an equation of the form 2.1.9.2:

$$y''_{tt} + (\alpha - 1)y'_t + \beta y = f(e^t).$$

16. $x^2 y''_{xx} + xy'_x + (x^2 - \nu^2)y = f.$

The nonhomogeneous Bessel equation.

The general solution is expressed in terms of Bessel functions:

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x) + \frac{\pi}{2}Y_\nu(x)\int xJ_\nu(x)f(x)\,dx - \frac{\pi}{2}J_\nu(x)\int xY_\nu(x)f(x)\,dx.$$

17. $x^2 y''_{xx} + x f y'_x + a(f - a - 1)y = 0.$

Particular solution: $y_0 = x^{-a}.$

18. $x^2 y''_{xx} + x(f + 2a)y'_x + [(bx + a)f - b^2x^2 + a(a - 1)]y = 0.$

Particular solution: $y_0 = x^{-a}e^{-bx}.$

19. $x^2 y''_{xx} + x f y'_x + [(ax^{2n+1} + n)f - a^2x^{4n+2} - n^2 - n]y = 0.$

Particular solution: $y_0 = x^{-n}\exp\Bigl(-\dfrac{a}{2n+1}x^{2n+1}\Bigr).$

20. $(ax^2 + bx + c)y''_{xx} + (x + k)f y'_x - f y = 0.$

Particular solution: $y_0 = x + k.$

21. $x^4 y''_{xx} + x^2 f y'_x + [(\lambda - x)f - \lambda^2]y = 0.$

Particular solution: $y_0 = x\exp\Bigl(\dfrac{\lambda}{x}\Bigr).$

22. $x^2(ax^2 + b)y''_{xx} + x(ax^2 + b)f y'_x - [(ax^2 - b)f + 2b]y = 0.$

Particular solution: $y_0 = ax + \dfrac{b}{x}.$

23. $(x^2 + a)^2 y''_{xx} + (x^2 + a)f y'_x - (xf + a)y = 0.$

Particular solution: $y_0 = \sqrt{x^2 + a}.$

24. $(x^2 + a)^2 y''_{xx} + (x^2 + a)f y'_x - m[xf + (m - 1)x^2 + a]y = 0.$

Particular solution: $y_0 = (x^2 + a)^{m/2}.$

25. $(ax^n + b)y''_{xx} + (ax^n + b)f y'_x - anx^{n-2}(xf + n - 1)y = 0.$

Particular solution: $y_0 = ax^n + b.$

26. $(ax^n + bx)y''_{xx} + (ax^n + bx)f y'_x - [(anx^{n-1} + b)f + an(n - 1)x^{n-2}]y = 0.$

Particular solution: $y_0 = ax^n + bx.$

27. $(x^n + a)^2 y''_{xx} + (x^n + a)f y'_x - x^{n-2}(xf + an - a)y = 0.$

Particular solution: $y_0 = (x^n + a)^{1/n}.$

28. $(ax^n + b)^2 y''_{xx} + (ax^n + b)f y'_x + (f - anx^{n-1} - 1)y = 0.$

Particular solution: $y_0 = \exp\Bigl(-\displaystyle\int\frac{dx}{ax^n + b}\Bigr).$

29. $f(x)y''_{xx} + [ax^2 + (ac + b)x + bc]y'_x - (ax + b)y = 0.$

Particular solution: $y_0 = x + c.$

30. $\boldsymbol{y''_{xx} - (f^2 + f'_x)y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(\int f\,dx\Bigr)$.

31. $\boldsymbol{y''_{xx} + fy'_x - [a(a+1)f^2 + af'_x]y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(a\int f\,dx\Bigr)$.

32. $\boldsymbol{y''_{xx} + 2fy'_x + (f^2 + f'_x)y = 0}$.

Solution: $\quad y = (C_2x + C_1)\exp\Bigl(-\int f\,dx\Bigr)$.

33. $\boldsymbol{y''_{xx} + (1-a)fy'_x - a(f^2 + f'_x)y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(a\int f\,dx\Bigr)$.

34. $\boldsymbol{y''_{xx} + fy'_x + (fg - g^2 + g'_x)y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(-\int g\,dx\Bigr)$.

35. $\boldsymbol{y''_{xx} + 2fy'_x + (f^2 + f'_x + a)y = 0}$.

The substitution $u = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a constant coefficient linear equation: $u''_{xx} + au = 0$.

36. $\boldsymbol{y''_{xx} + 2fy'_x + (f^2 + f'_x + ax^{2n} + bx^{n-1})y = 0}$.

The substitution $w = y\exp\Bigl(\int f\,dx\Bigr)$ leads to an equation of the form 2.1.2.10: $w''_{xx} + a(x^{2n} + bx^{n-1})w = 0$.

37. $\boldsymbol{y''_{xx} + (2f + a)y'_x + (f^2 + af + f'_x + b)y = 0}$.

The substitution $u = y\exp\Bigl(\int f\,dx\Bigr)$ leads to a constant coefficient linear equation: $u''_{xx} + au'_x + bu = 0$.

38. $\boldsymbol{y''_{xx} + (f + g)y'_x + (fg + f'_x)y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(-\int f\,dx\Bigr)$.

39. $\boldsymbol{xy''_{xx} + xfy'_x + (f + xf'_x)y = 0}$.

Particular solution: $\quad y_0 = x\exp\Bigl(-\int f\,dx\Bigr)$.

40. $\boldsymbol{xy''_{xx} + (xf + a)y'_x + (af + xf'_x)y = 0}$.

Particular solution: $\quad y_0 = \exp\Bigl(-\int f\,dx\Bigr)$.

41. $\boldsymbol{(x+a)y''_{xx} + (f+b)y'_x + f'_x y = 0}$.

Particular solution: $\displaystyle y_0 = \exp\biggl(\int \frac{1-b-f}{x+a}\,dx\biggr)$.

42. $\boldsymbol{x^2 y''_{xx} + x(2f+1)y'_x + (f^2 + x f'_x + x^2 - a)y = 0}$.

The substitution $\displaystyle y = w\exp\Bigl(-\int \frac{f}{x}\,dx\Bigr)$ leads to the Bessel equation 2.1.2.121:

$$x^2 w''_{xx} + x w'_x + (x^2 - a)w = 0.$$

43. $\boldsymbol{x^2 y''_{xx} + x(2f+a)y'_x + [f^2 + (a-1)f + x f'_x + b x^n + c]y = 0}$.

The substitution $\displaystyle w = y\exp\biggl(\int \frac{f}{x}\,dx\biggr)$ leads to an equation of the form 2.1.2.127:

$$x^2 w''_{xx} + a x w'_x + (b x^n + c)w = 0.$$

44. $\boldsymbol{x^2 y''_{xx} + x(2f + a x^n + b)y'_x + [f^2 + (a x^n + b - 1)f + x f'_x + \alpha x^{2n} + \beta x^n + \gamma]y = 0}$.

The substitution $\displaystyle w = y\exp\biggl(\int \frac{f}{x}\,dx\biggr)$ leads to an equation of the form 2.1.2.141:

$$x^2 w''_{xx} + (a x^n + b)x w'_x + (\alpha x^{2n} + \beta x^n + \gamma)w = 0.$$

45. $\boldsymbol{2f y''_{xx} + f'_x y'_x + a y = 0}$.

The substitution $\displaystyle \xi = \int \frac{dx}{\sqrt{f}}$ leads to a constant coefficient equation: $2y''_{\xi\xi} + a y = 0$.

46. $\boldsymbol{f y''_{xx} - f'_x y'_x - a f^3 y = 0}$.

Solution: $y = C_1 e^u + C_2 e^{-u}$, where $\displaystyle u = \sqrt{a}\int f\,dx$.

47. $\boldsymbol{f y''_{xx} - (f'_x + a f^2)y'_x + b f^3 y = 0}$.

Solution:

$$y = C_1\exp\Bigl(\lambda_1\int f\,dx\Bigr) + C_2\exp\Bigl(\lambda_2\int f\,dx\Bigr),$$

where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 - a\lambda + b = 0$.

48. $\boldsymbol{f y''_{xx} - (f'_x + a f)y'_x - b f^2(a + b f)y = 0}$.

Particular solution: $\displaystyle y_0 = \exp\Bigl(-b\int f\,dx\Bigr)$.

49. $\boldsymbol{f y''_{xx} - (f'_x + 2a f)y'_x + (a f'_x + a^2 f - b^2 f^3)y = 0}$.

Particular solution: $\displaystyle y_0 = e^{ax}\exp\Bigl(b\int f\,dx\Bigr)$.

50. $\boldsymbol{f^2 y''_{xx} + f(f'_x + a)y'_x + by = 0}$.

The substitution $\xi = \displaystyle\int \frac{dx}{f}$ leads to a constant coefficient linear equation: $y''_{\xi\xi} + ay'_\xi + by = 0$.

51. $\boldsymbol{f^2 y''_{xx} + f(f'_x + 2g + a)y'_x + (fg'_x + g^2 + ag + b)y = 0}$.

The transformation $\xi = \displaystyle\int \frac{dx}{f}$, $u = y\exp\biggl(\displaystyle\int \frac{g}{f}\,dx\biggr)$ leads to a constant coefficient equation: $u''_{\xi\xi} + au'_\xi + bu = 0$.

52. $\boldsymbol{fgy''_{xx} - (af'_x g + bfg'_x)y'_x - \lambda f^{2a+1} g^{2b+1} y = 0}$.

Solution: $y = C_1 e^u + C_2 e^{-u}$, where $u = \sqrt{\lambda}\displaystyle\int f^a g^b\,dx$.

53. $\boldsymbol{fy''_{xx} - f''_{xx} y = 0}$.

Solution: $y = C_1 f + C_2 f \displaystyle\int f^{-2}\,dx$.

54. $\boldsymbol{4f^2 y''_{xx} - [2ff''_{xx} - (f'_x)^2 + a]y = 0}$.

1°. Solution with $a = 0$:

$$y = \sqrt{f}\,\Bigl(C_1 + C_2\int \frac{dx}{f}\Bigr).$$

2°. Solution with $a > 0$:

$$y = \sqrt{f}\,(C_1 e^{\varphi} + C_2 e^{-\varphi}), \qquad \text{where} \quad \varphi = \frac{\sqrt{a}}{2}\int \frac{dx}{f}.$$

3°. Solution with $a < 0$:

$$y = \sqrt{f}\,(C_1\cos\varphi + C_2\sin\varphi), \qquad \text{where} \quad \varphi = \frac{\sqrt{-a}}{2}\int \frac{dx}{f}.$$

55. $\boldsymbol{y''_{xx} - \dfrac{f''_{xx}}{f'_x} y'_x + a^2 (f'_x)^2 f^{2n-2} y = 0}$.

Solution:

$$y = \sqrt{f}\left[C_1 J_{\frac{1}{2n}}\Bigl(\frac{a}{n} f^n\Bigr) + C_2 Y_{\frac{1}{2n}}\Bigl(\frac{a}{n} f^n\Bigr)\right],$$

where J_m and Y_m are Bessel functions.

56. $\boldsymbol{y''_{xx} + \left(\dfrac{ff'_x}{f^2 + a} - \dfrac{f''_{xx}}{f'_x}\right) y'_x - \dfrac{b^2 (f'_x)^2}{f^2 + a} y = 0}$.

Solution: $y = C_1\bigl(f + \sqrt{f^2 + a}\,\bigr)^b + C_2\bigl(f + \sqrt{f^2 + a}\,\bigr)^{-b}$.

57. $\boldsymbol{y''_{xx} + fy'_x + ae^{\lambda x}(f - ae^{\lambda x} + \lambda)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-\dfrac{a}{\lambda} e^{\lambda x}\Bigr)$.

58. $\boldsymbol{y''_{xx} + (f + ae^{\lambda x})y'_x + ae^{\lambda x}(f + \lambda)y = 0.}$

Particular solution: $\quad y_0 = \exp\Bigl(-\dfrac{a}{\lambda}e^{\lambda x}\Bigr).$

59. $\boldsymbol{y''_{xx} + (a + be^{2\lambda x})fy'_x + \lambda[(a - be^{2\lambda x})f - \lambda]y = 0.}$

Particular solution: $\quad y_0 = be^{\lambda x} + ae^{-\lambda x}.$

60. $\boldsymbol{y''_{xx} + 2fy'_x + (f^2 + f'_x + ae^{2\lambda x} + be^{\lambda x} + c)y = 0.}$

The substitution $u = y\exp\Bigl(\displaystyle\int f\,dx\Bigr)$ leads to an equation of the form 2.1.3.5: $u''_{xx} + (ae^{2\lambda x} + be^{\lambda x} + c)u = 0.$

61. $\boldsymbol{y''_{xx} + f\sinh(ax)y'_x - a[a + f\cosh(ax)]y = 0.}$

Particular solution: $\quad y_0 = \sinh(ax).$

62. $\boldsymbol{y''_{xx} + f\cosh(ax)y'_x - a[a + f\sinh(ax)]y = 0.}$

Particular solution: $\quad y_0 = \cosh(ax).$

63. $\boldsymbol{y''_{xx} - f'_x y'_x + a^2e^{2f}y = 0.}$

Solution: $\quad y = C_1\sin\Bigl(a\displaystyle\int e^f\,dx\Bigr) + C_2\cos\Bigl(a\displaystyle\int e^f\,dx\Bigr).$

64. $\boldsymbol{y''_{xx} - f'_x y'_x - a^2e^{2f}y = 0.}$

Solution: $\quad y = C_1\exp\Bigl(a\textstyle\int e^f\,dx\Bigr) + C_2\exp\Bigl(-a\textstyle\int e^f\,dx\Bigr).$

65. $\boldsymbol{(ae^{\lambda x} + b)^2y''_{xx} + (ae^{\lambda x} + b)fy'_x + ce^{\lambda x}(f - ce^{\lambda x} + \lambda b)y = 0.}$

Particular solution: $\quad y_0 = \bigl(ae^{\lambda x} + b\bigr)^{-\frac{c}{a\lambda}}.$

66. $\boldsymbol{xy''_{xx} + (1 - fx\ln x)y'_x + fy = 0.}$

Particular solution: $\quad y_0 = \ln x.$

67. $\boldsymbol{xy''_{xx} + (f + ax\ln x)y'_x + a(f\ln x + 1)y = 0.}$

Particular solution: $\quad y_0 = e^{ax}x^{-ax}.$

68. $\boldsymbol{x^2y''_{xx} + 2x(\ln x + a)fy'_x + [\tfrac{1}{4} - (\ln x + a + 2)f]y = 0.}$

Particular solution: $\quad y_0 = \sqrt{x}\,(\ln x + a).$

69. $\boldsymbol{x^2y''_{xx} + x(f + a\ln x)y'_x + a(f\ln x - \ln x - 1)y = 0.}$

Particular solution: $\quad y_0 = \exp(-\tfrac{1}{2}a\ln^2 x).$

70. $\boldsymbol{y''_{xx} + f\sin(ax)y'_x + a[a - f\cos(ax)]y = 0.}$

Particular solution: $\quad y_0 = \sin(ax).$

71. $\boldsymbol{y''_{xx} + f\cos(ax)y'_x + a[a + f\sin(ax)]y = 0}$.

Particular solution: $y_0 = \cos(ax)$.

72. $\boldsymbol{y''_{xx} + fy'_x + a[\lambda + f\tan(\lambda x) + (\lambda - a)\tan^2(\lambda x)]y = 0}$.

Particular solution: $y_0 = [\cos(\lambda x)]^{a/\lambda}$.

73. $\boldsymbol{y''_{xx} + fy'_x + a[\lambda - f\cot(\lambda x) + (\lambda - a)\cot^2(\lambda x)]y = 0}$.

Particular solution: $y_0 = [\sin(\lambda x)]^{a/\lambda}$.

74. $\boldsymbol{y''_{xx} + (f + a\sin x)y'_x + a(f\sin x + \cos x)y = 0}$.

Particular solution: $y_0 = \exp(a\cos x)$.

75. $\boldsymbol{y''_{xx} + (f + a\cos x)y'_x + a(f\cos x - \sin x)y = 0}$.

Particular solution: $y_0 = \exp(-a\sin x)$.

76. $\boldsymbol{y''_{xx} + (f + a\tan x)y'_x + (a + 1)(f\tan x + 1)y = 0}$.

Particular solution: $y_0 = \cos^{a+1} x$.

77. $\boldsymbol{y''_{xx} + (f + a\cot x)y'_x + (a - 1)(f\cot x - 1)y = 0}$.

Particular solution: $y_0 = \sin^{1-a} x$.

78. $\boldsymbol{y''_{xx} + \tan x\,(f + a - 1)y'_x + [(a\tan^2 x - 1)f + 2a + 2]y = 0}$.

Particular solution: $y_0 = \sin x\cos^a x$.

79. $\boldsymbol{y''_{xx} + (f + a\cos^n x)y'_x + a\cos^{n-1} x\,(f\cos x - n\sin x)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-a\displaystyle\int \cos^n x\,dx\Bigr)$.

80. $\boldsymbol{y''_{xx} + (f + a\sin^n x)y'_x + a\sin^{n-1} x\,(f\sin x + n\cos x)y = 0}$.

Particular solution: $y_0 = \exp\Bigl(-a\displaystyle\int \sin^n x\,dx\Bigr)$.

81. $\boldsymbol{\sin^2 x\,y''_{xx} + \sin x\,(f + a)y'_x + a(f - \cos x)y = 0}$.

Particular solution: $y_0 = \left(\cot\dfrac{x}{2}\right)^a$.

82. $\boldsymbol{\cos^2 x\,y''_{xx} + \cos x\,(a + f)y'_x + a(f + \sin x)y = 0}$.

Particular solution: $y_0 = \left[\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right]^a$.

2.1.10. Some Transformations

Notation: f, g, and h are arbitrary composite functions of their argument which is written in paretheses following the name of a function (the argument is a function of x).

1. $\boldsymbol{y''_{xx} + x^{-4} f\Big(\frac{1}{x}\Big) y = 0.}$

The transformation $\xi = \dfrac{1}{x}$, $w = \dfrac{y}{x}$ leads to the equation $w''_{\xi\xi} + f(\xi)w = 0$.

2. $\boldsymbol{y''_{xx} + (cx+d)^{-4} f\Big(\frac{ax+b}{cx+d}\Big) y = 0.}$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to a simpler equation: $w''_{\xi\xi} + \Delta^{-2} f(\xi)w = 0$, where $\Delta = ad - bc$.

3. $\boldsymbol{x^2 y''_{xx} + \Big[\frac{1-n^2}{4} + x^{2n} f(ax^n + b)\Big] y = 0.}$

The transformation $\xi = ax^n + b$, $w = yx^{\frac{n-1}{2}}$ leads to a simpler equation: $w''_{\xi\xi} + (an)^{-2} f(\xi)w = 0$.

4. $\boldsymbol{x^2 y''_{xx} + x(xf + a)y'_x + (xg + b)y = 0, \qquad f = f(x), g = g(x).}$

The substitution $y = x^k w$, where k is a root of the quadratic equation $k^2 + (a-1)k + b = 0$, leads to the equation $xw''_{xx} + (xf + a + 2k)w'_x + (g + kf)w = 0$.

5. $\boldsymbol{xP_n(x) y''_{xx} + Q_n(x) y'_x + R_{n-1}(x) y = 0,}$

where $P_n(x) = \sum_{m=0}^{n} a_m x^m$, $Q_n(x) = \sum_{m=0}^{n} b_m x^m$, $R_{n-1}(x) = \sum_{m=0}^{n-1} c_m x^m$.

The substitution $y = x^k w$, where $k = 1 - \dfrac{b_0}{a_0}$, leads to an equation of the similar form:

$$xP_n(x) w''_{xx} + [Q_n(x) + 2kP_n(x)]w'_x + [R_{n-1}(x) + F_{n-1}(x)]w = 0,$$

where $F_{n-1}(x) = \dfrac{k}{x}[Q_n(x) + (k-1)P_n(x)]$.

6. $\boldsymbol{x(x-1)P_{n-1}(x) y''_{xx} + Q_n(x) y'_x + R_{n-1}(x) y = 0,}$

where $P_{n-1}(x) = \sum_{m=0}^{n-1} a_m x^m$, $Q_n(x) = \sum_{m=0}^{n} b_m x^m$, $R_{n-1}(x) = \sum_{m=0}^{n-1} c_m x^m$.

The transformation $\xi = \dfrac{x}{x-1}$, $w = |x-1|^{-k} y$, where k is a root of the quadratic equation $a_{n-1}k^2 + (b_n - a_{n-1})k + c_{n-1} = 0$, leads to an equation of the similar form:

$$\xi(\xi-1)\widehat{P}_{n-1}(\xi) w''_{\xi\xi} + [2(1-k)\xi\widehat{P}_{n-1}(\xi) - \widehat{Q}_n(\xi)]w'_\xi + [k(k-1)\widehat{P}_{n-1}(\xi) + F_{n-1}(\xi)]w = 0,$$

where

$$\widehat{P}_{n-1}(\xi) = \sum_{m=0}^{n-1} a_m \xi^m (\xi-1)^{n-m-1}, \quad \widehat{Q}_n(\xi) = \sum_{m=0}^{n} b_m \xi^m (\xi-1)^{n-m},$$

$$\widehat{R}_{n-1}(\xi) = \sum_{m=0}^{n-1} c_m \xi^m (\xi-1)^{n-m-1}, \quad F_{n-1}(\xi) = \frac{\widehat{R}_{n-1}(\xi) + k\widehat{Q}_n(\xi) + k(k-1)\widehat{P}_{n-1}(\xi)}{\xi - 1}.$$

7. $\boldsymbol{y''_{xx} + \left[e^{2\lambda x} f(ae^{\lambda x} + b) - \frac{1}{4}\lambda^2\right] y = 0.}$

The transformation $\xi = ae^{\lambda x} + b$, $w = ye^{\lambda x/2}$ leads to a simpler equation: $w''_{\xi\xi} + (a\lambda)^{-2} f(\xi) w = 0$.

8. $\boldsymbol{y''_{xx} + f(e^{\lambda x}) y'_x + g(e^{\lambda x}) y = 0.}$

The substitution $z = e^{\lambda x}$ leads to the equation $\lambda^2 z^2 y''_{zz} + \lambda z[f(z) + \lambda] y'_z + g(z) y = 0$.

9. $\boldsymbol{y''_{xx} + [-\lambda^2 + \sinh^{-4}(\lambda x) f(\coth(\lambda x))] y = 0.}$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi) w = 0$.

10. $\boldsymbol{y''_{xx} + \bigl[-\lambda^2 + \cosh^{-4}(\lambda x) f\bigl(\tanh(\lambda x)\bigr)\bigr] y = 0.}$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi) w = 0$.

11. $\boldsymbol{y''_{xx} + \left[-\frac{1}{4}\lambda^2 + \frac{e^{2\lambda x}}{(ce^{\lambda x} + d)^4} f\left(\frac{ae^{\lambda x} + b}{ce^{\lambda x} + d}\right)\right] y = 0.}$

The transformation $\xi = \dfrac{ae^{\lambda x} + b}{ce^{\lambda x} + d}$, $w = \dfrac{ye^{\lambda x/2}}{ce^{\lambda x} + d}$ leads to a simpler equation: $w''_{\xi\xi} + (\Delta\lambda)^{-2} f(\xi) w = 0$, where $\Delta = ad - bc$.

12. $\boldsymbol{fy''_{xx} + (2f \tanh x + g) y'_x + (g \tanh x + h) y = 0, \quad f = f(x),\ g = g(x),\ h = h(x).}$

The substitution $u = y \cosh x$ leads to the simpler equation: $fu''_{xx} + gu'_x + (h - f)u = 0$.

13. $\boldsymbol{fy''_{xx} + (2f \coth x + g) y'_x + (g \coth x + h) y = 0, \quad f = f(x),\ g = g(x),\ h = h(x).}$

The substitution $u = y \sinh x$ leads to a simpler equation $fu''_{xx} + gu'_x + (h - f)u = 0$.

14. $\boldsymbol{x^2 y''_{xx} + [f(a \ln x + b) + \tfrac{1}{4}] y = 0.}$

The transformation $\xi = a \ln x + b$, $w = yx^{-1/2}$ leads to a simpler equation $w''_{\xi\xi} + a^{-2} f(\xi) w = 0$.

15. $\boldsymbol{(x^2 - 1)^2 y''_{xx} + f\left(\ln \frac{ax - a}{x + 1}\right) y = 0.}$

The transformation $\xi = \ln \dfrac{ax - a}{x + 1}$, $w = \dfrac{y}{\sqrt{|x^2 - 1|}}$ leads to a simpler equation: $4w''_{\xi\xi} + [f(\xi) - 1] w = 0$.

16. $\boldsymbol{x^2 f(\ln x) y''_{xx} + xg(\ln x) y'_x + h(\ln x) y = 0.}$

The substitution $\xi = \ln x$ leads to the equation $f(\xi) y''_{\xi\xi} + [g(\xi) - f(\xi)] y'_\xi + h(\xi) y = 0$.

17. $y''_{xx} + [\lambda^2 + \sin^{-4}(\lambda x) f(\cot(\lambda x))]y = 0.$

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi) w = 0$.

18. $y''_{xx} + [\lambda^2 + \cos^{-4}(\lambda x) f(\tan(\lambda x))]y = 0.$

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to a simpler equation: $w''_{\xi\xi} + \lambda^{-2} f(\xi) w = 0$.

19. $y''_{xx} + \left[\lambda^2 + \dfrac{1}{\sin^4(\lambda x + b)} f\left(\dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}\right)\right] y = 0.$

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{y}{\sin(\lambda x + b)}$ leads to a simpler equation: $w''_{\xi\xi} + [\lambda \sin(b - a)]^{-2} f(\xi) w = 0$.

20. $f y''_{xx} + (g - 2f \tan x) y'_x + (h - g \tan x) y = 0, \qquad f = f(x),\ g = g(x),\ h = h(x).$

The substitution $u = y \cos x$ leads to a simpler equation: $f u''_{xx} + g u'_x + (f + h) u = 0$.

21. $f y''_{xx} + (g + 2f \cot x) y'_x + (h + g \cot x) y = 0, \qquad f = f(x),\ g = g(x),\ h = h(x).$

The substitution $u = y \sin x$ leads to a simpler equation: $f u''_{xx} + g u'_x + (f + h) u = 0$.

22. $(x^2 + 1)^2 y''_{xx} + f(\arctan x + b) y = 0.$

The transformation $\xi = \arctan x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to a simpler equation: $w''_{\xi\xi} + [f(\xi) + 1] w = 0$.

23. $(x^2 + 1)^2 y''_{xx} + f(\operatorname{arccot} x + b) y = 0.$

The transformation $\xi = \operatorname{arccot} x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to a simpler equation: $w''_{\xi\xi} + [f(\xi) + 1] w = 0$.

24. $y''_{xx} + f(x) y = 0.$

The transformation $x = \varphi(\xi)$, $y = w\sqrt{|\varphi'_\xi|}$ leads to an equation of the similar form

$$w''_{\xi\xi} + \left[\frac{1}{2}\frac{\varphi'''_{\xi\xi\xi}}{\varphi'_\xi} - \frac{3}{4}\left(\frac{\varphi''_{\xi\xi}}{\varphi'_\xi}\right)^2 + (\varphi'_\xi)^2 f(\varphi)\right] w = 0.$$

2.1.11. Asymptotic Solutions

This subsection presents asymptotic solutions, as $\varepsilon \to 0$ ($\varepsilon > 0$), of some second-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being a real number.

1. Consider the equation

$$\varepsilon^2 y''_{xx} - f(x) y = 0 \qquad (1)$$

on a closed interval $a \le x \le b$.

Case 1. With the condition $f \neq 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by the formulae

$$
\begin{aligned}
&y_1 = f^{-1/4} \exp\biggl(-\frac{1}{\varepsilon}\int \sqrt{f}\,dx\biggr), &&y_2 = f^{-1/4} \exp\biggl(\frac{1}{\varepsilon}\int \sqrt{f}\,dx\biggr) &&\text{if } f > 0,\\
&y_1 = (-f)^{-1/4} \cos\biggl(\frac{1}{\varepsilon}\int \sqrt{-f}\,dx\biggr), &&y_2 = (-f)^{-1/4} \sin\biggl(\frac{1}{\varepsilon}\int \sqrt{-f}\,dx\biggr) &&\text{if } f < 0.
\end{aligned}
$$

Case 2. Discuss the asymptotic solution of equation (1) in the vicinity of the point $x = x_0$ where function $f(x)$ vanishes (such a point is referred to as a transition point). We assume that function f can be presented in the form

$$
f(x) = (x_0 - x)\psi(x), \qquad \text{where} \quad \psi(x) > 0.
$$

In this case, the asymptotic solution, as $\varepsilon \to 0$ ($\varepsilon > 0$), is described by three different formulae:

with $x > x_0$,

$$
y = [-f(x)]^{1/4}\biggl\{C_1 \cos\biggl[\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{-f(x)}\,dx + \frac{\pi}{4}\biggr] + C_2 \sin\biggl[\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{-f(x)}\,dx + \frac{\pi}{4}\biggr]\biggr\};
$$

with $x < x_0$,

$$
y = [f(x)]^{1/4}\biggl\{C_1 \exp\biggl[\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{f(x)}\,dx\biggr] + C_2 \exp\biggl[-\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{f(x)}\,dx\biggr]\biggr\};
$$

in the vicinity of the transition point $x = x_0$,

$$
y = \widetilde{C}_1 \operatorname{Bi}(z) + \widetilde{C}_2 \operatorname{Ai}(z), \qquad z = \varepsilon^{-2/3}[\psi(x_0)]^{1/3}(x_0 - x),
$$

where $\operatorname{Ai}(z)$ and $\operatorname{Bi}(z)$ are the Airy function of the first and second kind, respectively (see equation 2.1.2.2).

Constants C_1, C_2 and $\widetilde{C}_1$, $\widetilde{C}_2$ in the above asymptotic solutions are related by

$$
{}_1 = \frac{1}{\sqrt{\pi}}[\varepsilon\psi(x_0)]^{1/6}\widetilde{C}_1, \quad {}_2 = \frac{1}{\sqrt{\pi}}[\varepsilon\psi(x_0)]^{1/6}\widetilde{C}_2.
$$

2. The two-term asymptotic expansions of the solution of equation (1) with $f > 0$, as $\varepsilon \to 0$, on a closed interval $a \le x \le b$ has the form

$$
\begin{aligned}
y_1 &= f^{-1/4} \exp\biggl(-\frac{1}{\varepsilon}\int \sqrt{f}\,dx\biggr)\biggl\{1 - \varepsilon\int\biggl[\frac{1}{8}\frac{f''_{xx}}{f^{3/2}} - \frac{5}{32}\frac{(f'_x)^2}{f^{5/2}}\biggr]dx + O(\varepsilon^2)\biggr\},\\
y_2 &= f^{-1/4} \exp\biggl(\frac{1}{\varepsilon}\int \sqrt{f}\,dx\biggr)\biggl\{1 + \varepsilon\int\biggl[\frac{1}{8}\frac{f''_{xx}}{f^{3/2}} - \frac{5}{32}\frac{(f'_x)^2}{f^{5/2}}\biggr]dx + O(\varepsilon^2)\biggr\}.
\end{aligned}
\tag{2}
$$

where x_0 is an arbitrary number satisfying the inequality $1 \le x_0 \le b$.

The asymptotic expansions of the fundamental system of solutions of equation (1) with $f<0$ are derived by separating the real and imaginary parts in either formula (2).

3. Consider the equation

$$\varepsilon^2 y''_{xx} - x^{m-2} f(x) y = 0 \tag{3}$$

on a closed interval $a \le x \le b$, where $a<0$ and $b>0$, under the condition that m is a positive integer and $f(x) \ne 0$. In this case, the leading term of the asymptotic solution, as $\varepsilon \to 0$, in the vicinity of the point $x=0$ is expressed in terms of a simpler model equation which results from substituting function $f(x)$ in equation (3) by constant $f(x_0)$ (the solution of the model equation is expressed in terms of the Bessel functions of the order $1/m$, see equation 2.1.2.7).

We specify below the formulae by which the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (3) with $a<x<0$ and $0<x<b$ are related (excluding a small vicinity of the point $x=x_0$). Three different cases can be extracted:

1°. Let m be an even integer, and $f(x)>0$. Then,

$$y_1 = \begin{cases} [f(x)]^{-1/4} \exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x<0, \\ k^{-1}[f(x)]^{-1/4} \exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x>0; \end{cases}$$

$$y_2 = \begin{cases} [f(x)]^{-1/4} \exp\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x<0, \\ k[f(x)]^{-1/4} \exp\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x>0, \end{cases}$$

where $f=f(x)$, $k=\sin\left(\dfrac{\pi}{m}\right)$.

2°. Let m be an even integer, and $f(x)<0$. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\right] & \text{if } x<0, \\ k^{-1}|f(x)|^{-1/4} \cos\left[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\right] & \text{if } x>0; \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4} \cos\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\right] & \text{if } x<0, \\ k|f(x)|^{-1/4} \cos\left[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\right] & \text{if } x>0, \end{cases}$$

where $f=f(x)$, $k=\tan\left(\dfrac{\pi}{2m}\right)$.

3°. Let m be an odd integer. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\right] & \text{if } x<0, \\ \dfrac{1}{2}k^{-1}[f(x)]^{-1/4} \exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x>0; \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4} \cos\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\right] & \text{if } x < 0, \\ k[f(x)]^{-1/4} \exp\left[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\right] & \text{if } x > 0, \end{cases}$$

where $f = f(x)$, $k = \sin\left(\dfrac{\pi}{2m}\right)$.

4. Consider an equation of the form

$$\varepsilon^2 y''_{xx} - f(x,\varepsilon)y = 0 \tag{4}$$

on a closed interval $a \le x \le b$ under the condition that $f \neq 0$. Assume that the asymptotic relation holds

$$f(x,\varepsilon) = \sum_{k=0}^{\infty} f_k(x)\varepsilon^k, \qquad \varepsilon \to 0.$$

Then, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4) are given by the formulae

$$y_1 = f_0^{-1/4}(x) \exp\left[-\frac{1}{\varepsilon}\int \sqrt{f_0(x)}\,dx + \frac{1}{2}\int \frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\right]\bigl[1 + O(\varepsilon)\bigr],$$
$$y_2 = f_0^{-1/4}(x) \exp\left[\frac{1}{\varepsilon}\int \sqrt{f_0(x)}\,dx + \frac{1}{2}\int \frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\right]\bigl[1 + O(\varepsilon)\bigr].$$

5. Consider an equation of the form

$$\varepsilon y''_{xx} + g(x)y'_x + f(x)y = 0$$

on a closed interval $0 \le x \le 1$. With $g(x) > 0$, the asymptotic solution of this equation, satisfying the boundary conditions $y(0) = C_1$ and $y(1) = C_2$, can be presented in the following form:

$$y = (C_1 - kC_2)\exp\bigl[-\varepsilon^{-1}g(0)x\bigr] + C_2 \exp\left[\int_x^1 \frac{f(x)}{g(x)}\,dx\right] + O(\varepsilon),$$

where $k = \exp\left[\displaystyle\int_0^1 \frac{f(x)}{g(x)}\,dx\right]$.

6. Now let us take a look at an equation of the form

$$\varepsilon^2 y''_{xx} + \varepsilon g(x)y'_x + f(x)y = 0 \tag{5}$$

on a closed interval $a \le x \le b$. Assume

$$D(x) \equiv [g(x)]^2 - 4f(x) \neq 0.$$

Then, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (5), as $\varepsilon \to 0$, are described by the formulae

$$y_1 = |D(x)|^{-1/4} \exp\left[-\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\right]\bigl[1 + O(\varepsilon)\bigr],$$
$$y_2 = |D(x)|^{-1/4} \exp\left[\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\right]\bigl[1 + O(\varepsilon)\bigr].$$

7. The more general equation

$$\varepsilon^2 y''_{xx} + \varepsilon g(x,\varepsilon) y'_x + f(x,\varepsilon) y = 0. \tag{6}$$

is reducible, with the aid of the substitution $y = w \exp\Bigl(-\dfrac{1}{2\varepsilon}\displaystyle\int g\,dx\Bigr)$, to an equation of the form (3):

$$\varepsilon^2 w''_{xx} + \bigl(f - \tfrac{1}{4}g^2 - \tfrac{1}{2}\varepsilon g'_x\bigr) w = 0,$$

to which the asymptotic formulae given above in Paragraph 4 are applicable.

2.1.12. Series Solutions

Let us consider a homogeneous linear differential equation of the general form

$$y''_{xx} + f(x) y'_x + g(x) y = 0. \tag{1}$$

Case 1. Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of the point $x = x_0$, in the form

$$f(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=0}^{\infty} B_n (x - x_0)^n, \tag{2}$$

in the region $|x - x_0| < R$, where R stands for the minimum radius of convergence of the two series in (2). In this case, the point $x = x_0$ is referred to as an ordinary point, and equation (1) posesses two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n. \tag{3}$$

Coefficients a_n and b_n are determined by substituting the series (2) into equation (1) followed by extracting the coefficients with respect to identical powers of $(x - x_0)$.

Case 2. Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of the point $x = x_0$, in the form

$$f(x) = \sum_{n=-1}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=-2}^{\infty} B_n (x - x_0)^n, \tag{4}$$

in the region $|x - x_0| < R$. In this case, the point $x = x_0$ is referred to as a regular singular point. Let λ_1 and λ_2 be the roots of the quadratic equation

$$\lambda_1^2 + (A_{-1} - 1)\lambda + B_{-2} = 0.$$

There are three cases, depending on the values of the exponents of the singularity.

1) If $\lambda_1 \neq \lambda_2$ and $\lambda_1 - \lambda_2$ is not equal to an integer, equation (1) has two linearly independent solutions

$$\begin{aligned} y_1(x) &= |x - x_0|^{\lambda_1} \Bigl[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\Bigr], \\ y_2(x) &= |x - x_0|^{\lambda_2} \Bigl[1 + \sum_{n=1}^{\infty} b_n (x - x_0)^n\Bigr]. \end{aligned} \tag{5}$$

It is clear that in the general case, coefficients a_n and b_n of the series (4) and (5) will be different.

2) If $\lambda_1 = \lambda_2 = \lambda$, equation (1) possesses two linearly independent solutions in the forms

$$y_1(x) = |x - x_0|^{\lambda} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\right],$$

$$y_2(x) = y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda} \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

3) If $\lambda_1 = \lambda_2 + N$, where N is an integer greater than 0, equation (1) has two linearly independent solutions in the forms

$$y_1(x) = |x - x_0|^{\lambda_1} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\right],$$

$$y_2(x) = k y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

where k may be equal to zero.

To construct the solution in each of the three cases, the following procedure is to be done: Substitute the above expressions for y_1 and y_2 into the original equation (1) and equate the coefficients of $(x - x_0)^n$ and $(x - x_0)^n \ln|x - x_0|$ for different values of n to obtain recurrence relations for the unknown coefficients. From these recurrence relations the solution sought can be found.

2.2. Autonomous Equations $y''_{xx} = F(y, y'_x)$

Preliminary Comments. Equations of this form are often encountered in different areas of mechanics, applied mathematics, physics, and chemical engineering science.

1. The substitution $w(y) = y'_x$ leads to the first order equation

$$w'_y = w^{-1} F(y,\ w) \tag{1}$$

(see Chapter 1).

2. The solution of the original autonomous equation can be represented in the implicit form

$$x = \int \frac{dy}{w(y,\ C_1)} + C_2, \tag{2}$$

where $w = w(y,\ C_1)$ is the solution of the first order equation (1).

3. The solution of the original autonomous equation can be written in the parametric form

$$x = \int \frac{y'_\tau(\tau,\ C_1)}{w(\tau,\ C_1)}\, d\tau + C_2, \qquad y = y(\tau,\ C_1), \tag{3}$$

where $y = y(\tau,\ C_1)$, $w = w(\tau,\ C_1)$ is the parametric form of the solution of the first order equation (1).

Formula (2) is a special case of formula (3) with $y = \tau$.

2.2.1. Equations of the Form $y''_{xx} - y'_x = f(y)$

Preliminary Comments. Equations of this form are encountered in the theory of combustion and the theory of chemical reactors.

1. The substitution $w(y) = y'_x$ leads to the Abel equation $ww'_y - w = f(y)$ which is considered in Subsection 1.3.1 for some concrete functions f.

2. The solution of the original autonomous equation can be written in the parametric form (3), where $y = y(\tau,\ C_1)$, $w = w(\tau,\ C_1)$ is the parametric form of the solution to the Abel equation of the second kind $ww'_y - w = f(y)$.

1. $\quad \boldsymbol{y''_{xx} - y'_x = -\dfrac{2(m+1)}{(m+3)^2} y \pm \dfrac{m+1}{2a^2} y^m, \qquad m \neq \pm 1, \quad m \neq -3.}$

Solution in the parametric form:

$$x = \frac{m+3}{m-1} \ln\left[aC_1^{1-m}\frac{m-1}{m+3}\int\frac{d\tau}{\sqrt{1\pm\tau^{m+1}}} + C_2\right],$$
$$y = C_1^2\tau\left[aC_1^{1-m}\frac{m-1}{m+3}\int\frac{d\tau}{\sqrt{1\pm\tau^{m+1}}} + C_2\right]^{\frac{2}{m-1}}.$$

2. $\quad \boldsymbol{y''_{xx} - y'_x = \pm 2a^2 y^{-1}.}$

Solution in the parametric form:

$$x = -\ln\left[C_1\int\exp(\pm\tau^2)\,d\tau + C_2\right], \quad y = aC_1\exp(\pm\tau^2)\left[C_1\int\exp(\pm\tau^2)\,d\tau + C_2\right]^{-1}.$$

3. $\quad \boldsymbol{y''_{xx} - y'_x = -\tfrac{2}{9} y + \tfrac{16}{9} a^{3/2} y^{-1/2}.}$

Solution in the parametric form:

$$x = -3\ln\bigl\{C_1\exp(-\tau)\bigl[\exp(3\tau) + C_2\sin\bigl(\sqrt{3}\,\tau\bigr)\bigr]\bigr\},$$
$$y = a\exp(2\tau)\frac{\bigl[2\exp(3\tau) - C_2\sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2\cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}{\bigl[\exp(3\tau) + C_2\sin\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}.$$

4. $\quad \boldsymbol{y''_{xx} - y'_x = -\tfrac{9}{100} y \pm \tfrac{9}{100} a^{8/3} y^{-5/3}.}$

Solution in the parametric form:

$$x = -\tfrac{5}{4}\ln\bigl[\pm(\tau^4 - 6\tau^2 + 4C_1\tau - 3)\bigr] + C_2,$$
$$y = a(\tau^3 - 3\tau + C_1)^{3/2}\bigl[\pm(\tau^4 - 6\tau^2 + 4C_1\tau - 3)\bigr]^{-9/8}.$$

5. $\quad \boldsymbol{y''_{xx} - y'_x = -\tfrac{3}{16} y - \tfrac{3}{64} a^{8/3} y^{-5/3}.}$

Solution in the parametric form:

$$x = C_1 - 2\ln[\sin\tau\cosh(\tau+C_2) + \cos\tau\sinh(\tau+C_2)], \qquad y = a[\tan\tau + \tanh(\tau+C_2)]^{-3/2}.$$

► *In the solutions of equations 6–9, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \text{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \text{for the lower sign,} \end{cases}$$

where J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

6. $\boldsymbol{y''_{xx} - y'_x = Ay^{-1/2}}$.

Solution in the parametric form:

$$x = -2\int \tau^{-1} Z^{-1}(\tau Z'_\tau + \tfrac{1}{3}Z)\,d\tau + C_2, \qquad y = a\tau^{-4/3} Z^{-2}[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2],$$

where $\nu = \frac{1}{3}$, $A = \mp\frac{1}{3}a^{3/2}$.

7. $\boldsymbol{y''_{xx} - y'_x = Ay^{-2}}$.

Solution in the parametric form:

$$x = \mp\tfrac{2}{3}\int \tau Z^2 \big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-1}\,d\tau + C_2,$$
$$y = 2a\tau^{4/3} Z^2 \big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big]^{-1},$$

where $\nu = \frac{1}{3}$, $A = -36a^3$.

8. $\boldsymbol{y''_{xx} - y'_x = 2A^2 - Ay^{1/2}}$.

Solution in the parametric form:

$$x = \pm 2\int \tau^{-1}(Z'_\tau)^{-1}(\tau Z \pm 2Z'_\tau)\,d\tau + C_2, \quad y = a(Z'_\tau)^{-2}(\tau Z \pm 2Z'_\tau)^2,$$

where $\nu = 0$, $A = a^{1/2}$.

9. $\boldsymbol{y''_{xx} - y'_x = Ay^{-1/2} + 2B^2 + By^{1/2}}$.

Solution in the parametric form:

$$x = -2\int \tau^{-1} Z^{-1}(\tau Z'_\tau - Z)\,d\tau + C_2, \quad y = B^2 Z^{-2}(\tau Z'_\tau - Z)^2,$$

where $A = (1 - \nu^2)B^3$.

► *In the solutions of equations 10–14, the following notation is used: the function $\wp = \wp(\tau)$ is defined in the implicit form*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_1.$$

The upper sign in this formula corresponds to the classical elliptic Weirstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$.

10. $\boldsymbol{y''_{xx} - y'_x = Ay^2 - \frac{9}{625}A^{-1}}$.

Solution in the parametric form:

$$x = 5\ln\tau + C_2, \qquad y = 5a(\tau^2\wp \mp \tfrac{1}{2}), \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

11. $\boldsymbol{y''_{xx} - y'_x = Ay^2 - \frac{6}{25}y}$.

Solution in the parametric form:

$$x = 5\ln\tau + C_2, \qquad y = 5a\tau^2\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

12. $\boldsymbol{y''_{xx} - y'_x = Ay^2 + \frac{6}{25}y}$.

Solution in the parametric form:

$$x = 5\ln\tau + C_2, \qquad y = 5a(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{125}a^{-1}.$$

13. $\boldsymbol{y''_{xx} - y'_x = 12y + Ay^{-5/2}}$.

Solution in the parametric form:

$$x = \mp\tfrac{2}{7}\int \wp^{-1}\bigl(f \pm 2\tau\wp^2\bigr)^{-1}\,d\tau + C_2, \quad y = a\wp^{-6/7}\bigl(f \pm 2\tau\wp^2\bigr)^{-4/7},$$

where $f = \sqrt{\pm(4\wp^3 - 1)}$, $A = \mp 147a^{7/2}$.

14. $\boldsymbol{y''_{xx} - y'_x = \frac{63}{4}y + Ay^{-5/3}}$.

Solution in the parametric form:

$$x = -\tfrac{3}{4}\int (f \pm 2\tau\wp^2)(\tau f + 2\wp)^{-1}\,d\tau + C_2, \quad y = 2a\bigl(f \pm 2\tau\wp^2\bigr)^{3/2}(\tau f + 2\wp)^{-9/8},$$

where $f = \sqrt{\pm(4\wp^3 - 1)}$, $A = -\frac{128}{3}a^2(2a)^{2/3}$.

► *In the solutions of equations 15–18, the following notation is used:*

$$I = \int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_1$$

is the incomplete elliptic integral of the second kind in the Weierstrass form,

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad I_1 = 2\tau I \mp R, \quad I_2 = \tau^{-1}(2\tau RI \mp R^2 - 1).$$

15. $\boldsymbol{y''_{xx} - y'_x = Ay^{1/2} - \frac{12}{49}y}$.

Solution in the parametric form:

$$x = -7\int \tau R^{-1}I^{-1}\,d\tau + C_2, \quad y = 7a\tau^2 I^{-4}, \qquad \text{where} \quad A = \pm\tfrac{12}{49}(7a)^{1/2}.$$

16. $y''_{xx} - y'_x = 6y + Ay^{-4}$.

Solution in the parametric form:

$$x = -\tfrac{1}{5}\int \tau^{-1}R^{-1}I_1^{-1}\,d\tau + C_2, \quad y = a\tau^{-3/5}I_1^{-2/5}, \qquad \text{where} \quad A = \mp 150a^5.$$

17. $y''_{xx} - y'_x = 20y + Ay^{-1/2}$.

Solution in the parametric form:

$$x = \tfrac{1}{3}\int R^{-1}I_1^{-1}I_2\,d\tau + C_2, \quad y = aI_1^{-4/3}I_2^2, \qquad \text{where} \quad A = \pm 108a^{3/2}.$$

18. $y''_{xx} - y'_x = \tfrac{15}{4}y + Ay^{-7}$.

Solution in the parametric form:

$$x = \int R^{-1}I_1\bigl(4\tau I_1^2 \mp I_2^2\bigr)^{-1}\,d\tau + C_2, \quad y = aI_1^{1/2}\bigl(4\tau I_1^2 \mp I_2^2\bigr)^{-3/8}, \qquad \text{where} \quad A = \pm\tfrac{3}{4}a^8.$$

19. $y''_{xx} - y'_x = Ay + By^{-2} - B^2y^{-3}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.5:

$$ww'_y - w = Ay + By^{-2} - B^2y^{-3}.$$

20. $y''_{xx} - y'_x = -\tfrac{3}{16}y + Ay^{-1/3} + By^{-5/3}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.61:

$$ww'_y - w = -\tfrac{3}{16}y + Ay^{-1/3} + By^{-5/3}.$$

21. $y''_{xx} - y'_x = -\tfrac{5}{36}y + Ay^{-3/5} + By^{-7/5}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.62:

$$ww'_y - w = -\tfrac{5}{36}y + Ay^{-3/5} + By^{-7/5}.$$

22. $y''_{xx} - y'_x = \tfrac{4}{9}y + 2Ay^2 + 2A^2y^3$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.14:

$$ww'_y - w = \tfrac{4}{9}y + 2Ay^2 + 2A^2y^3.$$

23. $y''_{xx} - y'_x = Ay^{k-1} - kBy^k + kB^2y^{2k-1}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.6:

$$ww'_y - w = Ay^{k-1} - kBy^k + kB^2y^{2k-1}.$$

24. $y''_{xx} - y'_x = \pm\dfrac{2a^2}{\sqrt{y^2 \pm 8a^2}}.$

Solution in the parametric form:

$$x = \mp \int E^{-1}F^{-1}(F^2 \pm 2E^2)\,d\tau + C_2, \qquad y = \pm aE^{-1}F^{-1}(F^2 \mp 2E^2),$$

where $E = \displaystyle\int \exp(\mp\tau^2)\,d\tau + C_1$, $F = 2\tau E \pm \exp(\mp\tau^2)$.

25. $y''_{xx} - y'_x = A + B\exp(-2y/A).$

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.8:

$$ww'_y - w = A + B\exp(-2y/A).$$

26. $y''_{xx} - y'_x = a^2\lambda e^{2\lambda y} - a(b\lambda + 1)e^{\lambda y} + b.$

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.73:

$$ww'_y - w = a^2\lambda e^{2\lambda y} - a(b\lambda + 1)e^{\lambda y} + b.$$

27. $y''_{xx} - y'_x = a^2\lambda e^{2\lambda y} + a\lambda y e^{\lambda y} + be^{\lambda y}.$

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.1.74:

$$ww'_y - w = a^2\lambda e^{2\lambda y} + a\lambda y e^{\lambda y} + be^{\lambda y}.$$

2.2.2. Equations of the Form $y''_{xx} + f(y)y'_x + y = 0$

Preliminary Comments. Equation of this form are often encountered in the theory of nonlinear oscillations.

1. The transformation

$$z = -\tfrac{1}{2}y^2 + a, \qquad w = y'_x$$

leads to the Abel equation

$$ww'_z = g(z)w + 1, \qquad \text{where} \quad g(z) = f(y)/y, \quad y = \pm\sqrt{2(a - z)},$$

whose special cases are outlined in Subsection 1.3.2.

2. For oscillatory systems with a weak nonlinearity

$$y''_{xx} + \varepsilon F(y)y'_x + y = 0,$$

two leading terms of the asymptotic solution, as $\varepsilon \to 0$, are described by the formula

$$y = A\cos(x + B),$$

where functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon x$; they are determined from the autonomous system of the first order equations

$$A'_\xi = -\frac{A}{2\pi}\int_0^{2\pi} F(A\cos\varphi)\sin^2\varphi\,d\varphi, \quad B'_\xi = -\frac{1}{2\pi}\int_0^{2\pi} F(A\cos\varphi)\sin\varphi\cos\varphi\,d\varphi.$$

The right-hand sides of these equations depend only on function A. The system is solved consecutively starting from the first equation.

1. $\boldsymbol{y''_{xx} + ayy'_x + y = 0.}$

Solution in the parametric form:

$$x = -A \int \tau^{-1}(C_1 + 2A^2 \ln|\tau| - 2A\tau)^{-1/2}\, d\tau + C_2, \quad y = (C_1 + 2A^2 \ln|\tau| - 2A\tau)^{1/2},$$

where $A = a^{-1}$.

2. $\boldsymbol{y''_{xx} - \varepsilon(1 - y^2)y'_x + y = 0.}$

Van der Pol oscillator

Solution, as $\varepsilon \to 0$:

$$y = a\cos(x - \theta) - \tfrac{1}{32}\varepsilon a^3 \sin[3(x - \theta)] + O(\varepsilon^2),$$

where

$$a^2 = \frac{4}{1 + (4C_1^{-2} - 1)e^{-\varepsilon x}}, \qquad \theta = \tfrac{1}{8}\varepsilon \ln a - \tfrac{7}{64}\varepsilon a^2 + \tfrac{1}{16}\varepsilon^2 x + C_2.$$

In applications, x plays the role of time, C_1 is the initial oscillation amplitude, and C_2 is the initial phase with $\varepsilon = 0$.

As $\varepsilon \to +\infty$, the periodic solution of the Van der Pol equation consits of intervals with fast and slow vibrations and describes damping oscillations with the period $T = (3 - 2\ln 2)\varepsilon + O(\varepsilon^{-1/3})$.

3. $\boldsymbol{y''_{xx} + y(ay^2 + b)y'_x + y = 0.}$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.1: $ww'_z = (-2az + b)w + 1$.

Solution in the parametric form with $a < 0$:

$$x = \mp\frac{2}{3}k \int \tau^{-1/3}\left[\pm\frac{4}{3}k^2\tau^{-2/3}Z^{-1}\Bigl(\tau Z'_\tau + \frac{1}{3}Z\Bigr) - \frac{b}{a}\right]^{-1/2} d\tau + C_2,$$

$$y = \left[\pm\frac{4}{3}k^2\tau^{-2/3}Z^{-1}\Bigl(\tau Z'_\tau + \frac{1}{3}Z\Bigr) - \frac{b}{a}\right]^{1/2}, \qquad a = -\frac{9}{4}k^{-3},$$

where

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions.

4. $\boldsymbol{y''_{xx} + y(ay^2 + b)^{-2}y'_x + y = 0.}$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.2: $ww'_z = (-2az + b)^{-2}w + 1$.

5. $\boldsymbol{y''_{xx} + y(ay^2+b)^{-1/2}y'_x + y = 0.}$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.4: $ww'_z = (-2az+b)^{-1/2}w + 1$.

Solution in the parametric form:

$$x = -aC_1 \int \Bigl(aC_1^2E^2 - \frac{b}{a}\Bigr)^{-1/2} \frac{E\,d\tau}{\tau^2 - \tau + a} + C_2, \qquad y = \Bigl(aC_1^2E^2 - \frac{b}{a}\Bigr)^{1/2},$$

where $E = \exp\Bigl(-\int \frac{\tau\,d\tau}{\tau^2-\tau+a}\Bigr)$.

6. $\boldsymbol{y''_{xx} - y\Bigl(2a + \frac{1}{ay^2+b}\Bigr)y'_x + y = 0.}$

The transformation $z = -\dfrac{1}{2}y^2 - \dfrac{b}{2a}$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.3:

$$ww'_z = \Bigl(A - \frac{1}{Az}\Bigr)w + 1, \qquad \text{where} \quad A = -2a.$$

7. $\boldsymbol{y''_{xx} + ay\exp(\lambda y^2)y'_x + y = 0.}$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.7: $ww'_z = a\exp(-2\lambda z)w + 1$.

8. $\boldsymbol{y''_{xx} + y[a\exp(\lambda y^2) + b\exp(-\lambda y^2)]y'_x + y = 0.}$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.8: $ww'_z = [b\exp(2\lambda z) + a\exp(-2\lambda z)]w + 1$.

9. $\boldsymbol{y''_{xx} + 2ay\exp[a(b-y^2)]y'_x + y = 0.}$

Solution in the parametric form:

$$x = \mp 2k \int \bigl(b - 4k^2\tau^2 - \ln|kE_{\mp}^{-1}|\bigr)^{-1/2}\,d\tau + C_2, \quad y = \bigl(b - 4k^2\tau^2 - \ln|kE_{\mp}^{-1}|\bigr)^{1/2},$$

where $a = \mp\frac{1}{4}k^{-2}$, $E_{\mp} = \int \exp(\mp\tau^2)\,d\tau + C_1$.

10. $\boldsymbol{y''_{xx} + Ay\cosh(\lambda y^2)y'_x + y = 0.}$

This is a special case of equation 2.2.2.8 with $a = b = \frac{1}{2}A$.

11. $\boldsymbol{y''_{xx} + Ay\sinh(\lambda y^2)y'_x + y = 0.}$

This is a special case of equation 2.2.2.8 with $a = -b = \frac{1}{2}A$.

12. $\boldsymbol{y''_{xx} + 2Ayy'_x\sqrt{\sinh^2[A(B-y^2)] + 2A^{-1}} + y = 0.}$

Solution in the parametric form:

$$x = 2a\int (F^2+2E^2)G^{-1}Q^{-1}\,d\tau + C_2, \quad y = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$E = \int \exp(-\tau^2)\,d\tau + C_1, \quad F = 2\tau E + \exp(-\tau^2), \quad G = \sqrt{F^2 - 2E^2 + 8E^2F^2},$$

$$Q = \sqrt{B - 4a^2\operatorname{Arsinh}[aE^{-1}F^{-1}(F^2-2E^2)]}, \quad \operatorname{Arsinh} z = \ln\bigl(z + \sqrt{z^2+1}\,\bigr).$$

13. $y''_{xx} - 2Ayy'_x\sqrt{\cosh^2[A(y^2-B)] - 2A^{-1}} + y = 0.$

Solution in the parametric form:

$$x = 2a\int (F^2 - 2E^2)G^{-1}Q^{-1}\,d\tau + C_2, \quad y = Q; \qquad A = \tfrac{1}{4}a^{-2},$$

where

$$E = \int \exp(\tau^2)\,d\tau + C_1, \quad F = 2\tau E - \exp(\tau^2), \quad G = \sqrt{F^2 + 2E^2 - 8E^2F^2},$$

$$Q = \sqrt{B + 4a^2\operatorname{Arcosh}[aE^{-1}F^{-1}(F^2+2E^2)]}, \quad \operatorname{Arcosh} z = \pm\ln\bigl(z + \sqrt{z^2-1}\,\bigr).$$

14. $y''_{xx} + Ay\cos(\omega y^2)y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.11: $ww'_z = A\cos(2\omega z)w + 1$.

15. $y''_{xx} + Ay\sin(\omega y^2)y'_x + y = 0.$

The transformation $z = -\frac{1}{2}y^2$, $w = y'_x$ leads to the Abel equation of the form 1.3.2.12: $ww'_z = -A\sin(2\omega z)w + 1$.

2.2.3. Lienard Equations $y''_{xx} + f(y)y'_x + g(y) = 0$

1. $y''_{xx} + y + ay^3 = 0.$

Duffing equation.

Solution:

$$x = \pm\int \bigl(C_1 - y^2 - \tfrac{1}{2}ay^4\bigr)^{-1/2}\,dy + C_2.$$

The period of oscillations with the amplitude C is expressed in terms of the complete elliptic integral of the first kind:

$$T = \frac{4}{\sqrt{1+aC^2}}K\Bigl(\frac{aC^2}{2+2aC^2}\Bigr), \qquad \text{where} \quad K(m) = \int_0^{\pi/2}\frac{dt}{\sqrt{1-m\sin^2 t}}.$$

The asymptotic solution, as $a \to 0$, has the form

$$y = \widetilde{C}_1\cos[(1+\tfrac{3}{8}a\widetilde{C}_1^2)x + \widetilde{C}_2] + \tfrac{1}{32}a\widetilde{C}_1^3\cos[3(1+\tfrac{3}{8}a\widetilde{C}_1^2)x + 3\widetilde{C}_2] + O(a^2),$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. The corresponding asymptotics for the period of oscillations with the aplitude C is described by the formula

$$T = 2\pi(1 - \tfrac{3}{8}aC^2) + O(a^2).$$

2. $y''_{xx} = (ay + 3b)y'_x + cy^3 - aby^2 - 2b^2y.$

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.1:

$$ww'_y = (ay + 3b)w + cy^3 - aby^2 - 2b^2y.$$

3. $\boldsymbol{y''_{xx} = (3ay + b)y'_x - a^2y^3 - aby^2 + cy}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.2:

$$ww'_y = (3ay + b)w - a^2y^3 - aby^2 + cy.$$

4. $\boldsymbol{2y''_{xx} = (7ay + 5b)y'_x - 3a^2y^3 - 2cy^2 - 3b^2y}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y.$$

5. $\boldsymbol{y''_{xx} = y^{n-1}[(1 + 2n)y + an]y'_x - ny^{2n}(y + a)}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.8:

$$ww'_y = y^{n-1}[(1 + 2n)y + an]w - ny^{2n}(y + a).$$

6. $\boldsymbol{y''_{xx} = a(y - nb)y^{n-1}y'_x + c[y^2 - (2n + 1)by + n(n + 1)b^2]y^{2n-1}}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.9:

$$ww'_y = a(y - nb)y^{n-1}w + c[y^2 - (2n + 1)by + n(n + 1)b^2]y^{2n-1}.$$

7. $\boldsymbol{y''_{xx} = [a(2n + k)y^k + b]y^{n-1}y'_x + (-a^2ny^{2k} - aby^k + c)y^{2n-1}}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.10:

$$ww'_y = [a(2n + k)y^k + b]y^{n-1}w + (-a^2ny^{2k} - aby^k + c)y^{2n-1}.$$

8. $\boldsymbol{y''_{xx} = [a(2m + k)y^{2k} + b(2m - k)]y^{m-k-1}y'_x - (a^2my^{4k} + cy^{2k} + b^2m)y^{2m-2k-1}}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.11:

$$ww'_y = [a(2m + k)y^{2k} + b(2m - k)]y^{m-k-1}w - (a^2my^{4k} + cy^{2k} + b^2m)y^{2m-2k-1}.$$

9. $\boldsymbol{y''_{xx} = ae^{\lambda y}y'_x + be^{\lambda y}}$.

Solution in the parametric form:

$$x = -\frac{A}{\lambda}\int \tau^{-1}\left(C_1 + A^2\ln|\tau| - A\tau\right)^{-1} d\tau + C_2, \qquad y = \frac{1}{\lambda}\ln\Big[-\frac{\lambda}{b}(C_1 + A^2\ln|\tau| - A\tau)\Big],$$

where $A = b/a$.

10. $\boldsymbol{y''_{xx} = (ae^y + b)y'_x + ce^{2y} - abe^y - b^2}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.15:

$$ww'_y = (ae^y + b)w + ce^{2y} - abe^y - b^2.$$

11. $\boldsymbol{y''_{xx} = [a(2\mu+\lambda)e^{\lambda y} + b]e^{\mu y}y'_x + (-a^2\mu e^{2\lambda y} - abe^{\lambda y} + c)e^{2\mu y}}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.16:

$$ww'_y = [a(2\mu+\lambda)e^{\lambda y} + b]e^{\mu y}w + (-a^2\mu e^{2\lambda y} - abe^{\lambda y} + c)e^{2\mu y}.$$

12. $\boldsymbol{y''_{xx} = (ae^{\lambda y} + b)y'_x + c[a^2e^{2\lambda y} + ab(\lambda y + 1)e^{\lambda y} + b^2\lambda y]}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.17:

$$ww'_y = (ae^{\lambda y} + b)w + c[a^2e^{2\lambda y} + ab(\lambda y + 1)e^{\lambda y} + b^2\lambda y].$$

13. $\boldsymbol{y''_{xx} = e^{\lambda y}(2a\lambda y + a + b)y'_x - e^{2\lambda y}(a^2\lambda y^2 + aby + c)}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.18:

$$ww'_y = e^{\lambda y}(2a\lambda y + a + b)w - e^{2\lambda y}(a^2\lambda y^2 + aby + c).$$

14. $\boldsymbol{y''_{xx} = e^{ay}(2ay^2 + 2y + b)y'_x + e^{2ay}(-ay^4 - by^2 + c)}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.19:

$$ww'_y = e^{ay}(2ay^2 + 2y + b)w + e^{2ay}(-ay^4 - by^2 + c).$$

15. $\boldsymbol{y''_{xx} = (a\cosh y + b)y'_x - ab\sinh y + c}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.20:

$$ww'_y = (a\cosh y + b)w - ab\sinh y + c.$$

16. $\boldsymbol{y''_{xx} = (a\sinh y + b)y'_x - ab\cosh y + c}$.

The substitution $w(y) = y'_x$ leads to the Abel equation of the form 1.3.3.21:

$$ww'_y = (a\sinh y + b)w - ab\cosh y + c.$$

17. $\boldsymbol{y''_{xx} + a\sin y = 0}$.

This is the equation of oscillations of the mathematical pendulum.

Solution:

$$x = \pm\int (2a\cos y + C_1)^{-1/2}\,dy + C_2.$$

With $a > 0$ and initial conditions $y(0) = C > 0$, $y'_x(0) = 0$, where variable x takes the role of the time, and y is the angle deviation from the equillibrium state, the oscillations of the mathematical pendulum are described by the formula

$$\sin\frac{y}{2} = m\,\mathrm{sn}\bigl(\sqrt{a}\,x\bigr), \qquad m = \sin\frac{C}{2},$$

where $\mathrm{sn} = \mathrm{sn}(z)$ is the Jacobi elliptic function defined implicitly as follows:

$$\mathrm{sn}(z) = \sin\beta, \quad z = \int_0^\beta \frac{d\beta}{\sqrt{1 - m^2\sin^2\beta}}$$

(parameter β is to be eliminated from these relations).

The period of oscillations of the mathematical pendulum is expressed in terms of the complete elliptic integral of the second kind:

$$T = \frac{4}{\sqrt{a}}K(m), \qquad K(m) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - m^2\sin^2\beta}}.$$

At small amplitudes, as $C \to 0$, the following asymptotic formula holds for the period:

$$T = \frac{2\pi}{\sqrt{a}}\Bigl(1 + \frac{1}{16}C^2\Bigr) + O(C^4).$$

18. $y''_{xx} + a\sin(\omega y)y'_x + b\sin(\omega y) = 0.$

Solution in the parametric form:

$$x = -A\int \tau^{-1}\bigl(b^2 - \omega^2 F^2\bigr)^{-1/2}\,d\tau + C_2, \qquad y = \frac{1}{\omega}\arccos\Bigl(\frac{\omega}{b}F\Bigr),$$

where $A = b/a$, $F = A\tau - A^2\ln|\tau| + C_1$.

19. $y''_{xx} + a\cos(\omega y)y'_x + b\cos(\omega y) = 0.$

The substitution $\omega y = \omega u + \frac{\pi}{2}$ leads to an equation of the form 2.2.3.18:

$$u''_{xx} - a\sin(\omega u)u'_x - b\sin(\omega u) = 0.$$

2.2.4. Rayleigh Equations $y''_{xx} + f(y'_x) + g(y) = 0$

Preliminary Comments. Equations of this form are encountered in the theory of nonlinear oscillations.

1. Let us discuss the particular case $g(y) = y$ which corresponds to the equation

$$y''_{xx} + f(y'_x) + y = 0. \tag{1}$$

Differentiating equation (1) with respect to x and substituting $z(x) = y'_x$, we obtain the equation of nonlinear oscillations

$$z''_{xx} + \Phi(z)z'_x + z = 0, \qquad \text{where} \quad \Phi(z) = f'_z(z), \tag{2}$$

which is considered in Subsection 2.2.2.

The solution of equation (1) can be written in the parametric form:

$$x = x(\tau,\ C_1,\ C_2), \qquad y = -f(z) - \frac{z'_\tau}{x'_\tau},$$

where $x = x(\tau,\ C_1,\ C_2)$, $z = z(\tau,\ C_1,\ C_2)$ is the parametric form of the solution of equation (2).

2. The transformation

$$\xi = -\tfrac{1}{2}\bigl(y'_x\bigr)^2 + a, \qquad w = -y - f(y'_x),$$

reduces equation (1) to the Abel equation

$$ww'_\xi = H(\xi)w + 1, \qquad \text{where} \quad H(\xi) = z^{-1}\Phi(z), \quad z = \pm\sqrt{2(a-\xi)}, \tag{3}$$

where function $\Phi = \Phi(z)$ is defined above in equation (2). Concrete equations of the form (3) are outlined in Subsection 1.3.2.

3. The equation of the special form

$$y''_{xx} + a\bigl(y'_x\bigr)^2 + g(y) = 0 \tag{4}$$

is reduced, with the aid of the substitution $w(y) = (y'_x)^2$, to a first order linear equation $w'_y + 2aw + 2g(y) = 0$. Therefore, the solution of the equation (4) can be written in the implicit form

$$x = C_2 \pm \int \bigl[C_1 e^{-2ay} - G(y)\bigr]^{-1/2}\,dy, \qquad \text{where} \quad G(y) = 2e^{-2ay}\int e^{2ay} g(y)\,dy.$$

4. The equation of the special form

$$y''_{xx} + a(y'_x)^4 + b(y'_x)^2 + g(y) = 0 \tag{5}$$

is reduced, with the aid of the substitution $w(y) = (y'_x)^2$, to the Riccati equation $w'_y + 2aw^2 + 2bw + 2g(y) = 0$ which is outlined in Section 1.2.

5. For the oscillatory systems with a weak nonlinearity

$$y''_{xx} + \varepsilon F(y'_x) + y = 0,$$

two leading terms of the asymptotic solution, as $\varepsilon \to 0$, are described by the fromula

$$y = A\cos(x + B),$$

where functions $A = A(\xi)$ and $B = B(\xi)$ depend on the slow variable $\xi = \varepsilon x$ and are defined from the autonomous system of the first order equations

$$A'_\xi = \frac{1}{2\pi}\int_0^{2\pi} F(-A\sin\varphi)\sin\varphi\,d\varphi, \quad AB'_\xi = \frac{1}{2\pi}\int_0^{2\pi} F(-A\sin\varphi)\cos\varphi\,d\varphi.$$

The right-hand sides of these equations depend only on function A. The system is solved consecutively starting from the first equation.

1. $\boldsymbol{y''_{xx} + a(y'_x)^2 + by = 0.}$

This equation describes small oscillations when the drag force is proportional to the speed squared.

Solution in the implicit form:

$$x = C_2 \pm a\int \bigl[C_1 a^2 e^{-2ay} + b(\tfrac{1}{2} - ay)\bigr]^{-1/2}\,dy.$$

2. $\boldsymbol{y''_{xx} + \varepsilon[\frac{1}{3}(y'_x)^3 - y'_x] + y = 0.}$

Van der Pol equation.

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.2: $w''_{xx} - \varepsilon(1 - w^2)w'_x + w = 0$.

Solution, as $\varepsilon \to 0$:

$$y = \frac{2C_1}{\sqrt{1 - C_2 e^{-\varepsilon x}}}\cos x + \frac{2\sqrt{1 - C_1^2}}{\sqrt{1 - C_2 e^{-\varepsilon x}}}\sin x + O(\varepsilon^2).$$

3. $\boldsymbol{y''_{xx} + a(y'_x)^4 + b(y'_x)^2 + y = 0.}$

The transformation $\xi = -\frac{1}{2}(y'_x)^2$, $w = -y - a(y'_x)^4 - b(y'_x)^2$ leads to the Abel equation of the form 1.3.2.1: $ww'_\xi = (-8a\xi + 2b)w + 1$.

4. $\boldsymbol{y''_{xx} + (y'_x)^2\big[a(y'_x)^2 + b\big]^{-1} + y = 0.}$

The transformation $\xi = -\frac{1}{2}(y'_x)^2$, $w = -y - (y'_x)^2[a(y'_x)^2 + b]^{-1}$ leads to the Abel equation of the form 1.3.2.2: $ww'_\xi = 2b(b - 2a\xi)^{-2}w + 1$.

5. $\boldsymbol{y''_{xx} + A\exp[\lambda(y'_x)^2] + B + y = 0.}$

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.7:

$$w''_{xx} + 2A\lambda w\exp(\lambda w^2)w'_x + w = 0.$$

6. $\boldsymbol{y''_{xx} + A\cosh[\lambda(y'_x)^2] + B + y = 0.}$

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.11:

$$w''_{xx} + 2A\lambda w\sinh(\lambda w^2)w'_x + w = 0.$$

7. $\boldsymbol{y''_{xx} + A\sinh[\lambda(y'_x)^2] + B + y = 0.}$

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.10:

$$w''_{xx} + 2A\lambda w\cosh(\lambda w^2)w'_x + w = 0.$$

8. $\boldsymbol{y''_{xx} + a(y'_x)^2 + b\sin y = 0.}$

This equation describes the oscillations of the mathematical pendulum when the drag force is proportional to the speed squared.

Solution in the implicit form:

$$x = C_2 \pm \int\left[C_1e^{-2ay} + \frac{2b}{4a^2+1}(\cos y - 2a\sin y)\right]^{-1/2}dy.$$

9. $\boldsymbol{y''_{xx} + A\cos[\lambda(y'_x)^2] + B + y = 0.}$

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.15:

$$w''_{xx} - 2A\lambda w\sin(\lambda w^2)w'_x + w = 0.$$

10. $\boldsymbol{y''_{xx} + A\sin[\lambda(y'_x)^2] + B + y = 0.}$

Differentiating the equation with respect to x and passing on to a new variable $w(x) = y'_x$, we arrive at an equation of the form 2.2.2.14:

$$w''_{xx} + 2A\lambda w\cos(\lambda w^2)w'_x + w = 0.$$

2.3. Emden—Fowler Equation $y''_{xx} = Ax^n y^m$

2.3.1. Exact Solutions

Preliminary comments. The value of the insignificant parameter A is in many cases defined in the form of a function of two (one) auxiliary coefficients a and b:

$$A = \varphi(a,\ b) \tag{1}$$

and the corresponding solutions are represented in the parametric form

$$x = f_1(\tau,\ C_1,\ C_2, a), \qquad y = f_2(\tau,\ C_1,\ C_2, b), \tag{2}$$

where τ is a parameter, C_1 and C_2 are arbitrary constants, f_1 and f_2 are some functions.

Having fixed the auxiliary coefficient sign $a > 0$ (or $b > 0$) in (1), the coefficient b should be expressed in terms of both A and a with the help of

$$b = \psi(A,\ a).$$

Substituting this formula into (2), we obtain a solution of the equation under consideration (where the concrete numerical value of the coefficient a may be chosen arbitrarily). The case $a < 0$ (or $b < 0$), which may lead to the branch of the solution or to a different domain of determining the variables x and y in (2), should be considered in a similar manner.

One may also use a different approach by setting one of the auxiliary coefficients (e.g., a) equal to ± 1 in (1) and (2); then the other coefficients will be identically expressed in terms of A by means of (1).

Table 2.7 represents all solvable Emden—Fowler equations whose solutions are outlined in Subsection 2.3.1. The one-parameter families (in the space of parameters n, m) and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of m and the growth of n (for identical m). The number of the equation sought is indicated in the last column in this table.

1. $\boldsymbol{y''_{xx} = Ax^n}$.

Solution:

$$y = \begin{cases} \dfrac{Ax^{n+2}}{(n+1)(n+2)} + C_1 x + C_2 & \text{if } n \neq -1;\ -2, \\ -A \ln|x| + C_1 x + C_2 & \text{if } n = -2, \\ A \displaystyle\int \ln|x|\, dx + C_1 x + C_2 & \text{if } n = -1. \end{cases}$$

2. $\boldsymbol{y''_{xx} = Ay^m}$.

Solution:

$$x = \begin{cases} \pm \displaystyle\int \Bigl(\frac{2A}{m+1} y^{m+1} + C_1\Bigr)^{-1/2} dy + C_2 & \text{if } m \neq -1, \\ \pm \displaystyle\int (2A \ln|y| + C_1)^{-1/2}\, dy + C_2 & \text{if } m = -1. \end{cases}$$

TABLE 2.7
Solvable Cases of the Emden—Fowler Equation $y''_{xx} = Ax^n y^m$

No	m	n	Equation
	One-parameter families		
1	arbitrary	0	2.3.1.2
2	arbitrary	$-m-3$	2.3.1.3
3	arbitrary	$-\frac{1}{2}(m+3)$	2.3.1.4
4	0	arbitrary	2.3.1.1
5	1	arbitrary	2.3.1.5
	Isolated points		
6	-7	1	2.3.1.15
7	-7	3	2.3.1.16
8	$-5/2$	$-1/2$	2.3.1.22
9	-2	-2	2.3.1.28
10	-2	1	2.3.1.27
11	$-5/3$	$-10/3$	2.3.1.10
12	$-5/3$	$-7/3$	2.3.1.8
13	$-5/3$	$-5/6$	2.3.1.23
14	$-5/3$	$-1/2$	2.3.1.24
15	$-5/3$	1	2.3.1.7
16	$-5/3$	2	2.3.1.9
17	$-7/5$	$-13/5$	2.3.1.14
18	$-7/5$	1	2.3.1.13
19	$-1/2$	$-7/2$	2.3.1.12
20	$-1/2$	$-5/2$	2.3.1.6
21	$-1/2$	-2	2.3.1.26
22	$-1/2$	$-4/3$	2.3.1.17
23	$-1/2$	$-7/6$	2.3.1.18
24	$-1/2$	$-1/2$	2.3.1.25
25	$-1/2$	1	2.3.1.11
26	2	-5	2.3.1.19
27	2	$-20/7$	2.3.1.21
28	2	$-15/7$	2.3.1.20

Special cases:

1°. In the case $m = -1/2$, the solution can be written in the parametric form

$$x = aC_1^3(\tau^3 - 3\tau + C_2), \quad y = bC_1^4(\tau^2 - 1)^2, \qquad \text{where} \quad A = \pm\frac{4}{9}a^{-2}b^{3/2}.$$

2°. In the case $m = -4$, the solution can be written in the parametric form

$$x = aC_1^5\tau^{-1}\left(2\tau\int\frac{\tau\,d\tau}{R} + C_2\tau \mp R\right), \quad y = bC_1^2\tau^{-1},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $A = \mp 6a^{-2}b^5$.

3°. In the case $m = 2$, the solution can be written in the parametric form

$$x = aC_1^{-1}\tau, \quad y = bC_1^2\wp, \qquad \text{where} \quad A = \pm 6a^{-2}b^{-1},$$

the function $\wp$ of the parameter τ is defined in the implicit form:

$$\tau = \int\frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

The upper sign in this formula corresponds to the classical elliptic Weirstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$.

4°. In the case $m = -5/2$, the solution can be written in the parametric form

$$x = aC_1^7\wp^{-2}\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right], \quad y = bC_1^4\wp^{-2}, \qquad \text{where} \quad A = \mp 3a^{-2}b^{7/2}.$$

The function $\wp$ of the parameter τ is defined in the previous case.

3. $y''_{xx} = Ax^{-m-3}y^m$.

1°. Solution in the parametric form with $m \neq -1$:

$$x = aC_1^{m-1}\left[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\right]^{-1}, \quad y = bC_1^{m+1}\tau\left[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\right]^{-1},$$

where $A = \pm\dfrac{m+1}{2}a^{m+1}b^{1-m}$.

2°. Solution in the parametric form with $m = -1$:

$$x = C_1\left[\int \exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}, \quad y = b\exp(\mp\tau^2)\left[\int \exp(\mp\tau^2)\,d\tau + C_2\right]^{-1},$$

where $A = \mp 2b^2$.

4. $y''_{xx} = Ax^{-\frac{m+3}{2}}y^m$.

1°. Solution in the parametric form with $m \neq -1$:

$$x = aC_2^2\exp\left[\int \left(\frac{2}{m+1}\tau^{m+1} + \frac{1}{4}\tau^2 + C_1\right)^{-1/2} d\tau\right],$$
$$y = bC_2\tau\exp\left[\frac{1}{2}\int \left(\frac{2}{m+1}\tau^{m+1} + \frac{1}{4}\tau^2 + C_1\right)^{-1/2} d\tau\right],$$

where $A = \left(\dfrac{a}{b^2}\right)^{\frac{m-1}{2}}$.

2°. Solution in the parametric form with $m = -1$:

$$x = aC_2^2\exp\left[\int \left(2\ln|\tau| + \frac{1}{4}\tau^2 + C_1\right)^{-1/2} d\tau\right],$$
$$y = bC_2\tau\exp\left[\frac{1}{2}\int \left(2\ln|\tau| + \frac{1}{4}\tau^2 + C_1\right)^{-1/2} d\tau\right],$$

where $A = b^2/a$.

5. $y''_{xx} = Ax^n y$.

For $n \neq -2$, see equation 2.1.2.7. For $n = -2$, see equation 2.1.2.118.

6. $y''_{xx} = Ax^{-5/2}y^{-1/2}$.

Solution in the parametric form:

$$x = aC_1^{-3}(\tau^3 - 3\tau + C_2)^{-1}, \qquad y = bC_1(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \pm\frac{4}{9}a^{1/2}b^{3/2}$.

7. $y''_{xx} = Axy^{-5/3}$.

Solution in the parametric form:

$$x = \pm aC_1^8(\tau^4 - 6\tau^2 + 4C_2\tau - 3), \qquad y = bC_1^9(\tau^3 - 3\tau + C_2)^{3/2},$$

where $A = \pm\frac{9}{64}a^{-3}b^{8/3}$.

8. $y''_{xx} = Ax^{-7/3}y^{-5/3}$.

Solution in the parametric form:

$$x = \pm\frac{aC_1^{-8}}{\tau^4 - 6\tau^2 + 4C_2\tau - 3}, \qquad y = \pm\frac{bC_1(\tau^3 - 3\tau + C_2)^{3/2}}{\tau^4 - 6\tau^2 + 4C_2\tau - 3},$$

where $A = \pm\frac{9}{64}a^{1/3}b^{8/3}$.

9. $y''_{xx} = Ax^2y^{-5/3}$.

1°. Solution in the parametric form with $A < 0$:

$$x = aC_1^2 \cos\tau \cosh(\tau + C_2)[\tan\tau + \tanh(\tau + C_2)], \quad y = bC_1^3[\cos\tau \cosh(\tau + C_2)]^{3/2},$$

where $A = -\frac{3}{16}a^{-4}b^{8/3}$.

2°. Solution in the parametric form with $A > 0$:

$$x = aC_1^2[\sinh\tau + \cos(\tau + C_2)], \qquad y = bC_1^3[\cosh\tau - \sin(\tau + C_2)]^{3/2},$$

where $A = \frac{3}{4}a^{-4}b^{8/3}$.

10. $y''_{xx} = Ax^{-10/3}y^{-5/3}$.

1°. Solution in the parametric form with $A < 0$:

$$\begin{aligned} x &= aC_1^{-2}[\cos\tau \cosh(\tau + C_2)]^{-1}[\tan\tau + \tanh(\tau + C_2)]^{-1}, \\ y &= bC_1[\cos\tau \cosh(\tau + C_2)]^{1/2}[\tan\tau + \tanh(\tau + C_2)]^{-1}, \end{aligned}$$

where $A = -\frac{3}{16}a^{4/3}b^{8/3}$.

2°. Solution in the parametric form with $A > 0$:

$$\begin{aligned} x &= aC_1^{-2}[\sinh\tau + \cos(\tau + C_2)]^{-1}, \\ y &= bC_1[\cosh\tau - \sin(\tau + C_2)]^{3/2}[\sinh\tau + \cos(\tau + C_2)]^{-1}, \end{aligned}$$

where $A = \frac{3}{4}a^{4/3}b^{8/3}$.

11. $y''_{xx} = Axy^{-1/2}$.

Solution in the parametric form:

$$\begin{aligned} x &= aC_1 \exp(-\tau)\big[\exp(3\tau) + C_2\sin\big(\sqrt{3}\,\tau\big)\big], \\ y &= bC_1^2 \exp(-2\tau)\big[2\exp(3\tau) - C_2\sin\big(\sqrt{3}\,\tau\big) + \sqrt{3}C_2\cos\big(\sqrt{3}\,\tau\big)\big]^2, \end{aligned}$$

where $A = 16a^{-3}b^{3/2}$.

▶ *In the solutions of equations 12–14, the following notation is used:*

$$S_1 = \exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr), \qquad S_2 = 2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\; C_2 \cos\bigl(\sqrt{3}\,\tau\bigr),$$
$$S_3 = 2S_1(S_2)'_\tau - (S_1)'_\tau S_2 - S_1 S_2.$$

12. $y''_{xx} = Ax^{-7/2}y^{-1/2}$.

Solution in the parametric form:

$$x = aC_1^{-1}\exp(\tau)S_1^{-1}, \quad y = bC_1\exp(-\tau)S_1^{-1}S_2^2, \qquad \text{where} \quad A = 16(ab)^{3/2}.$$

13. $y''_{xx} = Axy^{-7/5}$.

Solution in the parametric form:

$$x = aC_1^4\exp(-2\tau)S_3, \quad y = bC_1^5\exp\bigl(-\tfrac{5}{2}\tau\bigr)S_1^{5/2}, \qquad \text{where} \quad A = \tfrac{5}{1024}a^{-3}b^{12/5}.$$

14. $y''_{xx} = Ax^{-13/5}y^{-7/5}$.

Solution in the parametric form:

$$x = aC_1^{-4}\exp(2\tau)S_3^{-1}, \quad y = bC_1\exp\bigl(-\tfrac{1}{2}\tau\bigr)S_1^{5/2}S_3^{-1}, \quad \text{where} \quad A = \tfrac{5}{1024}a^{3/5}b^{12/5}.$$

▶ *In the solutions of equations 15–18, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \qquad f = 2\tau I(\tau) + C_2\tau \mp R,$$

where $I(\tau) = \int \tau R^{-1}\,d\tau$ *is the incomplete elliptic integral of the second kind in the form of Weierstrass.*

15. $y''_{xx} = Axy^{-7}$.

Solution in the parametric form:

$$x = aC_1^8[4\tau f^2 \mp \tau^{-2}(fR - 1)^2], \quad y = bC_1^3 f^{1/2}, \qquad \text{where} \quad A = \pm\tfrac{3}{64}a^{-3}b^8.$$

16. $y''_{xx} = Ax^3y^{-7}$.

Solution in the parametric form:

$$x = aC_1^8[4\tau f^2 \mp \tau^{-2}(fR - 1)^2]^{-1}, \qquad y = bC_1^5 f^{1/2}[4\tau f^2 \mp \tau^{-2}(fR - 1)^2]^{-1},$$

where $A = \pm\frac{3}{64}a^{-5}b^8$.

17. $y''_{xx} = Ax^{-4/3}y^{-1/2}$.

Solution in the parametric form:

$$x = aC_1^9 f^3, \quad y = bC_1^4\tau^{-2}(fR - 1)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-2/3}b^{3/2}.$$

18. $y''_{xx} = Ax^{-7/6}y^{-1/2}$.

Solution in the parametric form:

$$x = aC_1^9 f^{-3}, \quad y = bC_1^5\tau^{-2}f^{-3}(fR - 1)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{3}a^{-5/6}b^{3/2}.$$

► *In the solutions of equations 19–24, the function $\wp$ of the parameter τ is defined in the implicit form*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

The upper sign in this formula corresponds to the classical elliptic Weirstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$.

19. $y''_{xx} = Ax^{-5}y^2.$

Solution in the parametric form:

$$x = aC_1\tau^{-1}, \quad y = bC_1^3\tau^{-1}\wp, \qquad \text{where} \quad A = \pm 6a^3b^{-1}.$$

20. $y''_{xx} = Ax^{-15/7}y^2.$

Solution in the parametric form:

$$x = aC_1^7\tau^7, \quad y = bC_1\tau(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{49}a^{1/7}b^{-1}.$$

21. $y''_{xx} = Ax^{-20/7}y^2.$

Solution in the parametric form:

$$x = aC_1^7\tau^{-7}, \quad y = bC_1^6\tau^{-6}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{49}a^{6/7}b^{-1}.$$

22. $y''_{xx} = Ax^{-1/2}y^{-5/2}.$

Solution in the parametric form:

$$x = aC_1^7\wp^2\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right]^{-1}, \qquad y = bC_1^3\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right]^{-1},$$

where $A = \mp 3a^{-3/2}b^{7/2}$.

23. $y''_{xx} = Ax^{-5/6}y^{-5/3}.$

Solution in the parametric form:

$$x = \frac{aC_1^{16}}{\left[\tau\sqrt{\pm(4\wp^3 - 1)} + 2\wp\right]^2}, \qquad y = \frac{bC_1^7\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right]^{3/2}}{\left[\tau\sqrt{\pm(4\wp^3 - 1)} + 2\wp\right]^2},$$

where $A = -\frac{1}{6}a^{-7/6}b^{8/3}$.

24. $y''_{xx} = Ax^{-1/2}y^{-5/3}.$

Solution in the parametric form:

$$x = aC_1^{16}\left[\tau\sqrt{\pm(4\wp^3 - 1)} + 2\wp\right]^2, \qquad y = bC_1^9\left[\sqrt{\pm(4\wp^3 - 1)} \pm 2\tau\wp^2\right]^{3/2},$$

where $A = -\frac{1}{6}a^{-3/2}b^{8/3}$.

► *In the solutions of equations 25–28, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions.

25. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-1/2}}$.

Solution in the parametric form:

$$x = a\tau^{2/3}Z^2, \quad y = b\tau^{-2/3}\Big(\tau Z'_\tau + \frac{1}{3}Z\Big)^2, \qquad \text{where} \quad A = \mp\frac{b}{3a}\Big(\mp\frac{b}{a}\Big)^{1/2}.$$

26. $\boldsymbol{y''_{xx} = Ax^{-2}y^{-1/2}}$.

Solution in the parametric form:

$$x = \tau^{-2/3}Z^{-2}, \quad y = b\tau^{-4/3}Z^{-2}\Big(\tau Z'_\tau + \frac{1}{3}Z\Big)^2, \qquad \text{where} \quad A = \mp\frac{1}{3}b^{3/2}.$$

27. $\boldsymbol{y''_{xx} = Axy^{-2}}$.

Solution in the parametric form:

$$x = a\tau^{-2/3}\Big[\Big(\tau Z'_\tau + \frac{1}{3}Z\Big)^2 \pm \tau^2 Z^2\Big], \quad y = b\tau^{2/3}Z^2, \qquad \text{where} \quad A = -\frac{9}{2}\Big(\frac{b}{a}\Big)^3.$$

28. $\boldsymbol{y''_{xx} = Ax^{-2}y^{-2}}$.

Solution in the parametric form:

$$x = \tau^{2/3}\Big[\Big(\tau Z'_\tau + \frac{1}{3}Z\Big)^2 \pm \tau^2 Z^2\Big]^{-1}, \qquad y = b\tau^{4/3}Z^2\Big[\Big(\tau Z'_\tau + \frac{1}{3}Z\Big)^2 \pm \tau^2 Z^2\Big]^{-1},$$

where $A = -\frac{9}{2}b^3$.

2.3.2. First Integrals (Conservations Laws)

In this subsection, first integrals of the form

$$\sum_{\alpha=0}^{k} f_\alpha(x,y)(y'_x)^\alpha = C, \qquad \text{where} \quad k = 2,\ 3,\ 4,\ 5,$$

for the Emden—Fowler equation $y''_{xx} = Ax^n y^m$ are given.

First integrals with $k = 2$

1. For $n = 0$ and arbitrary m $(m \neq -1)$,

$$(y'_x)^2 - \frac{2A}{m+1}y^{m+1} = C.$$

2. For $n=-\dfrac{m+3}{2}$ and arbitrary m $(m\neq -1)$,

$$x(y'_x)^2-yy'_x-\frac{2A}{m+1}x^{-\frac{m+1}{2}}y^{m+1}=C.$$

3. For $n=-m-3$ and arbitrary m $(m\neq -1)$,

$$x^2(y'_x)^2-2xyy'_x+y^2-\frac{2A}{m+1}x^{-m-1}y^{m+1}=C.$$

4. For $n=-\frac{20}{7}$, $m=2$,

$$\tfrac{343}{24}Ax^{8/7}(y'_x)^2-\left(\tfrac{49}{3}Ax^{1/7}y-x\right)y'_x-\tfrac{343}{36}A^2x^{-12/7}y^3+\tfrac{7}{6}Ax^{-6/7}y^2-y=C.$$

5. For $n=-\frac{15}{7}$, $m=2$,

$$\tfrac{343}{24}Ax^{6/7}(y'_x)^2-\left(\tfrac{49}{4}Ax^{-1/7}y+1\right)y'_x-\tfrac{343}{36}A^2x^{-9/7}y^3-\tfrac{7}{8}Ax^{-8/7}y^2=C.$$

First integrals with $k=3$

1. For $n=0$, $m=-\frac{1}{2}$,

$$(y'_x)^3-6Ay^{1/2}y'_x+6A^2x=C,$$
$$x(y'_x)^3-y(y'_x)^2-6Axy^{1/2}y'_x+\tfrac{16}{3}Ay^{3/2}+3A^2x^2=C.$$

2. For $n=1$, $m=-\frac{1}{2}$,

$$(y'_x)^3-6Axy^{1/2}y'_x+4Ay^{3/2}+2A^2x^3=C.$$

3. For $n=-\frac{4}{3}$, $m=-\frac{1}{2}$,

$$x(y'_x)^3-y(y'_x)^2-6Ax^{-1/3}y^{1/2}y'_x-9A^2x^{-2/3}=C.$$

4. For $n=-\frac{5}{2}$, $m=-\frac{1}{2}$,

$$x^2(y'_x)^3-2xy(y'_x)^2+(y^2-6Ax^{-1/2}y^{1/2})y'_x+\tfrac{2}{3}Ax^{-3/2}y^{3/2}-3A^2x^{-2}=C,$$
$$x^3(y'_x)^3-3x^2y(y'_x)^2+3(xy^2-2Ax^{1/2}y^{1/2})y'_x-y^3+6Ax^{-1/2}y^{3/2}-6A^2x^{-1}=C.$$

5. For $n=-\frac{7}{6}$, $m=-\frac{1}{2}$,

$$x^2(y'_x)^3-2xy(y'_x)^2+(y^2-6Ax^{5/6}y^{1/2})y'_x+6Ax^{-1/6}y^{3/2}+9A^2x^{2/3}=C.$$

6. For $n=-\frac{7}{2}$, $m=-\frac{1}{2}$,

$$x^3(y'_x)^3-3x^2y(y'_x)^2+3(xy^2-2Ax^{-1/2}y^{1/2})y'_x-y^3+2Ax^{-3/2}y^{3/2}-2A^2x^{-3}=C.$$

First integrals with $k = 4$

1. For $n = 1$, $m = -\frac{5}{3}$,

$$(y'_x)^4 + 6Axy^{-2/3}(y'_x)^2 - 18Ay^{1/3}y'_x + 9A^2x^2y^{-4/3} = C,$$
$$x(y'_x)^4 - y(y'_x)^3 + 6Ax^2y^{-2/3}(y'_x)^2 - 27Axy^{1/3}y'_x + \tfrac{81}{4}Ay^{4/3} + 9A^2x^3y^{-4/3} = C.$$

2. For $n = 2$, $m = -\frac{5}{3}$,

$$(y'_x)^4 + 6Ax^2y^{-2/3}(y'_x)^2 - 36Axy^{1/3}y'_x + 9A^2x^4y^{-4/3} = C.$$

3. For $n = 0$, $m = -\frac{5}{3}$,

$$x(y'_x)^4 - y(y'_x)^3 + 6Axy^{-2/3}(y'_x)^2 - 9Ay^{1/3}y'_x + 9A^2xy^{-4/3} = C,$$
$$x^2(y'_x)^4 - 2xy(y'_x)^3 + (y^2 + 6Ax^2y^{-2/3})(y'_x)^2 - 18Axy^{1/3}y'_x + 12Ay^{4/3} + 9A^2x^2y^{-4/3} = C.$$

4. For $n = -\frac{1}{2}$, $m = -\frac{5}{3}$,

$$x(y'_x)^4 - y(y'_x)^3 + 6Ax^{1/2}y^{-2/3}(y'_x)^2 + 9A^2y^{-4/3} = C.$$

5. For $n = -\frac{4}{3}$, $m = -\frac{5}{3}$,

$$x^2(y'_x)^4 - 2xy(y'_x)^3 + (y^2 + 6Ax^{2/3}y^{-2/3})(y'_x)^2 + 6Ax^{-1/3}y^{1/3}y'_x + 9A^2x^{-2/3}y^{-4/3} = C,$$
$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{2/3}y^{-2/3})(y'_x)^2 - (y^3 + 3Ax^{2/3}y^{1/3})y'_x - 3Ax^{-1/3}y^{4/3} + 9A^2x^{1/3}y^{-4/3} = C.$$

6. For $n = -\frac{7}{3}$, $m = -\frac{5}{3}$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{-1/3}y^{-2/3})(y'_x)^2 - (y^3 - 15Ax^{-1/3}y^{1/3})y'_x - \tfrac{3}{4}Ax^{-4/3}y^{4/3} + 9A^2x^{-5/3}y^{-4/3} = C,$$
$$x^4(y'_x)^4 - 4x^3y(y'_x)^3 + 6x^2(y^2 + Ax^{-1/3}y^{-2/3})(y'_x)^2 -2x(2y^3 - 3Ax^{-1/3}y^{1/3})y'_x + y^4 - 12Ax^{-1/3}y^{4/3} + 9A^2x^{-2/3}y^{-4/3} = C.$$

7. For $n = -\frac{5}{6}$, $m = -\frac{5}{3}$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + 2Ax^{7/6}y^{-2/3})(y'_x)^2 - (y^3 + 12Ax^{7/6}y^{1/3})y'_x + 6Ax^{1/6}y^{4/3} + 9A^2x^{4/3}y^{-4/3} = C.$$

8. For $n = -\frac{10}{3}$, $m = -\frac{5}{3}$,

$$x^4(y'_x)^4 - 4x^3y(y'_x)^3 + 6x^2(y^2 + Ax^{-4/3}y^{-2/3})(y'_x)^2 - 4x(y^3 - 6Ax^{-4/3}y^{1/3})y'_x + y^4 - 30Ax^{-4/3}y^{4/3} + 9A^2x^{-8/3}y^{-4/3} = C.$$

9. For $n = 1$, $m = -7$,

$$x(y'_x)^4 - y(y'_x)^3 + \tfrac{2}{3}Ax^2y^{-6}(y'_x)^2 - \tfrac{1}{3}Axy^{-5}y'_x - \tfrac{1}{12}Ay^{-4} + \tfrac{1}{9}A^2x^3y^{-12} = C.$$

10. For $n = 3$, $m = -7$,

$$x^3(y'_x)^4 - 3x^2y(y'_x)^3 + 3x(y^2 + \tfrac{2}{9}Ax^5y^{-6})(y'_x)^2 - y(y^2 + Ax^5y^{-6})y'_x + \tfrac{1}{4}Ax^4y^{-4} + \tfrac{1}{9}A^2x^9y^{-12} = C.$$

It should be noted that in the case $k = 4$ we omitted the first integrals of the form

$$\alpha F^2 + \beta F + \gamma = C,$$

where function $F = F(x, y, y'_x)$ is the left-hand side of the above intergals for $k = 2$, and α, β, and γ are some constants.

First integrals with $k=5$

1. For $n=0$, $m=-\frac{2}{3}$,

$$(y'_x)^5-15Ay^{1/3}(y'_x)^3+\tfrac{135}{2}A^2y^{2/3}y'_x-\tfrac{135}{2}A^3x=C.$$

2. For $n=-\frac{7}{3}$, $m=-\frac{2}{3}$,

$$\begin{aligned}&x^5(y'_x)^5-5x^4y(y'_x)^4+5x^3y(2y-3Ax^{-1/3}y^{-2/3})(y'_x)^3\\&\quad-5x^2y^2(2y-9Ax^{-1/3}y^{-2/3})(y'_x)^2+5x(y^4-9Ax^{-1/3}y^{7/3}+\tfrac{27}{2}A^2x^{-2/3}y^{2/3})y'_x\\&\qquad+15A(x^{-1/3}y^{10/3}-\tfrac{9}{2}Ax^{-2/3}y^{5/3}-\tfrac{9}{2}A^2x^{-1})=C.\end{aligned}$$

2.3.3. Some Formulas and Transformations

1. With $m\neq 1$, the Emden—Fowler equation has the particular solution

$$y=\lambda x^{\frac{n+2}{1-m}},\qquad\text{where}\quad\lambda=\left[\frac{(n+2)(n+m+1)}{A(m-1)^2}\right]^{\frac{1}{m-1}}.$$

2. The transformation $y=w/t$, $x=1/t$ leads to the Emden—Fowler equation with the independent variable to a different power:

$$w'_t=At^{-n-m-3}w^m.$$

3. Some more complicated transformations leading to the Emden—Fowler equation are outlined in Subsection 2.5.3 (see Fig. 3).

4. With $m\neq 1$ and $m\neq -2n-3$, the transformation

$$\xi=\frac{2n+m+3}{m-1}x^{\frac{n+2}{m-1}}y,\qquad u=x^{\frac{n+2}{m-1}}\Big(xy'_x+\frac{n+2}{m-1}y\Big)$$

leads to the Abel equation:

$$uu'_\xi-u=-\frac{(n+2)(n+m+1)}{(2n+m+3)^2}\xi+A\Big(\frac{m-1}{2n+m+3}\Big)^2\xi^m,$$

whose special cases are given in Subsection 1.3.1.

5. Some more complicated transformations leading to other Abel equations are outlined in Subsection 2.5.3.

2.4. Equations of the Form $y''_{xx}=A_1x^{n_1}y^{m_1}+A_2x^{n_2}y^{m_2}$

See Section 2.3 for the special cases $A_1=0$ and $A_2=0$.

2.4.1. Classification Table

Table 2.8 represents all solvable equations of the form $y''_{xx}=A_1x^{n_1}y^{m_1}+A_2x^{n_2}y^{m_2}$ whose solutions are outlined in Subsection 2.4.2. The two-parameter families (in the space of parameters m_1, m_2, n_1, and n_2), the one-parameter families, and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of m_1, the growth of m_2 (for identical m_1), the growth of n_1 (for identical m_1 and m_2), and the growth of n_2 (for identical m_1, m_2, and n_1). The number of the equation sought is indicated in the last column in this table.

TABLE 2.8
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
1	arbitrary	arbitrary	0	0	arbitrary	arbitrary	2.4.2.1
2	arbitrary	arbitrary	$-m_1-3$	$-m_2-3$	arbitrary	arbitrary	2.4.2.2
3	arbitrary	arbitrary	$-\frac{1}{2}(m_1+3)$	$-\frac{1}{2}(m_2+3)$	arbitrary	arbitrary	2.4.2.3
4	arbitrary	0	0	0	arbitrary	arbitrary	2.4.2.19
5	arbitrary	0	$-m_1-3$	-3	arbitrary	arbitrary	2.4.2.20
6	1	arbitrary	-2	-2	$-\dfrac{2(m_2+1)}{(m_2+3)^2}$	arbitrary	2.4.2.4
7	1	arbitrary	-2	$-m_2-1$	$-\dfrac{2(m_2+1)}{(m_2+3)^2}$	arbitrary	2.4.2.5
8	1	-3	arbitrary ($n_1 \neq -2$)	0	arbitrary	arbitrary	2.4.2.83
9	-7	-7	4	3	arbitrary	arbitrary	2.4.2.39
10	-5	-5	2	0	arbitrary	arbitrary	2.4.2.16
11	-3	-7	0	1	arbitrary	arbitrary	2.4.2.42
12	-3	-7	0	3	arbitrary	arbitrary	2.4.2.43
13	-3	-4	0	0	arbitrary	arbitrary	2.4.2.17
14	-3	-4	0	1	arbitrary	arbitrary	2.4.2.18
15	-2	-3	-2	0	arbitrary	arbitrary	2.4.2.88
16	-2	-3	1	0	arbitrary	arbitrary	2.4.2.87
17	-2	-2	-1	-2	arbitrary	arbitrary	2.4.2.28
18	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.48
19	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.49
20	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.24
21	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.90
22	$-\frac{5}{3}$	$-\frac{5}{3}$	0	$-\frac{2}{3}$	arbitrary	arbitrary	2.4.2.89
23	$-\frac{5}{3}$	$-\frac{5}{3}$	2	0	arbitrary	arbitrary	2.4.2.47
24	$-\frac{5}{3}$	$-\frac{5}{3}$	2	1	arbitrary	arbitrary	2.4.2.46
25	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2	arbitrary	arbitrary	2.4.2.81
26	$-\frac{3}{2}$	-2	0	1	arbitrary	arbitrary	2.4.2.80
27	$-\frac{7}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	2.4.2.25
28	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.102
29	$-\frac{4}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.101

TABLE 2.8 *Continued*
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
30	$-\frac{3}{5}$	$-\frac{7}{5}$	$-\frac{12}{5}$	$-\frac{13}{5}$	arbitrary	arbitrary	2.4.2.53
31	$-\frac{3}{5}$	$-\frac{7}{5}$	0	1	arbitrary	arbitrary	2.4.2.52
32	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	arbitrary	arbitrary	2.4.2.23
33	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{10}{3}$	arbitrary	arbitrary	2.4.2.55
34	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.59
35	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.57
36	$-\frac{1}{3}$	$-\frac{5}{3}$	0	0	arbitrary	arbitrary	2.4.2.56
37	$-\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.58
38	$-\frac{1}{3}$	$-\frac{5}{3}$	0	2	arbitrary	arbitrary	2.4.2.54
39	0	-2	-3	-2	arbitrary	arbitrary	2.4.2.108
40	0	-2	0	1	arbitrary	arbitrary	2.4.2.107
41	0	-1	-3	-2	arbitrary	arbitrary	2.4.2.22
42	0	-1	0	0	arbitrary	arbitrary	2.4.2.21
43	0	$-\frac{2}{3}$	-3	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.73
44	0	$-\frac{2}{3}$	0	0	arbitrary	arbitrary	2.4.2.72
45	0	$-\frac{1}{2}$	-4	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.96
46	0	$-\frac{1}{2}$	-3	$-\frac{7}{2}$	arbitrary	arbitrary	2.4.2.51
47	0	$-\frac{1}{2}$	-3	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.45
48	0	$-\frac{1}{2}$	-3	-2	arbitrary	arbitrary	2.4.2.106
49	0	$-\frac{1}{2}$	-3	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.85
50	0	$-\frac{1}{2}$	$-\frac{5}{3}$	$-\frac{7}{6}$	arbitrary	arbitrary	2.4.2.41
51	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	arbitrary	arbitrary	2.4.2.100
52	0	$-\frac{1}{2}$	$-\frac{3}{2}$	-2	arbitrary	arbitrary	2.4.2.79
53	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.78
54	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	arbitrary	arbitrary	2.4.2.99
55	0	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{4}{3}$	arbitrary	arbitrary	2.4.2.40
56	0	$-\frac{1}{2}$	0	-2	arbitrary	arbitrary	2.4.2.86
57	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	arbitrary	arbitrary	2.4.2.105
58	0	$-\frac{1}{2}$	0	0	arbitrary	arbitrary	2.4.2.44
59	0	$-\frac{1}{2}$	0	1	arbitrary	arbitrary	2.4.2.50

TABLE 2.8 *Continued*
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
60	0	$-\frac{1}{2}$	1	0	arbitrary	arbitrary	2.4.2.95
61	$\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{10}{3}$	$-\frac{7}{3}$	arbitrary	arbitrary	2.4.2.98
62	$\frac{1}{3}$	$-\frac{5}{3}$	0	1	arbitrary	arbitrary	2.4.2.97
63	1	-7	-2	-2	$\frac{15}{4}$	arbitrary	2.4.2.35
64	1	-7	-2	6	$\frac{15}{4}$	arbitrary	2.4.2.36
65	1	-4	-2	-2	6	arbitrary	2.4.2.31
66	1	-4	-2	3	6	arbitrary	2.4.2.32
67	1	-3	-5	0	arbitrary	arbitrary	2.4.2.84
68	1	-3	1	0	arbitrary	arbitrary	2.4.2.82
69	1	$-\frac{5}{2}$	-2	-2	12	arbitrary	2.4.2.64
70	1	$-\frac{5}{2}$	-2	$\frac{3}{2}$	12	arbitrary	2.4.2.65
71	1	-2	-2	-2	2	arbitrary	2.4.2.6
72	1	-2	-2	1	2	arbitrary	2.4.2.7
73	1	$-\frac{5}{3}$	-2	-2	$-\frac{3}{16}$	arbitrary	2.4.2.26
74	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$-\frac{3}{16}$	arbitrary	2.4.2.27
75	1	$-\frac{5}{3}$	-2	-2	$-\frac{9}{100}$	arbitrary	2.4.2.10
76	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$-\frac{9}{100}$	arbitrary	2.4.2.11
77	1	$-\frac{5}{3}$	-2	-2	$\frac{3}{4}$	arbitrary	2.4.2.12
78	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$\frac{3}{4}$	arbitrary	2.4.2.13
79	1	$-\frac{5}{3}$	-2	-2	$\frac{63}{4}$	arbitrary	2.4.2.66
80	1	$-\frac{5}{3}$	-2	$\frac{2}{3}$	$\frac{63}{4}$	arbitrary	2.4.2.67
81	1	$-\frac{7}{5}$	-2	-2	$-\frac{5}{36}$	arbitrary	2.4.2.29
82	1	$-\frac{7}{5}$	-2	$\frac{2}{5}$	$-\frac{5}{36}$	arbitrary	2.4.2.30
83	1	$-\frac{1}{2}$	-2	-2	$-\frac{2}{9}$	arbitrary	2.4.2.14
84	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	$-\frac{2}{9}$	arbitrary	2.4.2.15
85	1	$-\frac{1}{2}$	-2	-2	$-\frac{4}{25}$	arbitrary	2.4.2.8
86	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	$-\frac{4}{25}$	arbitrary	2.4.2.9
87	1	$-\frac{1}{2}$	-2	-2	20	arbitrary	2.4.2.33
88	1	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	20	arbitrary	2.4.2.34
89	1	0	-5	-3	arbitrary	arbitrary	2.4.2.77

TABLE 2.8 *Continued*
Solvable equations of the form $y''_{xx} = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$

No	m_1	m_2	n_1	n_2	A_1	A_2	Equation
90	1	0	1	0	arbitrary	arbitrary	2.4.2.76
91	1	$\frac{1}{2}$	-2	-2	$-\frac{12}{49}$	arbitrary	2.4.2.37
92	1	$\frac{1}{2}$	-2	$-\frac{3}{2}$	$-\frac{12}{49}$	arbitrary	2.4.2.38
93	2	0	-5	-4	arbitrary	arbitrary	2.4.2.92
94	2	0	-5	-3	arbitrary	arbitrary	2.4.2.69
95	2	0	$-\frac{20}{7}$	$-\frac{13}{7}$	arbitrary	arbitrary	2.4.2.94
96	2	0	$-\frac{20}{7}$	$-\frac{12}{7}$	arbitrary	arbitrary	2.4.2.71
97	2	0	$-\frac{15}{7}$	$-\frac{9}{7}$	arbitrary	arbitrary	2.4.2.70
98	2	0	$-\frac{15}{7}$	$-\frac{8}{7}$	arbitrary	arbitrary	2.4.2.93
99	2	0	0	0	arbitrary	arbitrary	2.4.2.68
100	2	0	0	1	arbitrary	arbitrary	2.4.2.91
101	2	1	-3	-2	arbitrary	$-\frac{6}{25}$	2.4.2.61
102	2	1	-3	-2	arbitrary	$\frac{6}{25}$	2.4.2.63
103	2	1	-2	-2	arbitrary	$-\frac{6}{25}$	2.4.2.60
104	2	1	-2	-2	arbitrary	$\frac{6}{25}$	2.4.2.62
105	3	1	-6	-5	arbitrary	arbitrary	2.4.2.104
106	3	1	0	1	arbitrary	arbitrary	2.4.2.103
107	3	2	$-\frac{18}{5}$	$-\frac{14}{5}$	arbitrary	arbitrary	2.4.2.74
108	3	2	$-\frac{12}{5}$	$-\frac{11}{5}$	arbitrary	arbitrary	2.4.2.75

2.4.2. Exact Solutions

1. $y''_{xx} = A_1 y^{m_1} + A_2 y^{m_2}, \qquad m_1 \neq -1,\ m_2 \neq -1.$

1°. Solution in the parametric form:

$$x = a \int \left(C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \frac{1}{2} a^{-2} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b^{1-m_2}(m_2+1)$.

2°. Solution in the parametric form:

$$x = a \int \left(C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = -\frac{1}{2} a^{-2} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b^{1-m_2}(m_2+1)$.

2. $y''_{xx} = A_1 x^{-m_1-3} y^{m_1} + A_2 x^{-m_2-3} y^{m_2}, \qquad m_1 \neq -1,\ m_2 \neq -1.$

1°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\tau\left[\int \left(C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \frac{1}{2} a^{1+m_1} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{1+m_2} b^{1-m_2}(m_2+1)$.

2°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\tau\left[\int \left(C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = -\frac{1}{2} a^{1+m_1} b^{1-m_1}(m_1+1)$, $A_2 = \pm\frac{1}{2} a^{1+m_2} b^{1-m_2}(m_2+1)$.

3. $y''_{xx} = A_1 x^{-\frac{m_1+3}{2}} y^{m_1} + A_2 x^{-\frac{m_2+3}{2}} y^{m_2}.$

1°. Solution in the parametric form with $m_1 \neq -1$ and $m_2 \neq -1$:

$$x = C_1^2 \exp\left[\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + \frac{2A_2}{m_2+1}\tau^{m_2+1}\right)^{-1/2} d\tau\right],$$
$$y = C_1\tau \exp\left[\frac{1}{2}\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + \frac{2A_2}{m_2+1}\tau^{m_2+1}\right)^{-1/2} d\tau\right].$$

2°. Solution in the parametric form with $m_1 \neq -1$ and $m_2 = -1$:

$$x = C_1^2 \exp\left[\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + 2A_2 \ln|\tau|\right)^{-1/2} d\tau\right],$$
$$y = C_1\tau \exp\left[\frac{1}{2}\int \left(C_2 + \frac{1}{4}\tau^2 + \frac{2A_1}{m_1+1}\tau^{m_1+1} + 2A_2 \ln|\tau|\right)^{-1/2} d\tau\right].$$

4. $y''_{xx} = -\dfrac{2(m+1)}{(m+3)^2} x^{-2} y + A x^{-2} y^m, \qquad m \neq -3,\ m \neq -1.$

Solution in the parametric form:

$$x = C_1\left[\int (1 \pm \tau^{m+1})^{-1/2} d\tau + C_2\right]^{\frac{m+3}{m-1}}, \quad y = b\tau\left[\int (1 \pm \tau^{m+1})^{-1/2} d\tau + C_2\right]^{\frac{2}{m-1}},$$

where $A = \pm\dfrac{(m+1)(m-1)^2}{2(m+3)^2} b^{1-m}$.

5. $\boldsymbol{y''_{xx} = -\dfrac{2(m+1)}{(m+3)^2} x^{-2} y + A x^{-m-1} y^m, \qquad m \neq -3,\ m \neq -1.}$

Solution in the parametric form:

$$x = C_1 \left[\int (1 \pm \tau^{m+1})^{-1/2}\, d\tau + C_2 \right]^{-\frac{m+3}{m-1}}, \quad y = bC_1 \tau \left[\int (1 \pm \tau^{m+1})^{-1/2}\, d\tau + C_2 \right]^{-\frac{m+1}{m-1}},$$

where $A = \pm \dfrac{(m+1)(m-1)^2}{2(m+3)^2} b^{1-m}$.

6. $\boldsymbol{y''_{xx} = 2x^{-2} y + A x^{-2} y^{-2}.}$

Solution in the parametric form:

$$x = C_1 \left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2 \right]^{-1/3},$$
$$y = b\tau \left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2 \right]^{-2/3},$$

where $A = -\frac{9}{2} b^3$.

7. $\boldsymbol{y''_{xx} = 2x^{-2} y + A x y^{-2}.}$

Solution in the parametric form:

$$x = C_1 \left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2 \right]^{1/3},$$
$$y = bC_1\tau \left[\sqrt{\tau(\tau+1)} - \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2 \right]^{-1/3},$$

where $A = -\frac{9}{2} b^3$.

8. $\boldsymbol{y''_{xx} = -\frac{4}{25} x^{-2} y + A x^{-2} y^{-1/2}.}$

Solution in the parametric form:

$$x = C_1 (\tau^3 - 3\tau + C_2)^{-5/3}, \quad y = b(\tau^2 - 1)^2 (\tau^3 - 3\tau + C_2)^{-4/3},$$

where $A = \pm \frac{4}{25} b^{3/2}$.

9. $\boldsymbol{y''_{xx} = -\frac{4}{25} x^{-2} y + A x^{-1/2} y^{-1/2}.}$

Solution in the parametric form:

$$x = C_1 (\tau^3 - 3\tau + C_2)^{5/3}, \quad y = bC_1(\tau^2 - 1)^2 (\tau^3 - 3\tau + C_2)^{1/3},$$

where $A = \pm \frac{4}{25} b^{3/2}$.

10. $\boldsymbol{y''_{xx} = -\frac{9}{100} x^{-2} y + A x^{-2} y^{-5/3}.}$

Solution in the parametric form:

$$x = C_1 \left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3) \right]^{-5/4},$$
$$y = b(\tau^3 - 3\tau + C_2)^{3/2} \left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3) \right]^{-9/8},$$

where $A = \pm \frac{9}{100} b^{8/3}$.

11. $y''_{xx} = -\frac{9}{100}x^{-2}y + Ax^{2/3}y^{-5/3}$.

Solution in the parametric form:

$$x = C_1[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{5/4}, \quad y = bC_1(\tau^3 - 3\tau + C_2)^{3/2}[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)]^{1/8},$$

where $A = \pm\frac{9}{100}b^{8/3}$.

12. $y''_{xx} = \frac{3}{4}x^{-2}y + Ax^{-2}y^{-5/3}$.

Solution in the parametric form:

$$x = C_1(\tau^3 \pm 3\tau + C_2)^{-1/2}, \quad y = b(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-3/4},$$

where $A = \pm\frac{4}{3}b^{8/3}$.

13. $y''_{xx} = \frac{3}{4}x^{-2}y + Ax^{2/3}y^{-5/3}$.

Solution in the parametric form:

$$x = C_1(\tau^3 \pm 3\tau + C_2)^{1/2}, \quad y = bC_1(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1/4},$$

where $A = \pm\frac{4}{3}b^{8/3}$.

14. $y''_{xx} = -\frac{2}{9}x^{-2}y + Ax^{-2}y^{-1/2}$.

Solution in the parametric form:

$$x = C_1(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)^{-3}, \quad \omega = \sqrt{3}\,k\tau,$$
$$y = bk^2(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)^{-2}\big[2C_1e^{2k\tau} + C_2e^{-k\tau}\big(\sqrt{3}\cos\omega - \sin\omega\big)\big]^2,$$

where $A = \frac{16}{9}bk^3$.

15. $y''_{xx} = -\frac{2}{9}x^{-2}y + Ax^{-1/2}y^{-1/2}$.

Solution in the parametric form:

$$x = C_1(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)^3, \quad \omega = \sqrt{3}\,k\tau,$$
$$y = bk^2C_1(C_1e^{2k\tau} + C_2e^{-k\tau}\sin\omega)\big[2C_1e^{2k\tau} + C_2e^{-k\tau}\big(\sqrt{3}\cos\omega - \sin\omega\big)\big]^2,$$

where $A = \frac{16}{9}bk^3$.

16. $y''_{xx} = A_1x^2y^{-5} + A_2y^{-5}$.

Solution in the parametric form:

$$x = \left(\frac{A_2}{A_1}\right)^{1/2}\tan\left[\int\left(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\right)^{-1/2}d\tau + C_2\right],$$
$$y = A_2^{1/2}\tau\left\{\cos\left[\int\left(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\right)^{-1/2}d\tau + C_2\right]\right\}^{-1}.$$

17. $y''_{xx} = A_1 y^{-3} + A_2 y^{-4}.$

1°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \mp a^{-2} b^4$, $A_2 = -\frac{3}{2} a^{-2} b^5$.

2°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \mp a^{-2} b^4$, $A_2 = \frac{3}{2} a^{-2} b^5$.

18. $y''_{xx} = A_1 y^{-3} + A_2 x y^{-4}.$

1°. Solution in the parametric form:

$$x=a\left[\int (C_1+\tau^{-3}\pm\tau^{-2})^{-1/2}\,d\tau+C_2\right]^{-1}, \quad y=b\tau\left[\int (C_1+\tau^{-3}\pm\tau^{-2})^{-1/2}\,d\tau+C_2\right]^{-1},$$

where $A_1 = \mp a^{-2} b^4$, $A_2 = -\frac{3}{2} a^{-3} b^5$.

2°. Solution in the parametric form:

$$x=a\left[\int (C_1-\tau^{-3}\pm\tau^{-2})^{-1/2}\,d\tau+C_2\right]^{-1}, \quad y=b\tau\left[\int (C_1-\tau^{-3}\pm\tau^{-2})^{-1/2}\,d\tau+C_2\right]^{-1},$$

where $A_1 = \mp a^{-2} b^4$, $A_2 = \frac{3}{2} a^{-3} b^5$.

19. $y''_{xx} = A_1 y^m + A_2, \qquad m \neq -1.$

1°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 + \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = \frac{1}{2} a^{-2} b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b$.

2°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 - \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right], \qquad y = b\tau,$$

where $A_1 = -\frac{1}{2} a^{-2} b^{1-m}(m+1)$, $A_2 = \pm\frac{1}{2} a^{-2} b$.

3°. See equation 2.4.2.21 for the case $m = -1$.

20. $y''_{xx} = A_1 x^{-m-3} y^m + A_2 x^{-3}, \qquad m \neq -1.$

1°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 + \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\tau\left[\int \left(C_1 + \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \frac{1}{2} a^{1+m} b^{1-m} (m+1)$, $A_2 = \pm\frac{1}{2} ab$.

2°. Solution in the parametric form:

$$x = a\left[\int \left(C_1 - \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\tau\left[\int \left(C_1 - \tau^{m+1} \pm \tau\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = -\frac{1}{2} a^{1+m} b^{1-m} (m+1)$, $A_2 = \pm\frac{1}{2} ab$.

21. $y''_{xx} = A_1 + A_2 y^{-1}.$

Solution: $\quad x = \displaystyle\int (C_1 + 2A_1 y + 2A_2 \ln|y|)^{-1/2} dy + C_2.$

22. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-1}.$

Solution in the parametric form:

$$x = \left[\int (C_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1/2}\, d\tau + C_2\right]^{-1},$$
$$y = \tau\left[\int (C_1 + 2A_1\tau + 2A_2 \ln|\tau|)^{-1/2}\, d\tau + C_2\right]^{-1}.$$

23. $y''_{xx} = A_1 x^{-5/2} y^{-1/2} + A_2 x^{-7/2} y^{-1/2}.$

Solution in the parametric form:

$$x = \frac{1}{F}, \qquad y = \frac{k^2}{F}\left\{2C_1 e^{2k\tau} + C_2 e^{-k\tau}\left[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\right]\right\}^2,$$

where $F = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau) - \dfrac{A_1}{A_2}$, $A_2 = 16k^3$, $\omega = k\sqrt{3}$.

24. $y''_{xx} = A_1 x^{-4/3} y^{-5/3} + A_2 x^{-7/3} y^{-5/3}.$

Solution in the parametric form:

$$x = \left(\tfrac{1}{36} A_2 \tau^4 + C_1 \tau^3 + C_2 \tau + C_3\right)^{-1},$$
$$y = \left(\tfrac{1}{9} A_2 \tau^3 + 3C_1 \tau^2 + C_2\right)^{3/2} \left(\tfrac{1}{36} A_2 \tau^4 + C_1 \tau^3 + C_2 \tau + C_3\right)^{-1},$$

where $9C_1 C_2 = A_1 + A_2 C_3$.

25. $y''_{xx} = A_1 x^{-8/5} y^{-7/5} + A_2 x^{-13/5} y^{-7/5}$.

Solution in the parametric form:

$$x = \Big(aC_1^4 F - \frac{A_1}{A_2}\Big)^{-1}, \qquad y = bC_1^5 S^{5/2}\Big(aC_1^4 F - \frac{A_1}{A_2}\Big)^{-1},$$

where $S = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\sqrt{3}\,k\tau)$, $F = (S'_\tau)^2 - 2SS''_{\tau\tau}$, $A_2 = -\dfrac{5}{1024}a^{-3}b^{12/5}k^{-6}$.

26. $y''_{xx} = -\frac{3}{16}x^{-2}y + Ax^{-2}y^{-5/3}$.

1°. Solution in the parametric form with $A < 0$:

$$x = C_1[\cosh(\tau + C_2)\cos\tau]^{-2}[\tanh(\tau + C_2) + \tan\tau]^{-2},$$
$$y = b[\tanh(\tau + C_2) + \tan\tau]^{-3/2},$$

where $A = -\frac{3}{64}b^{8/3}$.

2°. Solution in the parametric form with $A > 0$:

$$x = C_1[\sinh\tau + \cos(\tau + C_2)]^{-2},$$
$$y = b[\cosh\tau - \sin(\tau + C_2)]^{3/2}[\sinh\tau + \cos(\tau + C_2)]^{-3/2},$$

where $A = \frac{3}{16}b^{8/3}$.

27. $y''_{xx} = -\frac{3}{16}x^{-2}y + Ax^{2/3}y^{-5/3}$.

1°. Solution in the parametric form with $A < 0$:

$$x = C_1[\cosh(\tau + C_2)\cos\tau]^{2}[\tanh(\tau + C_2) + \tan\tau]^{2},$$
$$y = bC_1[\cosh(\tau + C_2)\cos\tau]^{2}[\tanh(\tau + C_2) + \tan\tau]^{1/2},$$

where $A = -\frac{3}{64}b^{8/3}$.

2°. Solution in the parametric form with $A > 0$:

$$x = C_1[\sinh\tau + \cos(\tau + C_2)]^{2},$$
$$y = bC_1[\cosh\tau - \sin(\tau + C_2)]^{3/2}[\sinh\tau + \cos(\tau + C_2)]^{1/2},$$

where $A = \frac{3}{16}b^{8/3}$.

28. $y''_{xx} = A_1 x^{-1} y^{-2} + A_2 x^{-2} y^{-2}$.

Solution in the parametric form:

$$x = \Big\{aC_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big] - \frac{A_1}{A_2}\Big\}^{-1},$$
$$y = bC_1\tau^{2/3}Z^2\Big\{aC_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big] - \frac{A_1}{A_2}\Big\}^{-1},$$

where

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

$J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions; $A_2 = -\frac{9}{2}a^{-3}b^3$.

▶ *In the solutions of equations 29–30, the following notation is used:*

$$S_1 = C_1 e^{2k\tau} + C_2 e^{-k\tau} \sin\bigl(\sqrt{3}\, k\tau\bigr),$$
$$S_2 = 2kC_1 e^{2k\tau} + kC_2 e^{-k\tau} \bigl[\sqrt{3} \cos\bigl(\sqrt{3}\, k\tau\bigr) - \sin\bigl(\sqrt{3}\, k\tau\bigr)\bigr],$$
$$S_3 = S_2^2 - 2S_1 (S_2)'_\tau.$$

29. $\boldsymbol{y''_{xx} = -\frac{5}{36} x^{-2} y + A x^{-2} y^{-7/5}}$.

Solution in the parametric form:

$$x = C_1 S_3^{-3/2}, \qquad y = b S_1^{5/2} S_3^{-5/4},$$

where $A = -\frac{5}{2304} b^{12/5} k^{-6}$.

30. $\boldsymbol{y''_{xx} = -\frac{5}{36} x^{-2} y + A x^{2/5} y^{-7/5}}$.

Solution in the parametric form:

$$x = C_1 S_3^{3/2}, \qquad y = b C_1 S_1^{5/2} S_3^{1/4},$$

where $A = -\frac{5}{2304} b^{12/5} k^{-6}$.

▶ *In the solutions of equations 31–39, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad I = \int \tau R^{-1}\, d\tau, \quad F_1 = 2\tau I + C_2 \tau \mp R, \quad F_2 = \tau^{-1}(R F_1 - 1),$$

where $I = I(\tau)$ is the incomplete elliptic integral of the second kind in the Weierstrass form.

31. $\boldsymbol{y''_{xx} = 6x^{-2} y + A x^{-2} y^{-4}}$.

Solution in the parametric form:

$$x = C_1 \tau^{-1/5} F_1^{1/5}, \qquad y = b \tau^{-3/5} F_1^{-2/5},$$

where $A = \mp 150 b^5$.

32. $\boldsymbol{y''_{xx} = 6x^{-2} y + A x^3 y^{-4}}$.

Solution in the parametric form:

$$x = C_1 \tau^{1/5} F_1^{-1/5}, \qquad y = b C_1 \tau^{-2/5} F_1^{-3/5},$$

where $A = \mp 150 b^5$.

33. $\boldsymbol{y''_{xx} = 20x^{-2} y + A x^{-2} y^{-1/2}}$.

Solution in the parametric form:

$$x = C_1 F_1^{1/3}, \qquad y = b F_1^{-4/3} F_2^2,$$

where $A = \pm 108 b^{3/2}$.

34. $y''_{xx} = 20x^{-2}y + Ax^{-1/2}y^{-1/2}$.

Solution in the parametric form:

$$x = C_1 F_1^{-1/3}, \qquad y = bC_1 F_1^{-5/3} F_2^2,$$

where $A = \pm 108 b^{3/2}$.

35. $y''_{xx} = \frac{15}{4}x^{-2}y + Ax^{-2}y^{-7}$.

Solution in the parametric form:

$$x = C_1 (4\tau F_1^2 \mp F_2^2)^{1/4}, \qquad y = bF_1^{1/2} (4\tau F_1^2 \mp F_2^2)^{-3/8},$$

where $A = \pm \frac{3}{4} b^8$.

36. $y''_{xx} = \frac{15}{4}x^{-2}y + Ax^{6}y^{-7}$.

Solution in the parametric form:

$$x = C_1 (4\tau F_1^2 \mp F_2^2)^{-1/4}, \qquad y = bC_1 F_1^{1/2} (4\tau F_1^2 \mp F_2^2)^{-5/8},$$

where $A = \pm \frac{3}{4} b^8$.

37. $y''_{xx} = -\frac{12}{49}x^{-2}y + Ax^{-2}y^{1/2}$.

Solution in the parametric form:

$$x = C_1 (I + C_2)^{-7}, \qquad y = b\tau^2 (I + C_2)^{-4},$$

where $A = \pm \frac{12}{49} b^{1/2}$.

38. $y''_{xx} = -\frac{12}{49}x^{-2}y + Ax^{-3/2}y^{1/2}$.

Solution in the parametric form:

$$x = C_1 (I + C_2)^{7}, \qquad y = bC_1\tau^2 (I + C_2)^{3},$$

where $A = \pm \frac{12}{49} b^{1/2}$.

39. $y''_{xx} = A_1 x^4 y^{-7} + A_2 x^3 y^{-7}$.

Solution in the parametric form:

$$x = \left[aC_1^8 (4\tau F_1^2 \mp F_2^2) - \frac{A_1}{A_2} \right]^{-1}, \qquad y = bC_1^3 F_1^{1/2} \left[aC_1^8 (4\tau F_1^2 \mp F_2^2) - \frac{A_1}{A_2} \right]^{-1},$$

where $A_2 = \pm \frac{3}{64} a^{-3} b^8$.

► *In the solutions of equations 40–43, the following notation is used:*

$$\begin{aligned}
R_1 &= \left(C_1+\tau^{-3}\pm\tau^{-2}\right)^{1/2}, & R_2 &= \left(C_1-\tau^{-3}\pm\tau^{-2}\right)^{1/2},\\
E_1 &= \int R_1^{-1}\,d\tau + C_2, & E_2 &= \int R_2^{-1}\,d\tau + C_2,\\
F_1 &= \tau - R_1E_1, & F_2 &= \tau - R_2E_2,\\
H_1 &= 3\tau^3F_1^2+3(1\pm\tau)E_1^2, & H_2 &= 3\tau^3F_2^2+3(-1\pm\tau)E_2^2.
\end{aligned}$$

40. $\boldsymbol{y''_{xx} = A_1x^{-4/3} + A_2x^{-4/3}y^{-1/2}}$.

Solutions in the parametric form:

$$x = a\tau^{-3}E_k^3, \qquad y = bF_k^2,$$

where $A_1 = \mp\frac{2}{9}a^{-2/3}b$, $A_2 = \frac{1}{3}a^{-2/3}b^{3/2}(-1)^k$; $k=1$ and $k=2$.

41. $\boldsymbol{y''_{xx} = A_1x^{-5/3} + A_2x^{-7/6}y^{-1/2}}$.

Solutions in the parametric form:

$$x = a\tau^3E_k^{-3}, \qquad y = b\tau^3E_k^{-3}F_k^2,$$

where $A_1 = \pm\frac{2}{9}a^{-1/3}b$, $A_2 = \frac{1}{3}a^{-5/6}b^{3/2}(-1)^{k+1}$; $k=1$ and $k=2$.

42. $\boldsymbol{y''_{xx} = A_1y^{-3} + A_2xy^{-7}}$.

Solutions in the parametric form:

$$x = a\tau^{-3}H_k, \qquad y = b\tau^{-1/2}E_k^{1/2},$$

where $A_1 = \mp\frac{1}{36}a^{-2}b^4$, $A_2 = -\frac{1}{36}a^{-3}b^8$; $k=1$ and $k=2$.

43. $\boldsymbol{y''_{xx} = A_1y^{-3} + A_2x^3y^{-7}}$.

Solutions in the parametric form:

$$x = a\tau^3H_k^{-1}, \qquad y = b\tau^{5/2}E_k^{1/2}H_k^{-1},$$

where $A_1 = \pm\frac{1}{36}a^{-2}b^4$, $A_2 = -\frac{1}{36}a^{-5}b^8$; $k=1$ and $k=2$.

► *In the solutions of equations 44–45, the following notation is used:*

$$f_1 = \begin{cases} C_1e^{k\tau}+C_2e^{-k\tau}-\dfrac{A_2}{A_1}\tau & \textit{if } A_1>0,\\[2ex] C_1\sin(k\tau)+C_2\cos(k\tau)-\dfrac{A_2}{A_1}\tau & \textit{if } A_1<0,\end{cases}$$

$$f_2 = \begin{cases} k(C_1e^{k\tau}-C_2e^{-k\tau})-\dfrac{A_2}{A_1} & \textit{if } A_1>0,\\[2ex] k[C_1\sin(k\tau)-C_2\cos(k\tau)]-\dfrac{A_2}{A_1} & \textit{if } A_1<0,\end{cases}$$

where $k=\sqrt{\frac{1}{2}|A_1|}$.

44. $\boldsymbol{y''_{xx} = A_1 + A_2 y^{-1/2}}$.

Solution in the parametric form:

$$x = f_1, \qquad y = f_2^2.$$

45. $\boldsymbol{y''_{xx} = A_1 x^{-3} + A_2 x^{-5/2} y^{-1/2}}$.

Solution in the parametric form:

$$x = f_1^{-1}, \qquad y = f_1^{-1} f_2^2.$$

► *In the solutions of equations 46–47, the following notation is used:*

For $A_1 > 0$,

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = \left(\tfrac{4}{3}A_1\right)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau);$$

For $A_1 < 0$,

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = \left(-\tfrac{1}{3}A_1\right)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

46. $\boldsymbol{y''_{xx} = A_1 x^2 y^{-5/3} + A_2 x y^{-5/3}}$.

Solution in the parametric form:

$$x = T_1 - \frac{A_2}{2A_1}, \qquad y = T_2^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &4C_1C_2 + C_3^2 = \tfrac{1}{4}A_1^{-2}A_2^2 &&\text{if } A_1 > 0,\\ &C_1C_3 = \tfrac{1}{16}A_1^{-2}A_2^2 &&\text{if } A_1 < 0. \end{aligned}$$

47. $\boldsymbol{y''_{xx} = A_1 x^2 y^{-5/3} + A_2 y^{-5/3}}$.

Solution in the parametric form:

$$x = T_1, \qquad y = T_2^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &4C_1C_2 + C_3^2 = -\tfrac{1}{2}A_1^{-1}A_2 &&\text{if } A_1 > 0,\\ &C_1C_3 = -\tfrac{1}{4}A_1^{-1}A_2 &&\text{if } A_1 < 0. \end{aligned}$$

► *In the solutions of equations 48–49, the following notation is used:*

For $A_2 > 0$,

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = (\tfrac{4}{3} A_2)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau);$$

For $A_2 < 0$,

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = (-\tfrac{1}{3} A_2)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

48. $\boldsymbol{y''_{xx} = A_1 x^{-7/3} y^{-5/3} + A_2 x^{-10/3} y^{-5/3}}$.

Solution in the parametric form:

$$x = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1}, \qquad y = T_2^{3/2}\left(T_1 - \frac{A_1}{2A_2}\right)^{-1},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &4C_1C_2 + C_3^2 = \tfrac{1}{4} A_1^2 A_2^{-2} && \text{if } A_2 > 0,\\ &C_1C_3 = \tfrac{1}{16} A_1^2 A_2^{-2} && \text{if } A_2 < 0. \end{aligned}$$

49. $\boldsymbol{y''_{xx} = A_1 x^{-4/3} y^{-5/3} + A_2 x^{-10/3} y^{-5/3}}$.

Solution in the parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1} T_2^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &4C_1C_2 + C_3^2 = -\tfrac{1}{2} A_1 A_2^{-1} && \text{if } A_2 > 0,\\ &C_1C_3 = -\tfrac{1}{4} A_1 A_2^{-1} && \text{if } A_2 < 0. \end{aligned}$$

► *In the solutions of equations 50–53, the following notation is used:*

$$\begin{aligned} R_1 &= C_1 \tau^{k_1} + C_2 \tau^{k_2} + C_3 \tau^{k_3},\\ R_2 &= (C_1 + C_2\tau)e^{k\tau} + C_3 e^{\omega\tau},\\ R_3 &= C_1 e^{k\tau} + e^{s\tau}(C_2 \sin\omega\tau + C_3 \cos\omega\tau),\\ Q_1 &= C_1 k_1 \tau^{k_1} + C_2 k_2 \tau^{k_2} + C_3 k_3 \tau^{k_3},\\ Q_2 &= (kC_1 + C_2 + kC_2\tau)e^{k\tau} + \omega C_3 e^{\omega\tau},\\ Q_3 &= kC_1 e^{k\tau} + e^{s\tau}[(sC_2 - \omega C_3)\sin\omega\tau + (sC_3 + \omega C_2)\cos\omega\tau],\\ S_1 &= \tau(Q_1)'_\tau, \quad S_2 = (Q_2)'_\tau, \quad S_3 = (Q_3)'_\tau, \end{aligned}$$

where k_1, k_2, *and* k_3 *(real numbers) or* k *and* $s \pm i\omega$ *(one real and two complex numbers) are the roots of the cubic equation* $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. *Subscripts of* *functions R_m, Q_m, and S_m $(m = 1,\ 2,\ 3)$ are selected depending on the sign of the following expression:*

$$2B_2^3 - 27B_1^2 \quad \begin{cases} > 0 & \textit{subscript 1}, \\ = 0 & \textit{subscript 2}, \\ < 0 & \textit{subscript 3}; \end{cases}$$

if $2B_2^3 = 27B_1^2$ *(subscript 2), then*

$$\begin{aligned} k &= (\tfrac{1}{6}B_2)^{1/2}, \quad \omega = -2(\tfrac{1}{6}B_2)^{1/2} \qquad \textit{if} \quad B_1 < 0, \\ k &= -(\tfrac{1}{6}B_2)^{1/2}, \quad \omega = 2(\tfrac{1}{6}B_2)^{1/2} \qquad \textit{if} \quad B_1 > 0. \end{aligned}$$

Remark. *The expressions for R_m, Q_m contain three constants C_1, C_2, and C_3. One of them may be arbitrarily fixed to set it equal to any nonzero number (for example, we may set $C_3 = \pm 1$), and the other constant may be arbitrary.*

50. $\boldsymbol{y''_{xx} = A_1 + A_2 x y^{-1/2}}$.

Solution in the parametric form:

$$x = R_m, \quad y = Q_m^2, \qquad A_1 = B_2, \quad A_2 = B_1.$$

51. $\boldsymbol{y''_{xx} = A_1 x^{-3} + A_2 x^{-7/2} y^{-1/2}}$.

Solution in the parametric form:

$$x = R_m^{-1}, \quad y = R_m^{-1} Q_m^2, \qquad A_1 = B_2, \quad A_2 = B_1.$$

52. $\boldsymbol{y''_{xx} = A_1 y^{-3/5} + A_2 x y^{-7/5}}$.

Solution in the parametric form:

$$x = a(2Q_m^2 - 4R_m S_m + B_2 R_m^2), \quad y = bR_m^{5/2},$$

where $A_1 = -ab^{-4/5} A_2 B_2$, $A_2 = -\frac{5}{32} a^{-3} b^{12/5} B_1^{-2}$.

53. $\boldsymbol{y''_{xx} = A_1 x^{-12/5} y^{-3/5} + A_2 x^{-13/5} y^{-7/5}}$.

Solution in the parametric form:

$$x = a\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-1}, \quad y = bR_m^{5/2}\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-1},$$

where $A_1 = \frac{5}{32} a^{2/5} b^{8/5} B_1^{-2} B_2$, $A_2 = -\frac{5}{32} a^{3/5} b^{12/5} B_1^{-2}$.

► *In the solutions of equations 54–55, the following notation is used:*

1°. *For $A_2 > 0$, $A_1 \neq 0$:*

$$\begin{aligned} T_1 &= C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin \omega\tau, \\ T_2 &= k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + \omega C_3 \cos \omega\tau, \end{aligned}$$

where $k = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1/2} + A_1]\}^{1/2}$, $\omega = \{\frac{2}{3}[(A_1^2 + 3A_2)^{1/2} - A_1]\}^{1/2}$; *arbitrary constants C_1, C_2, and C_3 are related by the constraint $4k^2 C_1 C_2 + \omega^2 C_3^2 = 0$.*

2°. *For* $-A_1^2 < 3A_2 < 0,\ \ A_1 > 0$:

$$T_1 = C_1\tau^{k_1} + C_2\tau^{-k_1} + C_3\tau^{k_2} + C_4\tau^{-k_2},$$
$$T_2 = k_1(C_1\tau^{k_1} - C_2\tau^{-k_1}) + k_2(C_3\tau^{k_2} - C_4\tau^{-k_2}),$$

where $k_1 = \{\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1/2}]\}^{1/2},\ \ k_2 = \{\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$; *arbitrary constants* C_1, C_2, *and* C_3 *are related by the constraint* $(C_1C_2 + C_3C_4)(A_1^2 + 3A_2)^{1/2} + (C_1C_2 - C_3C_4)A_1 = 0$.

3°. *For* $-A_1^2 < 3A_2 < 0,\ \ A_1 < 0$:

$$T_1 = C_1\sin\omega_1\tau + C_2\cos\omega_1\tau + C_3\sin\omega_2\tau,$$
$$T_2 = \omega_1(C_1\cos\omega_1\tau - C_2\sin\omega_1\tau) + \omega_2 C_3\cos\omega_2\tau,$$

where $\omega_1 = \{-\frac{2}{3}[A_1 + (A_1^2 + 3A_2)^{1/2}]\}^{1/2},\ \ \omega_2 = \{-\frac{2}{3}[A_1 - (A_1^2 + 3A_2)^{1/2}]\}^{1/2}$; *arbitrary constants* C_1, C_2, *and* C_3 *are related by* $\omega_1^2(C_1^2 + C_2^2) - \omega_2^2C_3^2 = 0$.

4°. *For* $A_1^2 + 3A_2 = 0,\ \ A_1 > 0$:

$$T_1 = (C_1 + C_2\tau)e^{k\tau} + (C_3 + C_4\tau)e^{-k\tau},$$
$$T_2 = (kC_1 + C_2 + kC_2\tau)e^{k\tau} - (kC_3 - C_4 + kC_4\tau)e^{-k\tau},$$

where $k = (\frac{2}{3}A_1)^{1/2}$; *arbitrary constants* C_1, C_2, *and* C_3 *are related by the constraint* $(C_1C_4 - C_2C_3)(\frac{2}{3}A_1)^{1/2} + 2C_2C_4 = 0$.

5°. *For* $A_1^2 + 3A_2 = 0,\ \ A_1 < 0$:

$$T_1 = (C_1 + C_2\tau)\sin\omega\tau + C_3\tau\cos\omega\tau,$$
$$T_2 = (\omega C_1 + C_3 + \omega C_2\tau)\cos\omega\tau + (C_2 - \omega C_3\tau)\sin\omega\tau,$$

where $\omega = (-\frac{2}{3}A_1)^{1/2}$; *arbitrary constants* C_1, C_2, *and* C_3 *are related by the constraint* $C_1C_3(-\frac{2}{3}A_1)^{1/2} + C_2^2 + C_3^2 = 0$.

6°. *For* $3A_2 < -A_1^2$:

$$T_1 = e^{k\tau}(C_1\sin\omega\tau + C_2\cos\omega\tau) + C_3e^{-k\tau}\sin\omega\tau,$$
$$T_2 = e^{k\tau}[(kC_2 + \omega C_1)\cos\omega\tau + (kC_1 - \omega C_2)\sin\omega\tau)] + C_3e^{-k\tau}(\omega\cos\omega\tau - k\sin\omega\tau),$$

where $k = \{\frac{1}{3}[A_1 + (-3A_2)^{1/2}]\}^{1/2},\ \ \omega = \{\frac{1}{3}[-A_1 + (-3A_2)^{1/2}]\}^{1/2}$; *arbitrary constants* C_1, C_2, *and* C_3 *are related by* $C_2A_1 + C_1(-A_1^2 - 3A_2)^{1/2} = 0$.

54. $\boldsymbol{y''_{xx} = A_1 y^{-1/3} + A_2 x^2 y^{-5/3}}$.

Solution in the parametric form:

$$x = T_1, \qquad y = T_2^{3/2}.$$

55. $\boldsymbol{y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-10/3} y^{-5/3}}$.

Solution in the parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1}T_2^{3/2}.$$

▶ *In the solutions of equations 56–59, the following notation is used:*

$$T_1 = \begin{cases} C_1 e^{\omega\tau} + C_2 e^{-\omega\tau} + C_3\tau & \text{if } A_1 > 0, \\ C_1 \sin\omega\tau + C_2\cos\omega\tau + C_3\tau & \text{if } A_1 < 0; \end{cases}$$

$$T_2 = \begin{cases} \omega(C_1 e^{\omega\tau} - C_2 e^{-\omega\tau}) + C_3 & \text{if } A_1 > 0, \\ \omega(C_1\cos\omega\tau - C_2\sin\omega\tau) + C_3 & \text{if } A_1 < 0; \end{cases}$$

where $\omega = |\frac{4}{3}A_1|^{1/2}$.

56. $y''_{xx} = A_1 y^{-1/3} + A_2 y^{-5/3}$.

Solution in the parametric form:

$$x = T_1, \qquad y = T_2^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3(A_1C_3^2 + A_2) + 16A_1^2C_1C_2 = 0 && \text{if } A_1 > 0, \\ &3(A_1C_3^2 + A_2) + 4A_1^2(C_1^2 + C_2^2) = 0 && \text{if } A_1 < 0. \end{aligned}$$

57. $y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-4/3} y^{-5/3}$.

Solution in the parametric form:

$$x = T_1^{-1}, \qquad y = T_1^{-1}T_2^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3(A_1C_3^2 + A_2) + 16A_1^2C_1C_2 = 0 && \text{if } A_1 > 0, \\ &3(A_1C_3^2 + A_2) + 4A_1^2(C_1^2 + C_2^2) = 0 && \text{if } A_1 < 0. \end{aligned}$$

58. $y''_{xx} = A_1 y^{-1/3} + A_2 x y^{-5/3}$.

Solution in the parametric form:

$$x = T_1 - \frac{A_2}{4A_1}\tau^2, \qquad y = \Big(T_2 - \frac{A_2}{2A_1}\tau\Big)^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3A_1C_3^2 + 16A_1^2C_1C_2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 > 0, \\ &3A_1C_3^2 + 4A_1^2(C_1^2 + C_2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

59. $y''_{xx} = A_1 x^{-8/3} y^{-1/3} + A_2 x^{-7/3} y^{-5/3}$.

Solution in the parametric form:

$$x = \Big(T_1 - \frac{A_2}{4A_1}\tau^2\Big)^{-1}, \qquad y = \Big(T_1 - \frac{A_2}{4A_1}\tau^2\Big)^{-1}\Big(T_2 - \frac{A_2}{2A_1}\tau\Big)^{3/2},$$

where arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3A_1C_3^2 + 16A_1^2C_1C_2 + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 > 0, \\ &3A_1C_3^2 + 4A_1^2(C_1^2 + C_2^2) + \tfrac{9}{16}A_1^{-2}A_2^2 = 0 && \text{if } A_1 < 0. \end{aligned}$$

▶ *In the solutions of equations 60–67, the following notation is used:*

$$f = \sqrt{\pm(4\wp^3 - 1)}, \qquad \tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2.$$

Function $\wp = \wp(\tau)$ is defined implicitly in terms of the above elliptic integral of the first kind. For the upper sign, function $\wp$ coincides with the classical elliptic Weirstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$. In the solution given below, one can take $\wp$ as the parameter instead of τ and use the explicit dependence $\tau = \tau(\wp)$.

60. $y''_{xx} = Ax^{-2}y^2 - \frac{6}{25}x^{-2}y.$

Solution in the parametric form:

$$x = C_1\tau^5, \quad y = b\tau^2\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

61. $y''_{xx} = Ax^{-3}y^2 - \frac{6}{25}x^{-2}y.$

Solution in the parametric form:

$$x = C_1\tau^{-5}, \quad y = bC_1\tau^{-3}\wp, \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

62. $y''_{xx} = Ax^{-2}y^2 + \frac{6}{25}x^{-2}y.$

Solution in the parametric form:

$$x = C_1\tau^5, \quad y = b(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

63. $y''_{xx} = Ax^{-3}y^2 + \frac{6}{25}x^{-2}y.$

Solution in the parametric form:

$$x = C_1\tau^{-5}, \quad y = bC_1\tau^{-5}(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm\tfrac{6}{25}b^{-1}.$$

64. $y''_{xx} = 12x^{-2}y + Ax^{-2}y^{-5/2}.$

Solution in the parametric form:

$$x = C_1\wp^{2/7}\bigl(f \pm 2\tau\wp^2\bigr)^{-1/7}, \qquad y = b\wp^{-6/7}\bigl(f \pm 2\tau\wp^2\bigr)^{-4/7},$$

where $A = \mp 147b^{7/2}$.

65. $y''_{xx} = 12x^{-2}y + Ax^{3/2}y^{-5/2}.$

Solution in the parametric form:

$$x = C_1\wp^{-2/7}\bigl(f \pm 2\tau\wp^2\bigr)^{1/7}, \qquad y = bC_1\wp^{-8/7}\bigl(f \pm 2\tau\wp^2\bigr)^{-3/7},$$

where $A = \mp 147b^{7/2}$.

66. $y''_{xx} = \frac{63}{4}x^{-2}y + Ax^{-2}y^{-5/3}.$

Solution in the parametric form:

$$x = C_1(\tau f + 2\wp)^{-1/4}, \qquad y = b(\tau f + 2\wp)^{-9/8}\bigl(f \pm 2\tau\wp^2\bigr)^{3/2},$$

where $A = -\frac{32}{3}b^{8/3}$.

67. $y''_{xx} = \frac{63}{4}x^{-2}y + Ax^{2/3}y^{-5/3}.$

Solution in the parametric form:

$$x = C_1(\tau f + 2\wp)^{1/4}, \qquad y = bC_1(\tau f + 2\wp)^{-7/8}\bigl(f \pm 2\tau\wp^2\bigr)^{3/2},$$

where $A = -\frac{32}{3}b^{8/3}$.

► *In the solutions of equations 68–73, the following notation is used:*

$$f_1 = \sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}, \qquad \tau = \int \frac{d\wp_1}{\sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}} - C_1,$$

and

$$f_2 = \sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}, \qquad \tau = \int \frac{d\wp_2}{\sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}} - C_1,$$

where functions $\wp_1 = \wp_1(\tau)$ *and* $\wp_2 = \wp_2(\tau)$ *are the inverse functions for the above elliptic integrals. For the upper signs, they are the classical Weierstrass functions* $\wp_1 = \wp(\tau + {}_1,\ 2,\ C_2)$ *and* $\wp_2 = \wp(\tau + {}_1,\ -2,\ C_2)$.

68. $\boldsymbol{y''_{xx} = A_1 y^2 + A_2}$.

Solutions in the parametric form:

$$x = a\tau, \qquad y = b\wp_k,$$

where $A_1 = \pm 6a^{-2}b^{-1}$, $A_2 = a^{-2}b(-1)^k$, $k = 1$ and $k = 2$.

69. $\boldsymbol{y''_{xx} = A_1 x^{-5} y^2 + A_2 x^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{-1}, \qquad y = b\tau^{-1}\wp_k,$$

where $A_1 = \pm 6a^3b^{-1}$, $A_2 = ab(-1)^k$, $k = 1$ and $k = 2$.

70. $\boldsymbol{y''_{xx} = A_1 x^{-15/7} y^2 + A_2 x^{-9/7}}$.

Solutions in the parametric form:

$$x = a\tau^7, \qquad y = b\tau(\tau^2\wp_k \mp 1),$$

where $A_1 = \pm\frac{6}{49}a^{1/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-5/7}b(-1)^k$, $k = 1$ and $k = 2$.

71. $\boldsymbol{y''_{xx} = A_1 x^{-20/7} y^2 + A_2 x^{-12/7}}$.

Solutions in the parametric form:

$$x = a\tau^{-7}, \qquad y = b\tau^{-6}(\tau^2\wp_k \mp 1),$$

where $A_1 = \pm\frac{6}{49}a^{6/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-2/7}b(-1)^k$, $k = 1$ and $k = 2$.

72. $\boldsymbol{y''_{xx} = A_1 + A_2 y^{-2/3}}$.

Solutions in the parametric form:

$$x = a[f_k - (-1)^k\tau], \qquad y = b\wp_k^3,$$

where $A_1 = \pm\frac{1}{2}a^{-2}b$, $A_2 = \frac{1}{12}a^{-2}b^{5/3}(-1)^k$, $k = 1$ and $k = 2$.

73. $\boldsymbol{y''_{xx} = A_1 x^{-3} + A_2 x^{-7/3} y^{-2/3}}$.

Solutions in the parametric form:

$$x = a[f_k - (-1)^k\tau]^{-1}, \qquad y = b\wp_k^3[f_k - (-1)^k\tau]^{-1},$$

where $A_1 = \pm\frac{1}{2}ab$, $A_2 = \frac{1}{12}a^{1/3}b^{5/3}(-1)^k$, $k = 1$ and $k = 2$.

► *In the solutions of equations 74–75, the following notation is used:*

$$E = \int \left(1 \pm \tau^4\right)^{-1/2} d\tau + C_2, \qquad k^2 = \pm 1.$$

Function E can be expressed in terms of elliptic integrals or lemniscate functions.

74. $\boldsymbol{y''_{xx} = A_1 x^{-18/5} y^3 + A_2 x^{-14/5} y^2}$.

Solutions in the parametric form:

$$x = aC_1^5 E^{-5}, \qquad y = bC_1^4 E^{-4}(\tau E - k),$$

where $A_1 = \pm\frac{2}{25}a^{8/5}b^{-2}$, $A_2 = \pm\frac{6}{25}a^{4/5}b^{-1}k$.

75. $\boldsymbol{y''_{xx} = A_1 x^{-12/5} y^3 + A_2 x^{-11/5} y^2}$.

Solutions in the parametric form:

$$x = aC_1^5 E^5, \qquad y = bC_1 E(\tau E - k),$$

where $A_1 = \pm\frac{2}{25}a^{2/5}b^{-2}$, $A_2 = \pm\frac{6}{25}a^{1/5}b^{-1}k$.

► *In the solutions of equations 76–81, the following notation is used:*

$$f = \begin{cases} J_{1/3}(\tau) & \textit{for the upper sign (Bessel function),} \\ I_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} Y_{1/3}(\tau) & \textit{for the upper sign (Bessel function),} \\ K_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$H = C_1 f + C_2 g + \beta\omega\left(g\int f\,d\tau - f\int g\,d\tau\right), \qquad \omega = \begin{cases} \frac{1}{2}\pi & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign.} \end{cases}$$

76. $\boldsymbol{y''_{xx} = A_1 xy + A_2}$.

Solutions in the parametric form:

$$x = a\tau^{2/3}, \qquad y = \tau^{1/3}H,$$

where $A_1 = \mp\frac{9}{4}a^{-3}$, $A_2 = \frac{9}{4}a^{-2}\beta$.

77. $\boldsymbol{y''_{xx} = A_1 x^{-5} y + A_2 x^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3}, \qquad y = \tau^{-1/3}H,$$

where $A_1 = \mp\frac{9}{4}a^3$, $A_2 = \frac{9}{4}a\beta$.

78. $y''_{xx} = A_1 x^{-3/2} + A_2 x^{-1/2} y^{-1/2}$.

Solutions in the parametric form:

$$x = a\tau^{2/3} H^2, \qquad y = b\tau^{-2/3}\big(\tau H'_\tau + \tfrac{1}{3}H\big)^2,$$

where $A_1 = -\frac{1}{2}a^{-1/2}b\beta$, $A_2 = \mp\frac{1}{3}a^{-3/2}b^{3/2}$.

79. $y''_{xx} = A_1 x^{-3/2} + A_2 x^{-2} y^{-1/2}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3} H^{-2}, \qquad y = b\tau^{-4/3} H^{-2}\big(\tau H'_\tau + \tfrac{1}{3}H\big)^2,$$

where $A_1 = -\frac{1}{2}a^{-1/2}b\beta$, $A_2 = \mp\frac{1}{3}b^{3/2}$.

80. $y''_{xx} = A_1 y^{-3/2} + A_2 x y^{-2}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3}\big[\mp\tau^2 H^2 + 2\beta\tau H - \big(\tau H'_\tau + \tfrac{1}{3}H\big)^2\big], \qquad y = b\tau^{2/3} H^2,$$

where $A_1 = -ab^{-1/2}\beta A_2$, $A_2 = \frac{9}{2}a^{-3}b^3$.

81. $y''_{xx} = A_1 x^{-3/2} y^{-3/2} + A_2 x^{-2} y^{-2}$.

Solutions in the parametric form:

$$x = a\tau^{2/3}\big[\mp\tau^2 H^2 + 2\beta\tau H - \big(\tau H'_\tau + \tfrac{1}{3}H\big)^2\big]^{-1},$$
$$y = b\tau^{4/3} H^2\big[\mp\tau^2 H^2 + 2\beta\tau H - \big(\tau H'_\tau + \tfrac{1}{3}H\big)^2\big]^{-1},$$

where $A_1 = -\frac{9}{2}a^{-1/2}b^{5/2}\beta$, $A_2 = \frac{9}{2}b^3$.

► *In the solutions of equations 82–88, the following notation is used:*

$$U_\nu = \begin{cases} C_1 J_\nu(\tau) & \textit{for the upper sign (Bessel function),} \\ C_1 I_\nu(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$V_\nu = \begin{cases} C_2 Y_\nu(\tau) & \textit{for the upper sign (Bessel function),} \\ C_2 K_\nu(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$Z_\nu = \alpha_1 U_\nu + \alpha_2 V_\nu, \qquad X_\nu = \beta_1 U_\nu + \beta_2 V_\nu,$$

$$N = \begin{cases} Z_\nu X_\nu & \textit{for } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2; \\ \alpha U_\nu^2 + \beta U_\nu V_\nu + \gamma V_\nu^2 & \textit{for } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$F_\nu = \tau Z'_\nu + \nu Z_\nu, \qquad G_\nu = \tau X'_\nu + \nu X_\nu,$$

$$N_1 = \begin{cases} Z_\nu G_\nu + X_\nu F_\nu & \textit{for } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2, \\ \tau N' + 2\nu N & \textit{for } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$N_2 = N_1^2 \pm 4\tau^2 N^2 + \omega^2\Delta, \qquad \omega = \begin{cases} \frac{2}{\pi} & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign.} \end{cases}$$

Prime denotes differentiation with respect to τ.

82. $\boldsymbol{y''_{xx} = A_1 xy + A_2 y^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{2/3}, \qquad y = b\tau^{1/3}N^{1/2},$$

where $\nu = \frac{1}{3}$, $A_1 = \mp\frac{9}{4}a^{-3}$, $A_2 = \frac{9}{16}a^{-2}b^4\omega^2\Delta$.

83. $\boldsymbol{y''_{xx} = A_1 x^n y + A_2 y^{-3}, \qquad n \neq -2}$.

Solutions in the parametric form:

$$x = a\tau^{2\nu}, \qquad y = b\tau^{\nu}N^{1/2},$$

where $\nu = \dfrac{1}{n+2}$, $A_1 = \mp\dfrac{1}{4\nu^2}a^{-n-2}$, $A_2 = \dfrac{1}{16\nu^2}a^{-2}b^4\omega^2\Delta$.

84. $\boldsymbol{y''_{xx} = A_1 x^{-5} y + A_2 y^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3}, \qquad y = b\tau^{-1/3}N^{1/2},$$

where $\nu = \frac{1}{3}$, $A_1 = \mp\frac{9}{4}a^{3}$, $A_2 = \frac{9}{16}a^{-2}b^4\omega^2\Delta$.

85. $\boldsymbol{y''_{xx} = A_1 x^{-3} + A_2 x^{-1/2} y^{-1/2}}$.

Solutions in the parametric form:

$$x = a\tau^{2/3}N, \qquad y = b\tau^{-2/3}N^{-1}N_1^2,$$

where $\nu = \frac{1}{3}$, $A_1 = -2ab\omega^2\Delta$, $A_2 = \mp\frac{8}{3}a^{-3/2}b^{3/2}$.

86. $\boldsymbol{y''_{xx} = A_1 + A_2 x^{-2} y^{-1/2}}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3}N^{-1}, \qquad y = b\tau^{-4/3}N^{-2}N_1^2,$$

where $\nu = \frac{1}{3}$, $A_1 = -2a^{-2}b\omega^2\Delta$, $A_2 = \mp\frac{8}{3}b^{3/2}$.

87. $\boldsymbol{y''_{xx} = A_1 x y^{-2} + A_2 y^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{-2/3}N^{-1}N_2, \qquad y = b\tau^{2/3}N,$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{128}a^{-3}b^3$, $A_2 = \frac{9}{64}a^{-2}b^4\omega^2\Delta$.

88. $\boldsymbol{y''_{xx} = A_1 x^{-2} y^{-2} + A_2 y^{-3}}$.

Solutions in the parametric form:

$$x = a\tau^{2/3}NN_2^{-1}, \qquad y = b\tau^{4/3}N^2N_2^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{128}b^3$, $A_2 = \frac{9}{64}a^{-2}b^4\omega^2\Delta$.

► *In the solutions of equations 89–90, the following notation is used:*

$$\Delta = C_2^2 - 2C_1, \qquad R = (36\Delta + 54B\tau - 2\tau^3)^{1/2}, \qquad z = 3\int \tau^{-1}R^{-1}\,d\tau,$$

$$W(z) = \begin{cases} \dfrac{\sqrt{-\Delta}}{C_1}\tan\bigl(\pm\sqrt{-\Delta}\,z\bigr) + \dfrac{C_2}{C_1} & \textit{if } \Delta < 0; \\ \dfrac{\sqrt{\Delta}}{C_1}\tanh\bigl(\mp\sqrt{\Delta}\,z\bigr) + \dfrac{C_2}{C_1} & \textit{if } \Delta > 0; \\ \mp\dfrac{1}{C_1 z} - \sqrt{\dfrac{2}{|C_1|}} & \textit{if } \Delta = 0,\ \ C_2 < 0; \\ \mp\dfrac{1}{C_1 z} + \sqrt{\dfrac{2}{|C_1|}} & \textit{if } \Delta = 0,\ \ C_2 > 0. \end{cases}$$

89. $\boldsymbol{y''_{xx} = A_1 y^{-5/3} + A_2 x^{-2/3} y^{-5/3}}$.

Solutions in the parametric form:

$$x = a\tau^{-3/2}(C_1W^2 - 2C_2W + 2)^{3/2},$$
$$y = b\tau^{-9/4}(C_1W^2 - 2C_2W + 2)^{3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$

where $A_1 = 24a^{-2}b^{8/3}C_1$, $A_2 = -36a^{-4/3}b^{8/3}B$.

90. $\boldsymbol{y''_{xx} = A_1 x^{-2/3} y^{-5/3} + A_2 x^{-4/3} y^{-5/3}}$.

Solutions in the parametric form:

$$x = a\tau^{3/2}(C_1W^2 - 2C_2W + 2)^{-3/2},$$
$$y = b\tau^{-3/4}(C_1W^2 - 2C_2W + 2)^{-3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$

where $A_1 = -36a^{-4/3}b^{8/3}B$, $A_2 = 24a^{-2/3}b^{8/3}C_1$.

► *In the solutions of equations 91–102, the following notation is used: functions P_1 and P_2 are the general solution of four modifications of the first Painlevé equation:*

$$P_1'' = \pm 6P_1^2 + \tau, \qquad P_2'' = \pm 6P_2^2 - \tau$$

(in the case of the upper sign, the equation for P_1 is the canonical form of the first Painlevé equation, see Subsection 2.8.2). In addition,

$$\begin{aligned} Q_1 &= \pm 6P_1^2 + \tau, & Q_2 &= \pm 6P_2^2 - \tau, \\ T_1 &= \tau^2 P_1 \mp 1, & T_2 &= \tau^2 P_2 \mp 1, \\ U_1 &= (P_1')^2 - 2P_1Q_1 \pm 8P_1^3, & U_2 &= (P_2')^2 - 2P_2Q_2 \pm 8P_2^3, \\ V_1 &= P_1'Q_1' + P_1' - Q_1^2, & V_2 &= P_2'Q_2' - P_2' - Q_2^2, \end{aligned}$$

where primes denote differentiation with respect to τ.

91. $\boldsymbol{y''_{xx} = A_1 y^2 + A_2 x}$.

Solutions in the parametric form:

$$x = a\tau, \qquad y = bP_k,$$

where $A_1 = \pm 6a^{-2}b^{-1}$, $A_2 = a^{-3}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

92. $\boldsymbol{y''_{xx} = A_1 x^{-5} y^2 + A_2 x^{-4}}$.

Solutions in the parametric form:

$$x = a\tau^{-1}, \qquad y = b\tau^{-1}P_k,$$

where $A_1 = \pm 6a^3b^{-1}$, $A_2 = a^2b(-1)^{k+1}$; $k = 1$ and $k = 2$.

93. $\boldsymbol{y''_{xx} = A_1 x^{-15/7} y^2 + A_2 x^{-8/7}}$.

Solutions in the parametric form:

$$x = a\tau^7, \qquad y = b\tau T_k,$$

where $A_1 = \pm\frac{6}{49}a^{1/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-6/7}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

94. $\boldsymbol{y''_{xx} = A_1 x^{-20/7} y^2 + A_2 x^{-13/7}}$.

Solutions in the parametric form:

$$x = a\tau^{-7}, \qquad y = b\tau^{-6}T_k,$$

where $A_1 = \pm\frac{6}{49}a^{6/7}b^{-1}$, $A_2 = \frac{1}{49}a^{-1/7}b(-1)^{k+1}$; $k = 1$ and $k = 2$.

95. $\boldsymbol{y''_{xx} = A_1 x + A_2 y^{-1/2}}$.

Solutions in the parametric form:

$$x = aP_k, \qquad y = b(P'_k)^2,$$

where $A_1 = \pm 24a^{-3}b$, $A_2 = 2a^{-2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

96. $\boldsymbol{y''_{xx} = A_1 x^{-4} + A_2 x^{-5/2} y^{-1/2}}$.

Solutions in the parametric form:

$$x = aP_k^{-1}, \qquad y = bP_k^{-1}(P'_k)^2,$$

where $A_1 = \pm 24a^2b$, $A_2 = 2a^{1/2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

97. $\boldsymbol{y''_{xx} = A_1 y^{1/3} + A_2 x y^{-5/3}}$.

Solutions in the parametric form:

$$x = aU_k, \qquad y = bP_k^{3/2},$$

where $A_1 = \mp 8ab^{-2}A_2$, $A_2 = -\frac{3}{16}a^{-3}b^{8/3}$; $k = 1$ and $k = 2$.

98. $y''_{xx} = A_1 x^{-10/3} y^{1/3} + A_2 x^{-7/3} y^{-5/3}$.

Solutions in the parametric form:

$$x = aU_k^{-1}, \qquad y = bP_k^{3/2} U_k^{-1},$$

where $A_1 = \mp 8ab^{-2}A_2$, $A_2 = -\frac{3}{16}a^{1/3}b^{8/3}$; $k=1$ and $k=2$.

99. $y''_{xx} = A_1 x^{-3/2} + A_2 y^{-1/2}$.

Solutions in the parametric form:

$$x = a(P'_k)^2, \qquad y = bQ_k^2,$$

where $A_1 = \frac{1}{2}a^{-1/2}b(-1)^k$, $A_2 = \pm 6a^{-2}b^{3/2}$; $k=1$ and $k=2$.

100. $y''_{xx} = A_1 x^{-3/2} + A_2 x^{-5/2} y^{-1/2}$.

Solutions in the parametric form:

$$x = a(P'_k)^{-2}, \qquad y = b(P'_k)^{-2}Q_k^2,$$

where $A_1 = \frac{1}{2}a^{-1/2}b(-1)^k$, $A_2 = \pm 6a^{1/2}b^{3/2}$; $k=1$ and $k=2$.

101. $y''_{xx} = A_1 y^{-4/3} + A_2 x y^{-5/3}$.

Solutions in the parametric form:

$$x = aV_k, \qquad y = b(P'_k)^3,$$

where $A_1 = ab^{-1/3}A_2(-1)^k$, $A_2 = \frac{1}{36}a^{-3}b^{8/3}$; $k=1$ and $k=2$.

102. $y''_{xx} = A_1 x^{-5/3} y^{-4/3} + A_2 x^{-7/3} y^{-5/3}$.

Solutions in the parametric form:

$$x = aV_k^{-1}, \qquad y = b(P'_k)^3 V_k^{-1},$$

where $A_1 = \frac{1}{36}a^{-1/3}b^{7/3}(-1)^k$, $A_2 = \frac{1}{36}a^{1/3}b^{8/3}$; $k=1$ and $k=2$.

► *In the solutions of equations 103–108, the following notation is used: functions P_1 and P_2 are the general solution of four modifications of the second Painlevé equation (with parameter $a=0$):*

$$P''_1 = \tau P_1 \pm 2P_1^3, \qquad P''_2 = -\tau P_2 \pm 2P_2^3.$$

where primes denote differentiation with respect to τ. In the case of the upper sign, the equation for P_1 is the canonical form of the second Painlevé equation (with parameter $a=0$, see Subsection 2.8.2).

103. $y''_{xx} = A_1 y^3 + A_2 xy.$

Solutions in the parametric form:

$$x = a\tau, \qquad y = bP_k,$$

where $A_1 = \pm 2a^{-2}b^{-2}$, $A_2 = a^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

104. $y''_{xx} = A_1 x^{-6} y^3 + A_2 x^{-5} y.$

Solutions in the parametric form:

$$x = a\tau^{-1}, \qquad y = b\tau^{-1}P_k,$$

where $A_1 = \pm 2a^4 b^{-2}$, $A_2 = a^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

105. $y''_{xx} = A_1 + A_2 x^{-1/2} y^{-1/2}.$

Solutions in the parametric form:

$$x = aP_k^2, \quad y = b(P_k')^2, \qquad P_k' = (P_k)_\tau',$$

where $A_1 = \pm 2a^{-2}b$, $A_2 = \frac{1}{2}a^{-3/2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

106. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-1/2}.$

Solutions in the parametric form:

$$x = aP_k^{-2}, \quad y = bP_k^{-2}(P_k')^2, \qquad P_k' = (P_k)_\tau',$$

where $A_1 = \pm 2ab$, $A_2 = \frac{1}{2}b^{3/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

107. $y''_{xx} = A_1 + A_2 xy^{-2}.$

Solutions in the parametric form:

$$x = a[\tau P_k^2 \pm P_k^4 - (P_k')^2], \quad y = bP_k^2, \qquad P_k' = (P_k)_\tau',$$

where $A_1 = \mp 2a^{-2}b(-1)^k$, $A_2 = 2a^{-3}b^3(-1)^{k+1}$; $k = 1$ and $k = 2$.

108. $y''_{xx} = A_1 x^{-3} + A_2 x^{-2} y^{-2}.$

Solutions in the parametric form:

$$x = a\bigl[\tau P_k^2 \pm P_k^4 - (P_k')^2\bigr]^{-1}, \quad y = bP_k^2\bigl[\tau P_k^2 \pm P_k^4 - (P_k')^2\bigr]^{-1},$$

where $A_1 = \pm 2ab$, $A_2 = 2b^3$; $k = 1$ and $k = 2$.

2.5. Generalized Emden—Fowler Equation $y''_{xx} = Ax^n y^m (y'_x)^l$

2.5.1. The Classification Table

The case $l = 0$ corresponding to the classical Emden—Fowler equation is outlined in Section 2.3. In this section, the case $l \neq 0$ is considered.

Table 2.9 represents all solvable equations of the form $y''_{xx} = Ax^n y^m (y'_x)^l$ whose solutions are outlined in Subsection 2.5.2. The two-parameter families (in the space of parameters n, m, and l), one-parameter families, and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of l, the growth of m (for identical l), and the growth of n (for identical m and l). The number of the equation sought is indicated in the last column in this table.

TABLE 2.9
Solvable cases of the generalized Emden—Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
Two-parameter families				
1	arbitrary	arbitrary	0	2.5.2.1
2	arbitrary	0	arbitrary	2.5.2.2
3	$\dfrac{2n+m+3}{n+m+2}$	arbitrary $(m \neq -1)$	arbitrary $(n \neq -1)$	2.5.2.3
One-parameter families				
4	arbitrary $(l \neq 1, 2)$	-1	-1	2.5.2.6
5	arbitrary $(l \neq \frac{3}{2})$	$-\frac{1}{2}$	$-\frac{1}{2}$	2.5.2.97
6	$\dfrac{3m+5}{2m+3}$	arbitrary $(m \neq -\frac{3}{2})$	$-\frac{1}{2}$	2.5.2.13
7	$\dfrac{3m+5}{2m+3}$	arbitrary $(m \neq -\frac{3}{2})$	1	2.5.2.10
8	$\dfrac{3n+4}{2n+3}$	$-\frac{1}{2}$	arbitrary $(n \neq -\frac{3}{2})$	2.5.2.11
9	$\dfrac{3n+4}{2n+3}$	1	arbitrary $(n \neq -\frac{3}{2})$	2.5.2.12
10	$\dfrac{3n+4}{2n+3}$	$-n-3$	arbitrary $(n \neq -\frac{3}{2})$	2.5.2.107
11	1	arbitrary $(m \neq -1, 0)$	-1	2.5.2.5
12	2	-1	arbitrary $(n \neq -1, 0)$	2.5.2.4
13	3	arbitrary $(m \neq -2)$	1	2.5.2.96
14	3	$-n-3$	arbitrary	2.5.2.9
Isolated points				
15	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.33
16	$\frac{1}{2}$	1	$-\frac{15}{8}$	2.5.2.83
17	$\frac{1}{2}$	1	$-\frac{20}{13}$	2.5.2.86
18	$\frac{1}{2}$	1	$-\frac{5}{4}$	2.5.2.80
19	$\frac{1}{2}$	1	0	2.5.2.78
20	$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{7}{6}$	2.5.2.53
21	$\frac{4}{5}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.76

TABLE 2.9 *Continued*

Solvable cases of the generalized Emden—Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
22	1	-2	1	2.5.2.14
23	1	-1	-1	2.5.2.8
24	$\frac{8}{7}$	1	$-\frac{3}{4}$	2.5.2.54
25	$\frac{8}{7}$	1	$-\frac{1}{2}$	2.5.2.52
26	$\frac{6}{5}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.68
27	$\frac{5}{4}$	1	$-\frac{1}{2}$	2.5.2.58
28	$\frac{5}{4}$	1	0	2.5.2.56
29	$\frac{9}{7}$	$-\frac{13}{8}$	1	2.5.2.39
30	$\frac{9}{7}$	$-\frac{1}{2}$	1	2.5.2.38
31	$\frac{13}{10}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.47
32	$\frac{27}{20}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.72
33	$\frac{18}{13}$	$-\frac{1}{2}$	$-\frac{7}{2}$	2.5.2.40
34	$\frac{7}{5}$	$-\frac{7}{4}$	1	2.5.2.18
35	$\frac{7}{5}$	$-\frac{10}{7}$	1	2.5.2.46
36	$\frac{7}{5}$	$-\frac{2}{3}$	1	2.5.2.32
37	$\frac{7}{5}$	$-\frac{1}{2}$	1	2.5.2.17
38	$\frac{7}{5}$	1	0	2.5.2.89
39	$\frac{7}{5}$	1	1	2.5.2.91
40	$\frac{7}{5}$	5	1	2.5.2.75
41	$\frac{10}{7}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.19
42	$\frac{22}{15}$	$-\frac{1}{2}$	$-\frac{2}{3}$	2.5.2.70
43	$\frac{3}{2}$	-2	$-\frac{1}{2}$	2.5.2.106
44	$\frac{3}{2}$	-2	1	2.5.2.99
45	$\frac{3}{2}$	$-\frac{1}{2}$	-2	2.5.2.100
46	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	2.5.2.29
47	$\frac{3}{2}$	$-\frac{1}{2}$	1	2.5.2.98
48	$\frac{3}{2}$	1	-2	2.5.2.105
49	$\frac{3}{2}$	1	$-\frac{1}{2}$	2.5.2.103
50	$\frac{23}{15}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.84
51	$\frac{11}{7}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.27

TABLE 2.9 *Continued*

Solvable cases of the generalized Emden—Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
52	$\frac{8}{5}$	0	1	2.5.2.73
53	$\frac{8}{5}$	1	$-\frac{7}{4}$	2.5.2.26
54	$\frac{8}{5}$	1	$-\frac{10}{7}$	2.5.2.48
55	$\frac{8}{5}$	1	$-\frac{2}{3}$	2.5.2.35
56	$\frac{8}{5}$	1	$-\frac{1}{2}$	2.5.2.24
57	$\frac{8}{5}$	1	1	2.5.2.74
58	$\frac{8}{5}$	1	5	2.5.2.94
59	$\frac{21}{13}$	$-\frac{7}{2}$	$-\frac{1}{2}$	2.5.2.45
60	$\frac{33}{20}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.87
61	$\frac{17}{10}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.49
62	$\frac{12}{7}$	1	$-\frac{13}{8}$	2.5.2.44
63	$\frac{12}{7}$	1	$-\frac{1}{2}$	2.5.2.42
64	$\frac{7}{4}$	$-\frac{1}{2}$	1	2.5.2.51
65	$\frac{7}{4}$	0	1	2.5.2.50
66	$\frac{9}{6}$	$-\frac{2}{3}$	$-\frac{1}{2}$	2.5.2.81
67	$\frac{13}{7}$	$-\frac{3}{4}$	1	2.5.2.65
68	$\frac{13}{7}$	$-\frac{1}{2}$	1	2.5.2.61
69	2	-1	-1	2.5.2.7
70	2	1	-2	2.5.2.16
71	$\frac{11}{5}$	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.95
72	$\frac{7}{3}$	$-\frac{7}{6}$	$-\frac{1}{2}$	2.5.2.64
73	$\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.36
74	$\frac{5}{2}$	$-\frac{15}{8}$	1	2.5.2.69
75	$\frac{5}{2}$	$-\frac{20}{13}$	1	2.5.2.71
76	$\frac{5}{2}$	$-\frac{5}{4}$	1	2.5.2.67
77	$\frac{5}{2}$	0	1	2.5.2.66
78	3	-5	2	2.5.2.79
79	3	$-\frac{7}{2}$	$-\frac{1}{2}$	2.5.2.31
80	3	$-\frac{10}{3}$	$-\frac{5}{3}$	2.5.2.37
81	3	$-\frac{20}{7}$	2	2.5.2.85

TABLE 2.9 *Continued*

Solvable cases of the generalized Emden—Fowler equation $y''_{xx} = Ax^n y^m (y'_x)^l$

No	l	m	n	Equation
82	3	$-\frac{5}{2}$	$-\frac{1}{2}$	2.5.2.22
83	3	$-\frac{13}{5}$	$-\frac{7}{5}$	2.5.2.43
84	3	$-\frac{7}{3}$	$-\frac{5}{3}$	2.5.2.25
85	3	$-\frac{15}{7}$	2	2.5.2.82
86	3	-2	-2	2.5.2.104
87	3	-2	-1	2.5.2.15
88	3	-2	$-\frac{1}{2}$	2.5.2.101
89	3	-2	1	2.5.2.28
90	3	$-\frac{4}{3}$	$-\frac{1}{2}$	2.5.2.59
91	3	$-\frac{7}{6}$	$-\frac{1}{2}$	2.5.2.60
92	3	$-\frac{5}{6}$	$-\frac{5}{3}$	2.5.2.93
93	3	$-\frac{1}{2}$	$-\frac{5}{2}$	2.5.2.90
94	3	$-\frac{1}{2}$	$-\frac{5}{3}$	2.5.2.92
95	3	0	-4	2.5.2.55
96	3	0	$-\frac{5}{2}$	2.5.2.88
97	3	0	$-\frac{1}{2}$	2.5.2.20
98	3	0	2	2.5.2.77
99	3	1	-7	2.5.2.62
100	3	1	-4	2.5.2.57
101	3	1	-2	2.5.2.102
102	3	1	$-\frac{5}{3}$	2.5.2.23
103	3	1	$-\frac{7}{5}$	2.5.2.41
104	3	1	$-\frac{1}{2}$	2.5.2.30
105	3	1	0	2.5.2.21
106	3	2	$-\frac{5}{3}$	2.5.2.34
107	3	3	-7	2.5.2.63

2.5.2. Exact Solutions

1. $y''_{xx} = Ay^m (y'_x)^l$.

1°. Solution in the parametric form with $m \neq -1$, $l \neq 2$:

$$x = aC_1^{1-m-l} \int (1 \pm \tau^{m+1})^{\frac{1}{l-2}} \, d\tau + C_2, \qquad y = bC_1^{2-l}\tau,$$

where $A = \pm \dfrac{m+1}{2-l} a^{l-2} b^{1-m-l}$.

2°. Solution in the parametric form with $m = -1$, $l \neq 2$:

$$x = aC_1 \int \tau^{\frac{l}{l-2}} \exp(\mp\tau^2) \, d\tau + C_2, \qquad y = bC_1 \exp(\mp\tau^2),$$

where $A = \mp \dfrac{4b^2}{a^2(2-l)} \left(\mp \dfrac{a}{2b}\right)^l$.

3°. Solution in the parametric form with $m \neq -1$, $l = 2$:

$$x = aC_1 \int \tau^{\frac{1-m}{1+m}} \exp(\mp\tau^2) \, d\tau + C_2, \qquad y = b\tau^{\frac{2}{m+1}},$$

where $A = \pm(m+1)b^{-1-m}$.

4°. Solution with $m = -1$, $l = 2$:

$$y = \begin{cases} (C_1 x + C_2)^{\frac{1}{1-A}} & \text{if } A \neq 1, \\ C_2 \exp(C_1 x) & \text{if } A = 1. \end{cases}$$

2. $y''_{xx} = Ax^n (y'_x)^l$.

1°. Solution in the parametric form with $n \neq -1$, $l \neq 1$:

$$x = aC_1^{1-l}\tau, \qquad y = bC_1^{2+n-l} \int (1 \pm \tau^{n+1})^{\frac{1}{1-l}} \, d\tau + C_2,$$

where $A = \pm \dfrac{n+1}{1-l} a^{l-n-2} b^{1-l}$.

2°. Solution in the parametric form with $n = -1$, $l \neq 1$:

$$x = aC_1 \exp(\mp\tau^2), \qquad y = bC_1 \int \tau^{\frac{3-l}{1-l}} \exp(\mp\tau^2) \, d\tau + C_2,$$

where $A = \mp \dfrac{4a^2}{b^2(1-l)} \left(\mp \dfrac{b}{2a}\right)^{3-l}$.

3°. Solution in the parametric form with $n \neq -1$, $l = 1$:

$$x = a\tau^{\frac{2}{n+1}}, \qquad y = bC_1 \int \tau^{\frac{1-n}{1+n}} \exp(\mp\tau^2) \, d\tau + C_2,$$

where $A = \mp(n+1)b^{1-n}$.

4°. Solution with $n = -1$, $l = 1$:

$$y = \begin{cases} C_1 x^{A+1} + C_2 & \text{if } A \neq 1, \\ C_1 \ln|x| + C_2 & \text{if } A = 1. \end{cases}$$

3. $y''_{xx} = Ax^n y^m (y'_x)^{\frac{2n+m+3}{n+m+2}}$.

Solution in the parametric form with $n \neq -1$, $m \neq -1$:

$$x = \exp\left[\int \frac{d\tau}{f(\tau)} + C_2\right], \qquad y = \tau \exp\left[-\frac{n+1}{m+1}\int \frac{d\tau}{f(\tau)} - \frac{n+1}{m+1}C_2\right],$$

where function $f = f(\tau)$ is defined impicitly by the formula

$$[f + (\sigma-1)\tau](f+\sigma\tau)^{\frac{\sigma}{1-\sigma}} = C_1 + \frac{A\tau^{m+2}}{n+m+2}, \qquad \sigma = -\frac{n+1}{m+1}.$$

See equation 2.5.2.5 for the case $n=-1$. See equation 2.5.2.4 for the case $m=-1$.

4. $y''_{xx} = Ax^n y^{-1}(y'_x)^2$.

Solution in the parametric form with $n \neq -1$, $n \neq 0$:

$$x = a\tau^{\frac{1}{n}}, \qquad y = \pm\exp\left[\int \tau^{\frac{1-n}{n}}\left(\frac{n}{n+1}\tau^{\frac{n+1}{n}} + n\tau^{\frac{1}{n}} + C_1\right)^{-1} d\tau + C_2\right],$$

where $A = -a^{-n}$.

See equation 2.5.2.7 for the case $n=-1$. See equation 2.5.2.1 for the case $n=0$.

5. $y''_{xx} = Ax^{-1}y^m y'_x$.

Solution in the parametric form with $m \neq -1$, $m \neq 0$:

$$x = \pm\exp\left[\int \tau^{\frac{1-m}{m}}\left(\frac{m}{m+1}\tau^{\frac{m+1}{m}} + m\tau^{\frac{1}{m}} + C_1\right)^{-1} d\tau + C_2\right], \qquad y = (A\tau)^{\frac{1}{m}}.$$

See equation 2.5.2.8 for the case $m=-1$. See equation 2.5.2.2 for the case $m=0$.

6. $y''_{xx} = Ax^{-1}y^{-1}(y'_x)^l$.

Solution in the parametric form with $l \neq 1$, $l \neq 2$:

$$x = \pm\left(f - \frac{\tau}{\lambda}\right)^{-1}\exp\left[\frac{1}{\lambda}\int \frac{d\tau}{f(\tau)} + C_2\right], \quad y = \pm\exp\left[\int \frac{d\tau}{f(\tau)} + \lambda C_2\right], \qquad \lambda = \frac{l-1}{l-2},$$

where the function $f = f(\tau)$ is defined implicitly by the formula

$$\ln\left(\frac{f}{\tau} - \frac{1}{\lambda}\right) - \frac{\tau}{\lambda f - \tau} = \pm\frac{A}{\lambda}\tau^\lambda - \ln\tau + C_1, \qquad \lambda = \frac{l-1}{l-2}.$$

See equation 2.5.2.7 for the case $l=2$. See equation 2.5.2.2 for the case $l=1$.

7. $y''_{xx} = Ax^{-1}y^{-1}(y'_x)^2$.

Solution in the parametric form:

$$x = \pm e^\tau, \qquad y = C_2(\mp A\tau + e^\tau + C_1)\exp\left[\pm A\int(\mp A\tau + e^\tau + C_1)^{-1}\,d\tau\right].$$

8. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-1}y'_x}$.

Solution in the parametric form:

$$x = C_2(\pm A\tau + e^\tau + C_1)\exp\left[\mp A\int(\pm A\tau + e^\tau + C_1)^{-1}\,d\tau\right], \qquad y = \pm e^\tau.$$

9. $\boldsymbol{y''_{xx} = Ax^n y^{-n-3}(y'_x)^3}$.

Solution in the parametric form with $n \neq -1$:

$$x = aC_1^{n+1}\tau\left[\int(1\pm\tau^{n+1})^{-1/2}\,d\tau + C_2\right]^{-1}, \quad y = bC_1^{n-1}\left[\int(1\pm\tau^{n+1})^{-1/2}\,d\tau + C_2\right]^{-1},$$

where $A = \mp\dfrac{n+1}{2}a^{1-n}b^{n+1}$.

See equation 2.5.2.15 for the case $n = -1$.

10. $\boldsymbol{y''_{xx} = Axy^m(y'_x)^{\frac{3m+5}{2m+3}}}$.

Solution in the parametric form with $m \neq -3/2$:

$$x = aC_1^{-2}\left\{(1\pm\tau^{\mu+1})^{1/2}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right] - \tau\right\},$$

$$y = bC_1^{(\mu+1)(\mu-2)}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right]^{\mu+2},$$

where $\mu = -\dfrac{2m+3}{m+1}$, $A = -\dfrac{\mu b^{\frac{1}{\mu+2}}}{(\mu+2)a}\left[\pm\dfrac{(\mu+1)a}{2(\mu+2)b}\right]^{\frac{1}{\mu}}$.

11. $\boldsymbol{y''_{xx} = Ax^n y^{-\frac{1}{2}}(y'_x)^{\frac{3n+4}{2n+3}}}$.

Solution in the parametric form with $n \neq -3/2$:

$$x = aC_1^{(\mu+1)^2}\tau^{\mu+1}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right]^{-\mu-1},$$

$$y = bC_1^4\left\{(1\pm\tau^{\mu+1})^{1/2}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right] - \tau\right\}^2,$$

where $\mu = -\dfrac{n}{n+1}$, $A = \dfrac{\mu+3}{a(\mu+1)}a^{\frac{\mu}{\mu+1}}b^{\frac{1}{2}}\left(\pm\dfrac{a}{b}\right)^{\frac{1}{\mu+3}}$.

12. $\boldsymbol{y''_{xx} = Ax^n y(y'_x)^{\frac{3n+4}{2n+3}}}$.

Solution in the parametric form with $n \neq -3/2$:

$$x = aC_1^{(\mu-1)(\mu+2)}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right]^{\mu+2},$$

$$y = bC_1^{-2}\left\{(1\pm\tau^{\mu+1})^{1/2}\left[\int(1\pm\tau^{\mu+1})^{-1/2}\,d\tau + C_2\right] - \tau\right\},$$

where $\mu = -\dfrac{2n+3}{n+1}$, $A = \dfrac{\mu a^{\frac{1}{\mu+2}}}{(\mu+2)b}\left[\pm\dfrac{(\mu+1)b}{2(\mu+2)a}\right]^{\frac{1}{\mu}}$.

13. $\boldsymbol{y''_{xx} = Ax^{-\frac{1}{2}} y^m (y'_x)^{\frac{3m+5}{2m+3}}}.$

Solution in the parametric form with $m \neq -3/2$:

$$x = aC_1^4 \left\{ (1 \pm \tau^{\mu+1})^{1/2} \left[\int (1 \pm \tau^{\mu+1})^{-1/2} \, d\tau + C_2 \right] - \tau \right\}^2,$$
$$y = bC_1^{(\mu+1)^2} \tau^{\mu+1} \left[\int (1 \pm \tau^{\mu+1})^{-1/2} \, d\tau + C_2 \right]^{-\mu-1},$$

where $\mu = -\dfrac{m}{m+1}$, $A = -\dfrac{\mu+3}{b(\mu+1)} a^{\frac{1}{2}} b^{\frac{\mu}{\mu+1}} \left(\pm \dfrac{b}{a} \right)^{\frac{1}{\mu+3}}$.

14. $\boldsymbol{y''_{xx} = Axy^{-2} y'_x}.$

Solution in the parametric form:

$$x = aC_1 \left\{ 2\tau \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right] \pm \exp(\mp\tau^2) \right\}, \qquad y = bC_1 \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right],$$

where $A = \mp \frac{1}{2} a^{-2} b^2$.

15. $\boldsymbol{y''_{xx} = Ax^{-1} y^{-2} (y'_x)^3}.$

Solution in the parametric form:

$$x = a \exp(\mp\tau^2) \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right]^{-1}, \qquad y = C_1 \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right]^{-1},$$

where $A = \pm 2a^2$.

16. $\boldsymbol{y''_{xx} = Ax^{-2} y (y'_x)^2}.$

Solution in the parametric form:

$$x = aC_1 \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right], \qquad y = bC_1 \left\{ 2\tau \left[\int \exp(\mp\tau^2) \, d\tau + C_2 \right] \pm \exp(\mp\tau^2) \right\},$$

where $A = \pm a^2 b^{-2}$.

17. $\boldsymbol{y''_{xx} = Axy^{-1/2} (y'_x)^{7/5}}.$

Solution in the parametric form:

$$x = \pm aC_1 (\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-1/2}, \qquad y = bC_1^{16} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^2,$$

where $A = \pm 15 a^{-2} b^{1/2} \left(\dfrac{a}{16b} \right)^{2/5}$.

18. $\boldsymbol{y''_{xx} = Axy^{-7/4} (y'_x)^{7/5}}.$

Solution in the parametric form:

$$x = \pm aC_1^{27} (\tau^3 - 3\tau + C_2)^{-1/2} (\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2),$$
$$y = bC_1^{32} (\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3},$$

where $A = \pm \dfrac{5}{12} a^{-2} b^{7/4} \left(\dfrac{a}{9b} \right)^{2/5}$.

19. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{10/7}$.

Solution in the parametric form:

$$x = aC_1^{-1}(\tau^3 - 3\tau + C_2)^{-1}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{2/3},$$
$$y = bC_1^{27}(\tau^3 - 3\tau + C_2)^{-1}(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2)^2,$$

where $A = 28a(ab)^{1/2}\left(\dfrac{a}{27b}\right)^{3/7}$.

20. $y''_{xx} = Ax^{-1/2}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^4(\tau^2 - 1)^2, \qquad y = bC_1^3(\tau^3 - 3\tau + C_2),$$

where $A = \pm\frac{4}{9}a^{3/2}b^{-2}$.

21. $y''_{xx} = Ay(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2), \quad y = bC_1\tau, \qquad \text{where} \quad A = -6ab^{-3}.$$

22. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-1}, \qquad y = bC_1^{-3}(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \mp\frac{4}{9}a^{3/2}b^{1/2}$.

23. $y''_{xx} = Ax^{-5/3}y(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^9(\tau^3 - 3\tau + C_2)^{3/2}, \qquad y = \pm bC_1^8(\tau^4 - 6\tau^2 + 4C_2\tau - 3),$$

where $A = \mp\frac{9}{64}a^{8/3}b^{-3}$.

24. $y''_{xx} = Ax^{-1/2}y(y'_x)^{8/5}$.

Solution in the parametric form:

$$x = aC_1^{16}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^2, \qquad y = \pm bC_1(\tau^2 - 1)(\tau^3 - 3\tau + C_2)^{-1/2},$$

where $A = \mp 15a^{1/2}b^{-2}\left(\dfrac{b}{16a}\right)^{2/5}$.

25. $y''_{xx} = Ax^{-5/3}y^{-7/3}(y'_x)^3$.

Solution in the parametric form:

$$x = \pm aC_1(\tau^3 - 3\tau + C_2)^{3/2}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1}, \qquad y = \pm bC_1^{-8}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{-1},$$

where $A = \mp\frac{9}{64}a^{8/3}b^{1/3}$.

26. $\boldsymbol{y''_{xx} = Ax^{-7/4}y(y'_x)^{8/5}}$.

Solution in the parametric form:

$$x = aC_1^{32}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{4/3},$$
$$y = \pm bC_1^{27}(\tau^3 - 3\tau + C_2)^{-1/2}(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2),$$

where $A = \mp\dfrac{5}{12}a^{7/4}b^{-2}\Bigl(\dfrac{b}{9a}\Bigr)^{2/5}$.

27. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{11/7}}$.

Solution in the parametric form:

$$x = aC_1^{27}(\tau^3 - 3\tau + C_2)^{-1}(\tau^6 - 15\tau^4 + 20C_2\tau^3 - 45\tau^2 + 12C_2\tau + 27 - 8C_2^2)^2,$$
$$y = bC_1^{-1}(\tau^3 - 3\tau + C_2)^{-1}(\tau^4 - 6\tau^2 + 4C_2\tau - 3)^{2/3},$$

where $A = -28b(ab)^{1/2}\Bigl(\dfrac{b}{27a}\Bigr)^{3/7}$.

28. $\boldsymbol{y''_{xx} = Axy^{-2}(y'_x)^3}$.

1°. Solution in the parametric form with $A < \frac{1}{4}$:

$$x = \tau(C_1\tau^\nu + C_2\tau^{-\nu}), \quad y = \tau^2, \qquad \text{where} \quad \nu = \sqrt{1 - 4A}.$$

2°. Solution in the parametric form with $A = \frac{1}{4}$:

$$x = \tau(C_1 \ln|\tau| + C_2), \quad y = \tau^2.$$

3°. Solution in the parametric form with $A > \frac{1}{4}$:

$$x = \tau C_1 \sin(\nu \ln \tau + C_2), \quad y = \tau^2, \qquad \text{where} \quad \nu = \sqrt{4A - 1}.$$

29. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-1/2}(y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = \pm\tau^2(C_1\tau^\nu + C_2\tau^{-\nu})^2, \qquad y = \tfrac{1}{4}\tau^{-2}[(1+\nu)C_1\tau^\nu + (1-\nu)C_2\tau^{-\nu}],$$

where $A = \mp k^2$, $\nu = k^{-2}(k^4 + 4)^{1/2}$.

30. $\boldsymbol{y''_{xx} = Ax^{-1/2}y(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^2 \exp(-2\tau)\bigl[2\exp(3\tau) - C_2\sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2\cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2,$$
$$y = bC_1\exp(-\tau)\bigl[\exp(3\tau) + C_2\sin\bigl(\sqrt{3}\,\tau\bigr)\bigr],$$

where $A = -16a^{3/2}b^{-3}$.

31. $y''_{xx} = Ax^{-1/2}y^{-7/2}(y'_x)^3.$

Solution in the parametric form:

$$x = \frac{aC_1 \exp(-\tau)\bigl[2\exp(3\tau) - C_2 \sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2 \cos\bigl(\sqrt{3}\,\tau\bigr)\bigr]^2}{\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)},$$

$$y = \frac{bC_1^{-1}\exp(\tau)}{\exp(3\tau) + C_2 \sin\bigl(\sqrt{3}\,\tau\bigr)},$$

where $A = -16(ab)^{3/2}$.

32. $y''_{xx} = Axy^{-2/3}(y'_x)^{7/5}.$

1°. Solution in the parametric form with $A < 0$:

$$x = aC_1[\cosh(\tau + C_2)\cos\tau]^{1/2}[\tanh(\tau + C_2) - \tan\tau],$$
$$y = bC_1^6 \cosh^3(\tau + C_2)\cos^3\tau\,[\tanh(\tau + C_2) + \tan\tau]^3,$$

where $A = -5a^{-2}b^{2/3}\Bigl(\dfrac{a}{12b}\Bigr)^{2/5}$.

2°. Solution in the parametric form with $A > 0$:

$$x = aC_1[\cosh\tau - \sin(\tau + C_2)]^{-1/2}[\sinh\tau - \cos(\tau + C_2)],$$
$$y = bC_1^6[\sinh\tau + \cos(\tau + C_2)]^3,$$

where $A = 5a^{-2}b^{2/3}\Bigl(\dfrac{a}{6b}\Bigr)^{2/5}$.

33. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = aC_1^{-1}[\cosh(\tau + C_2)\cos\tau]^{-1},$$
$$y = bC_1 \cosh(\tau + C_2)\cos\tau\,[\tanh(\tau + C_2) - \tan\tau]^2,$$

where $A = -4ab$.

34. $y''_{xx} = Ax^{-5/3}y^2(y'_x)^3.$

1°. Solution in the parametric form with $A > 0$:

$$x = aC_1^3[\cosh(\tau + C_2)\cos\tau]^{3/2},$$
$$y = bC_1^2 \cosh(\tau + C_2)\cos\tau\,[\tanh(\tau + C_2) + \tan\tau],$$

where $A = \frac{3}{16}a^{8/3}b^{-4}$.

2°. Solution in the parametric form with $A < 0$:

$$x = aC_1^3[\cosh\tau - \sin(\tau + C_2)]^{3/2}, y = bC_1^2[\sinh\tau + \cos(\tau + C_2)],$$

where $A = -\frac{3}{4}a^{8/3}b^{-4}$.

35. $y''_{xx} = Ax^{-2/3}y(y'_x)^{8/5}$.

1°. Solution in the parametric form with $A > 0$:

$$x = aC_1^6 \cosh^3(\tau + C_2)\cos^3\tau[\tanh(\tau + C_2) + \tan\tau]^3,$$
$$y = bC_1[\cosh(\tau + C_2)\cos\tau]^{1/2}[\tanh(\tau + C_2) - \tan\tau],$$

where $A = 5a^{2/3}b^{-2}\left(\dfrac{b}{12a}\right)^{2/5}$.

2°. Solution in the parametric form with $A < 0$:

$$x = aC_1^6[\sinh\tau + \cos(\tau + C_2)]^3,$$
$$y = bC_1[\cosh\tau - \sin(\tau + C_2)]^{-1/2}[\sinh\tau - \cos(\tau + C_2)],$$

where $A = -5a^{2/3}b^{-2}\left(\dfrac{b}{6a}\right)^{2/5}$.

36. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aC_1\cosh(\tau + C_2)\cos\tau\,[\tanh(\tau + C_2) - \tan\tau]^2,$$
$$y = bC_1^{-1}[\cosh(\tau + C_2)\cos\tau]^{-1},$$

where $A = 4ab$.

37. $y''_{xx} = Ax^{-5/3}y^{-10/3}(y'_x)^3$.

1°. Solution in the parametric form with $A > 0$:

$$x = aC_1[\cosh(\tau + C_2)\cos\tau]^{1/2}[\tanh(\tau + C_2) + \tan\tau]^{-1},$$
$$y = bC_1^{-2}[\cosh(\tau + C_2)\cos\tau]^{-1}[\tanh(\tau + C_2) + \tan\tau]^{-1},$$

where $A = \frac{3}{16}a^{8/3}b^{4/3}$.

2°. Solution in the parametric form with $A < 0$:

$$x = aC_1[\cosh\tau - \sin(\tau + C_2)]^{3/2}[\sinh\tau + \cos(\tau + C_2)]^{-1},$$
$$y = bC_1^{-2}[\sinh\tau + \cos(\tau + C_2)]^{-1},$$

where $A = -\frac{3}{4}a^{8/3}b^{4/3}$.

► *In the solutions of equations 38–45, the following notation is used:*

$$E = \exp(3\tau), \quad S_1 = E + C_2\sin\bigl(\sqrt{3}\,\tau\bigr), \quad S_2 = 2E - C_2\sin\bigl(\sqrt{3}\,\tau\bigr) + \sqrt{3}\,C_2\cos\bigl(\sqrt{3}\,\tau\bigr),$$
$$S_3 = 2S_1(S_2)'_\tau - (S_1)'_\tau S_2 - S_1S_2, \quad S_4 = 2S_1(S_3)'_\tau - 5(S_1)'_\tau S_3 + S_1S_3.$$

38. $y''_{xx} = Axy^{-1/2}(y'_x)^{9/7}$.

Solution in the parametric form:

$$x = aC_1E^{-1/6}S_1^{-1/2}S_2, \quad y = bC_1^8E^{-4/3}S_3^2,$$

where $A = 7a^{-2}b^{1/2}\left(\dfrac{a}{64b}\right)^{2/7}$.

39. $\boldsymbol{y''_{xx} = Axy^{-13/8}(y'_x)^{9/7}}$.

Solution in the parametric form:

$$x = aC_1^{25}E^{-5/6}S_1^{-1/2}S_4, \qquad y = bC_1^{32}E^{-16/15}S_3^{8/5},$$

where $A = 7a^{-2}b^{13/8}\left(\dfrac{25a}{256b}\right)^{2/7}$.

40. $\boldsymbol{y''_{xx} = Ax^{-7/2}y^{-1/2}(y'_x)^{18/13}}$.

Solution in the parametric form:

$$x = aC_1^{-1}E^{1/15}S_1^{-1}S_3^{2/5}, \qquad y = bC_1^{25}E^{-5/3}S_1^{-1}S_4^2,$$

where $A = -208a^{5/2}b^{1/2}\left(\dfrac{a}{25b}\right)^{5/13}$.

41. $\boldsymbol{y''_{xx} = Ax^{-7/5}y(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^5E^{-5/6}S_1^{5/2}, \qquad y = bC_1^4E^{-2/3}S_3,$$

where $A = -\frac{5}{1024}a^{12/5}b^{-3}$.

42. $\boldsymbol{y''_{xx} = Ax^{-1/2}y(y'_x)^{12/7}}$.

Solution in the parametric form:

$$x = aC_1^8E^{-4/3}S_3^2, \qquad y = bC_1E^{-1/6}S_1^{-1/2}S_2,$$

where $A = -7a^{1/2}b^{-2}\left(\dfrac{b}{64a}\right)^{2/7}$.

43. $\boldsymbol{y''_{xx} = Ax^{-7/5}y^{-13/5}(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1E^{-1/6}S_1^{5/2}S_3^{-1}, \qquad y = bC_1^{-4}E^{2/3}S_3^{-1},$$

where $A = -\frac{5}{1024}a^{12/5}b^{3/5}$.

44. $\boldsymbol{y''_{xx} = Ax^{-13/8}y(y'_x)^{12/7}}$.

Solution in the parametric form:

$$x = aC_1^{32}E^{-16/15}S_3^{8/5}, \qquad y = bC_1^{25}E^{-5/6}S_1^{-1/2}S_4,$$

where $A = -7a^{13/8}b^{-2}\left(\dfrac{25b}{256a}\right)^{2/7}$.

45. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-7/2}(y'_x)^{21/13}}$.

Solution in the parametric form:

$$x = aC_1^{25}E^{-5/3}S_1^{-1}S_4^2, \qquad y = bC_1^{-1}E^{-1/15}S_1^{-1}S_3^{2/5},$$

where $A = 208a^{1/2}b^{5/2}\left(\dfrac{b}{25a}\right)^{5/13}$.

► *In the solutions of equations 46–49, the following notation is used:*

$$
\begin{aligned}
T_1 &= \cosh(\tau + C_2)\cos\tau, & \theta_1 &= \cosh\tau - \sin(\tau + C_2),\\
T_2 &= \tanh(\tau + C_2) + \tan\tau, & \theta_2 &= \sinh\tau + \cos(\tau + C_2),\\
T_3 &= \tanh(\tau + C_2) - \tan\tau, & \theta_3 &= \sinh\tau - \cos(\tau + C_2),\\
T_4 &= 3T_2T_3 - 4; & \theta_4 &= 3\theta_2\theta_3 - 2\theta_1^2.
\end{aligned}
$$

46. $y''_{xx} = Axy^{-10/7}(y'_x)^{7/5}$.

1°. Solution in the parametric form with $A < 0$:

$$x = aC_1^9 T_1^{3/2} T_4, \quad y = bC_1^{14} T_1^{7/3} T_2^{7/3}, \qquad \text{where} \quad A = -\frac{5}{9}a^{-2}b^{10/7}\left(\frac{9a}{28b}\right)^{2/5}.$$

2°. Solution in the parametric form with $A > 0$:

$$x = aC_1^9 \theta_1^{-1/2}\theta_4, \quad y = bC_1^{14}\theta_2^{7/3}, \qquad \text{where} \quad A = \frac{5}{9}a^{-2}b^{10/7}\left(\frac{9a}{14b}\right)^{2/5}.$$

47. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{13/10}$.

Solution in the parametric form:

$$x = aC_1^{-1} T_1^{-1/3} T_2^{2/3}, \quad y = bC_1^9 T_1^3 T_4^2, \qquad \text{where} \quad A = -20a(ab)^{1/2}\left(\frac{a}{9b}\right)^{3/10}.$$

48. $y''_{xx} = Ax^{-10/7}y(y'_x)^{8/5}$.

1°. Solution in the parametric form with $A > 0$:

$$x = aC_1^{14} T_1^{7/3} T_2^{7/3}, \quad y = bC_1^9 T_1^{3/2} T_4, \qquad \text{where} \quad A = \frac{5}{9}a^{10/7}b^{-2}\left(\frac{9b}{28a}\right)^{2/5}.$$

2°. Solution in the parametric form with $A < 0$:

$$x = aC_1^{14}\theta_2^{7/3}, \quad y = bC_1^9\theta_1^{-1/2}\theta_4, \qquad \text{where} \quad A = -\frac{5}{9}a^{10/7}b^{-2}\left(\frac{9b}{14a}\right)^{2/5}.$$

49. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{17/10}$.

Solution in the parametric form:

$$x = aC_1^9 T_1^3 T_4^2, \quad y = bC_1^{-1} T_1^{-1/3} T_2^{2/3}, \qquad \text{where} \quad A = 20b(ab)^{1/2}\left(\frac{b}{9a}\right)^{3/10}.$$

► *In the solutions of equations 50–65, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad F_1 = 2\tau I(\tau) + C_2\tau \mp R, \quad F_2 = \tau^{-1}(RF_1 - 1), \quad F_3 = 4\tau F_1^2 \mp F_2^2,$$

where $I(\tau) = \displaystyle\int \frac{\tau\,d\tau}{R}$ *is the incomplete elliptic integral of the second kind in the Weierstrass form.*

50. $\boldsymbol{y''_{xx} = Ax(y'_x)^{7/4}}$.

Solution in the parametric form:

$$x = aC_1^{-3}R, \quad y = bC_1^5\tau^{-1}F_1, \qquad \text{where} \quad A = \mp\frac{2}{3}a^{-2}(\mp\frac{6a}{b})^{3/4}.$$

51. $\boldsymbol{y''_{xx} = Axy^{-1/2}(y'_x)^{7/4}}$.

Solution in the parametric form:

$$x = aC_1^{-1}F_2, \quad y = bC_1^5\tau^2F_1^{-2}, \qquad \text{where} \quad A = \mp\frac{2}{3}a^{-2}b^{1/2}\Bigl(\pm\frac{3a}{b}\Bigr)^{3/4}.$$

52. $\boldsymbol{y''_{xx} = Ax^{-1/2}y(y'_x)^{8/7}}$.

Solution in the parametric form:

$$x = aC_1^{-16}F_3^2, \quad y = bC_1^5F_1^{-3/2}F_2, \qquad \text{where} \quad A = \mp\frac{7}{16}a^{-1/2}b^{-1}\Bigl(\frac{16a}{3b}\Bigr)^{1/7}.$$

53. $\boldsymbol{y''_{xx} = Ax^{-7/6}y^{-1/2}(y'_x)^{2/3}}$.

Solution in the parametric form:

$$x = aC_1^5F_1^{-3}F_3^6, \quad y = bC_1F_1^{-3}(F_2F_3 - 8F_1^2)^2, \qquad \text{where} \quad A = \mp 4a^{-5/6}b^{3/2}\Bigl(\frac{a}{b}\Bigr)^{2/3}.$$

54. $\boldsymbol{y''_{xx} = Ax^{-3/4}y(y'_x)^{8/7}}$.

Solution in the parametric form:

$$x = aC_1^{-32}F_3^{-4}, \qquad y = bC_1^3F_1^{-3/2}(F_2F_3 - 8F_1^2),$$

where $A = \mp\dfrac{7}{32}a^{-1/4}b^{-1}\Bigl(\dfrac{32a}{3b}\Bigr)^{1/7}$.

55. $\boldsymbol{y''_{xx} = Ax^{-4}(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^2\tau^{-1}, \quad y = bC_1^5\tau^{-1}F_1, \qquad \text{where} \quad A = \pm 6a^5b^{-2}.$$

56. $\boldsymbol{y''_{xx} = Ay(y'_x)^{5/4}}$.

Solution in the parametric form:

$$x = aC_1^5\tau^{-1}F_1, \quad y = bC_1^{-3}R, \qquad \text{where} \quad A = \pm\frac{2}{3}b^{-2}\Bigl(\mp\frac{6b}{a}\Bigr)^{3/4}.$$

57. $\boldsymbol{y''_{xx} = Ax^{-4}y(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^3F_1^{-1}, \quad y = bC_1^5\tau F_1^{-1}, \qquad \text{where} \quad A = \pm 6a^5b^{-3}.$$

58. $\boldsymbol{y''_{xx} = Ax^{-1/2}y(y'_x)^{5/4}}$.

Solution in the parametric form:

$$x = aC_1^5\tau^2F_1^{-2}, \quad y = bC_1^{-1}F_2, \qquad \text{where} \quad A = \pm\frac{2}{3}a^{1/2}b^{-2}\Bigl(\pm\frac{3b}{a}\Bigr)^{3/4}.$$

59. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-4/3}(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^4F_2^2, \quad y = bC_1^9F_1^3, \qquad \text{where} \quad A = \mp\frac{4}{3}a^{3/2}b^{-2/3}.$$

60. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-7/6}(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^5F_1^{-3}F_2^2, \quad y = bC_1^9F_1^{-3}, \qquad \text{where} \quad A = \mp\frac{4}{3}a^{3/2}b^{-5/6}.$$

61. $\boldsymbol{y''_{xx} = Axy^{-1/2}(y'_x)^{13/7}}$.

Solution in the parametric form:

$$x = aC_1^5F_1^{-3/2}F_2, \quad y = bC_1^{-16}F_3^2, \qquad \text{where} \quad A = \pm\frac{7}{16}a^{-1}b^{-1/2}\Bigl(\frac{16b}{3a}\Bigr)^{1/7}.$$

62. $\boldsymbol{y''_{xx} = Ax^{-7}y(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^3F_1^{1/2}, \quad y = bC_1^8F_3, \qquad \text{where} \quad A = \mp\frac{3}{64}a^8b^{-3}.$$

63. $\boldsymbol{y''_{xx} = Ax^{-7}y^3(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^5F_1^{1/2}F_3^{-1}, \quad y = bC_1^8F_3^{-1}, \qquad \text{where} \quad A = \mp\frac{3}{64}a^8b^{-5}.$$

64. $\boldsymbol{y''_{xx} = Ax^{-1/2}y^{-7/6}(y'_x)^{7/3}}$.

Solution in the parametric form:

$$x = aC_1F_1^{-3}(F_2F_3 - 8F_1^2)^2, \quad y = bC_1^5F_1^{-3}F_3^6, \qquad \text{where} \quad A = \pm 4a^{3/2}b^{-5/6}\Bigl(\frac{b}{a}\Bigr)^{2/3}.$$

65. $\boldsymbol{y''_{xx} = Axy^{-3/4}(y'_x)^{13/7}}$.

Solution in the parametric form:

$$x = aC_1^3F_1^{-3/2}(F_2F_3 - 8F_1^2), \quad y = bC_1^{-32}F_3^{-4}, \qquad \text{where} \quad A = \pm\frac{7}{32}a^{-1}b^{-1/4}\Bigl(\frac{32b}{3a}\Bigr)^{1/7}.$$

▶ *In the solutions of equations 66–95, the following notation is used:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2, \qquad f = \sqrt{\pm(4\wp^3 - 1)}.$$

Function $\wp = \wp(\tau)$ is defined implicitly. The upper sign in the formulae corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$. The solutions given below are written in the parametric form. One can assume as the parameter either τ, hence $\wp = \wp(\tau)$, or $\wp$, hence $\tau = \tau(\wp)$.

66. $y''_{xx} = Ax(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aC_1^{-3} f, \quad y = bC_1\tau, \qquad \text{where} \quad A = \mp\frac{2}{b}\Bigl(\pm\frac{6}{ab}\Bigr)^{1/2}.$$

67. $y''_{xx} = Axy^{-5/4}(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aC_1^{-1}(\tau f - \wp), \quad y = bC_1^2\tau^4, \qquad \text{where} \quad A = -\frac{1}{2}a^{-1}b^{1/4}\Bigl(\pm\frac{3a}{2b}\Bigr)^{1/2}.$$

68. $y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{6/5}$.

Solution in the parametric form:

$$x = aC_1^9\tau^{-3}\wp^3, \quad y = bC_1^4(\tau f - \wp)^2, \qquad \text{where} \quad A = \pm\frac{5}{3}a^{-1/3}b^{1/2}\Bigl(\frac{a}{4b}\Bigr)^{1/5}.$$

69. $y''_{xx} = Axy^{-15/8}(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aC_1^3\tau^{-6}(\tau^3 f + 3\tau^2\wp \mp 1), \quad y = bC_1^4\tau^{-8}, \qquad \text{where} \quad A = \frac{1}{8}a^{-1}b^{7/8}\Bigl(\mp\frac{3a}{b}\Bigr)^{1/2}.$$

70. $y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{22/15}$.

Solution in the parametric form:

$$x = aC_1^{-1}\tau^3(\tau^2\wp \mp 1)^3, \quad y = bC_1^4\tau^{-12}(\tau^3 f + 3\tau^2\wp \mp 1)^2,$$

where $A = -5a^{-1/3}b^{1/2}\Bigl(\pm\dfrac{a}{4b}\Bigr)^{7/15}$.

71. $y''_{xx} = Axy^{-20/13}(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aC_1\tau(\tau^3 f - 4\tau^2\wp \pm 6), \quad y = bC_1^{13}\tau^{13}, \qquad \text{where} \quad A = \mp\frac{2}{13}a^{-1}b^{7/13}\Bigl(\pm\frac{6a}{13b}\Bigr)^{1/2}.$$

72. $y''_{xx} = Ax^{-2/3}y^{-1/2}(y'_x)^{27/20}$.

Solution in the parametric form:

$$x = aC_1^{-9}\tau^{-18}(\tau^2\wp \mp 1)^3, \quad y = bC_1\tau^2(\tau^3 f - 4\tau^2\wp \pm 6)^2,$$

where $A = \dfrac{20}{3}a^{-1/3}b^{1/2}\left(\pm\dfrac{a}{4b}\right)^{7/20}$.

73. $y''_{xx} = Ax(y'_x)^{8/5}$.

Solution in the parametric form:

$$x = aC_1^{-3}f, \quad y = bC_1^7\wp^{-2}(f \pm 2\tau\wp^2), \qquad \text{where} \quad A = \mp\frac{5}{6}a^{-2}\left(\frac{3a}{b}\right)^{3/5}.$$

74. $y''_{xx} = Axy(y'_x)^{8/5}$.

Solution in the parametric form:

$$x = aC_1^{-8}(\tau f + 2\wp), \quad y = bC_1^7\wp(f \pm 2\tau\wp^2)^{-1/2},$$

where $A = \dfrac{10}{3}a^{-2}b^{-1}\left(\dfrac{3a}{b}\right)^{3/5}$.

75. $y''_{xx} = Axy^5(y'_x)^{7/5}$.

Solution in the parametric form:

$$x = aC_1^{-27}(\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-1/2}, \quad y = bC_1^8(\tau f + 2\wp)^{-1/3},$$

where $A = -10a^{-2}b^{-5}\left(\dfrac{a}{b}\right)^{2/5}$.

76. $y''_{xx} = Ax^{-1/2}y^{-5/2}(y'_x)^{4/5}$.

Solution in the parametric form:

$$x = aC_1^{27}(\tau^2\wp \mp 1)^2(f \pm 2\tau\wp^2)^{-1}, \quad y = bC_1^7(f \pm 2\tau\wp^2)^{-1}(\tau f + 2\wp)^{4/3},$$

where $A = -5a^{-3/2}b^{7/2}\left(\dfrac{a}{2b}\right)^{4/5}$.

77. $y''_{xx} = Ax^2(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^2\wp, \quad y = bC_1^{-1}\tau, \qquad \text{where} \quad A = \mp 6a^{-1}b^{-2}.$$

78. $y''_{xx} = Ay(y'_x)^{1/2}$.

Solution in the parametric form:

$$x = aC_1\tau, \quad y = bC_1^{-3}f, \qquad \text{where} \quad A = \pm\frac{2}{a}\left(\pm\frac{6}{ab}\right)^{1/2}.$$

79. $y''_{xx} = Ax^2y^{-5}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1^3\tau^{-1}\wp, \quad y = bC_1\tau^{-1}, \qquad \text{where} \quad A = \mp 6a^{-1}b^3.$$

80. $y''_{xx} = Ax^{-5/4}y(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = aC_1^2\tau^4, \quad y = bC_1^{-1}(\tau f - \wp), \qquad \text{where} \quad A = \frac{1}{2}a^{1/4}b^{-1}\Big(\pm\frac{3b}{2a}\Big)^{1/2}.$$

81. $y''_{xx} = Ax^{-1/2}y^{-2/3}(y'_x)^{9/5}.$

Solution in the parametric form:

$$x = aC_1^4(\tau f - \wp)^2, \quad y = bC_1^9\tau^{-3}\wp^3, \qquad \text{where} \quad A = \mp\frac{5}{3}a^{1/2}b^{-1/3}\Big(\frac{b}{4a}\Big)^{1/5}.$$

82. $y''_{xx} = Ax^2y^{-15/7}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1\tau(\tau^2\wp \mp 1), \quad y = bC_1^7\tau^7, \qquad \text{where} \quad A = \mp\frac{6}{49}a^{-1}b^{1/7}.$$

83. $y''_{xx} = Ax^{-15/8}y(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = aC_1^4\tau^{-8}, \quad y = bC_1^3\tau^{-6}(\tau^3 f + 3\tau^2\wp \mp 1), \qquad \text{where} \quad A = -\frac{1}{8}a^{7/8}b^{-1}\Big(\mp\frac{3b}{a}\Big)^{1/2}.$$

84. $y''_{xx} = Ax^{-1/2}y^{-2/3}(y'_x)^{23/15}.$

Solution in the parametric form:

$$x = aC_1^4\tau^{-12}(\tau^3 f + 3\tau^2\wp \mp 1)^2, \quad y = bC_1^{-1}\tau^3(\tau^2\wp \mp 1)^3,$$

where $A = 5a^{1/2}b^{-1/3}\Big(\pm\dfrac{b}{4a}\Big)^{7/15}.$

85. $y''_{xx} = Ax^2y^{-20/7}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1^6\tau^{-6}(\tau^2\wp \mp 1), \quad y = bC_1^7\tau^{-7}, \qquad \text{where} \quad A = \mp\frac{6}{49}a^{-1}b^{6/7}.$$

86. $y''_{xx} = Ax^{-20/13}y(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = aC_1^{13}\tau^{13}, \quad y = bC_1\tau(\tau^3 f - 4\tau^2\wp \pm 6), \qquad \text{where} \quad A = \pm\frac{2}{13}a^{7/13}b^{-1}\Big(\pm\frac{6b}{13a}\Big)^{1/2}.$$

87. $y''_{xx} = Ax^{-1/2}y^{-2/3}(y'_x)^{33/20}$.

Solution in the parametric form:

$$x = aC_1\tau^2(\tau^3 f - 4\tau^2\wp \mp 6)^2, \quad y = bC_1^{-9}\tau^{-18}(\tau^2\wp \mp 1)^3,$$

where $A = -\dfrac{20}{3}a^{1/2}b^{-1/3}\Big(\pm\dfrac{b}{4a}\Big)^{7/20}$.

88. $y''_{xx} = Ax^{-5/2}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^4\wp^{-2}, \quad y = bC_1^7\wp^{-2}(f \pm 2\tau\wp^2), \qquad \text{where} \quad A = \pm 3a^{7/2}b^{-2}.$$

89. $y''_{xx} = Ay(y'_x)^{7/5}$.

Solution in the parametric form:

$$x = aC_1^7\wp^{-2}(f \pm 2\tau\wp^2), \quad y = bC_1^{-3}f, \qquad \text{where} \quad A = \pm\frac{5}{6}b^{-2}\Big(\frac{3b}{a}\Big)^{3/5}.$$

90. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^3(f \pm 2\tau\wp^2)^{-1}, \quad y = bC_1^7\wp^2(f \pm 2\tau\wp^2)^{-1}, \qquad \text{where} \quad A = \pm 3a^{7/2}b^{-3/2}.$$

91. $y''_{xx} = Axy(y'_x)^{7/5}$.

Solution in the parametric form:

$$x = aC_1^7\wp(f \pm 2\tau\wp^2)^{-1/2}, \quad y = bC_1^{-8}(\tau f + 2\wp), \quad \text{where} \quad A = -\frac{10}{3}a^{-1}b^{-2}\Big(\frac{3b}{a}\Big)^{3/5}.$$

92. $y''_{xx} = Ax^{-5/3}y^{-1/2}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^9(f \pm 2\tau\wp^2)^{3/2}, \quad y = bC_1^{16}(\tau f + 2\wp)^2, \qquad \text{where} \quad A = \tfrac{1}{6}a^{8/3}b^{-3/2}.$$

93. $y''_{xx} = Ax^{-5/3}y^{-5/6}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^7(f \pm 2\tau\wp^2)^{3/2}(\tau f + 2\wp)^{-2}, \quad y = bC_1^{16}(\tau f + 2\wp)^{-2},$$

where $A = \frac{1}{6}a^{8/3}b^{-7/6}$.

94. $y''_{xx} = Ax^5y(y'_x)^{8/5}$.

Solution in the parametric form:

$$x = aC_1^8(\tau f + 2\wp)^{-1/3}, \quad y = bC_1^{-27}(\tau^2\wp \mp 1)(f \pm 2\tau\wp^2)^{-1/2},$$

where $A = 10a^{-5}b^{-2}\Big(\dfrac{b}{a}\Big)^{2/5}$.

95. $y''_{xx} = Ax^{-5/2}y^{-1/2}(y'_x)^{11/5}$.

Solution in the parametric form:

$$x = aC_1^7(f \pm 2\tau\wp^2)^{-1}(\tau f + 2\wp)^{4/3}, \quad y = bC_1^{27}(\tau^2\wp \mp 1)^2(f \pm 2\tau\wp^2)^{-1},$$

where $A = 5a^{7/2}b^{-3/2}\Big(\dfrac{b}{2a}\Big)^{4/5}$.

► *In the solutions of equations 96–97, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \textit{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \textit{for the lower sign,} \end{cases}$$

where J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

96. $\boldsymbol{y''_{xx} = Axy^m (y'_x)^3}$.

Solution in the parametric form with $m \neq -2$:

$$x = \tau^\nu Z, \quad y = b\tau^{2\nu}, \qquad \text{where} \quad \nu = \frac{1}{m+2}, \quad A = \pm\Bigl(\frac{m+2}{2b}\Bigr)^2.$$

See 2.5.2.28 for the case $m = -2$.

97. $\boldsymbol{y''_{xx} = Ax^{-1/2} y^{-1/2} (y'_x)^l}$.

Solution in the parametric form with $l \neq 3/2$:

$$x = a\tau^{2\nu} Z^2, \quad y = b\tau^{-2\nu}(\tau Z'_\tau + \nu Z)^2,$$

where $\nu = \dfrac{1-l}{3-2l}$, $A = \dfrac{1}{3-2l}\Bigl(\mp\dfrac{b}{a}\Bigr)^{\frac{3}{2}-l}$. See 2.5.2.29 for the case $l = 3/2$.

► *In the solutions of equations 98–106, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions;

$$U_1 = \tau Z'_\tau + \tfrac{1}{3} Z, \quad U_2 = U_1^2 \pm \tau^2 Z^2, \quad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1 U_2.$$

98. $\boldsymbol{y''_{xx} = Axy^{-1/2} (y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = a\tau^{-2/3} Z^{-1} U_1, \quad y = b\tau^{-4/3} U_2^2, \qquad \text{where} \quad A = -\frac{2}{a}\Bigl(\mp\frac{3}{a}\Bigr)^{1/2}.$$

99. $\boldsymbol{y''_{xx} = Axy^{-2} (y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-1} U_3, \quad y = b\tau^{-2/3} U_2, \qquad \text{where} \quad A = -\frac{b}{a^2}\Bigl(\pm 3ab\Bigr)^{1/2}.$$

100. $\boldsymbol{y''_{xx} = Ax^{-2} y^{-1/2} (y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_2, \quad y = b\tau^{-8/3} Z^{-2} U_3^2, \qquad \text{where} \quad A = \pm\tfrac{2}{3} a^{3/2}.$$

101. $y''_{xx} = Ax^{-1/2}y^{-2}(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_1^2, \quad y = b\tau^{-2/3}Z^{-2}, \qquad \text{where} \quad A = \pm\tfrac{1}{3}a^{3/2}.$$

102. $y''_{xx} = Ax^{-2}y(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^{2/3}Z^2, \quad y = b\tau^{-2/3}U_2, \qquad \text{where} \quad A = \frac{9}{2}\left(\frac{a}{b}\right)^3.$$

103. $y''_{xx} = Ax^{-1/2}y(y'_x)^{3/2}$.

Solution in the parametric form:

$$x = a\tau^{-4/3}U_2^2, \quad y = b\tau^{-2/3}Z^{-1}U_1, \qquad \text{where} \quad A = \frac{2}{b}\left(\mp\frac{3}{b}\right)^{1/2}.$$

104. $y''_{xx} = Ax^{-2}y^{-2}(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = b\tau^{2/3}Z^{-1}U_2^{-1}, \qquad \text{where} \quad A = \tfrac{9}{2}a^3.$$

105. $y''_{xx} = Ax^{-2}y(y'_x)^{3/2}$.

Solution in the parametric form:

$$x = a\tau^{-2/3}U_2, \quad y = b\tau^{-4/3}Z^{-1}U_3, \qquad \text{where} \quad A = \frac{a}{b^2}\left(\pm 3ab\right)^{1/2}.$$

106. $y''_{xx} = Ax^{-1/2}y^{-2}(y'_x)^{3/2}$.

Solution in the parametric form:

$$x = a\tau^{-8/3}Z^{-2}U_3^2, \quad y = b\tau^{-4/3}Z^{-2}U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}b^{3/2}.$$

107. $y''_{xx} = Ax^ny^{-n-3}(y'_x)^{\frac{3n+4}{2n+3}}$.

In the books by Zaitsev & Polyanin (1993, 1994), it was shown that this equation is reducible to the Riccati equation whose solution is expressed in terms of associated Legendre functions.

2.5.3. Some Formulae and Transformations

For the sake of visualization, we use the symbolic notation

$$\{n,\ m,\ l\}$$

to denote the generalized Emden—Fowler equation

$$y''_{xx} = Ax^ny^m(y'_x)^l.$$

Hereinafter we omit the insignificant parameter A (which can be reduced to ± 1 by scaling the variables in accordance with the rule $x \to ax$, $y \to by$, selecting appropriate constants a and b).

1. With $m+l \neq 1$, the generalized Emden—Fowler equation has a particular solution:

$$y = Bx^{\frac{n+2-l}{1-m-l}}, \qquad \text{where} \quad B = \left(\frac{n+2-l}{1-m-l}\right)^{\frac{1-l}{m+l-1}} \left[\frac{n+m+1}{A(1-m-l)}\right]^{\frac{1}{m+l-1}}.$$

2. Assuming y as the independent variable and x as the dependent one, we obtain the generalized Emden—Fowler equation for function $x = x(y)$ with the parameters changed:

$$x''_{yy} = -Ay^m x^n (x'_y)^{3-l}.$$

Denote this transformation as $\mathcal{F}$ and represent it as follows:

$$\{n,\ m,\ l\} \quad \longleftarrow - - - - \longrightarrow \quad \{m,\ n,\ 3-l\} \qquad \textit{transformation } \mathcal{F}.$$

Twofold transformation $\mathcal{F}$ yields the original equation.

3. With $m \neq 0$, $n \neq -1$, and $l \neq 1$, the transformation

$$t = (y'_x)^{1-l}, \qquad w = x^{n+1}$$

leads to the generalized Emden—Fowler equation for function $w = w(t)$ with the parameters changed:

$$w''_{tt} = Bt^{\frac{1}{1-l}} w^{-\frac{n}{n+1}} (w'_t)^{\frac{2m+1}{m}},$$

where $B = -\dfrac{m}{n+1}\left[\dfrac{A(1-l)}{n+1}\right]^{\frac{1}{m}}$. Denote this transformation as $\mathcal{G}$ and represent it as follows:

$$\{n,\ m,\ l\} \quad \longmapsto \quad \left\{\frac{1}{1-l},\ -\frac{n}{n+1},\ \frac{2m+1}{m}\right\} \qquad \textit{transformation } \mathcal{G}.$$

Threefold transformation $\mathcal{G}$ yields the original equation.

When obtained the solution of the transformed equation in the form $w = w(t)$, the solution of the original equation can be written in the parametric form as

$$x = w^{\frac{1}{n+1}}, \quad y = k(w'_t)^{-\frac{1}{m}},$$

where $k = \left[\dfrac{n+1}{A(1-l)}\right]^{\frac{1}{m}}$.

Different compositions of transformations $\mathcal{F}$ and $\mathcal{G}$ generate six different generalized Emden—Fowler equations whose parameters are shown in Figure 1.

4. In the particular case $l = 0$, the transformation $y = w/t$, $x = 1/t$ leads to the Emden—Fowler equation with the independent variable to a different power:

$$w''_{tt} = At^{-n-m-3} w^m.$$

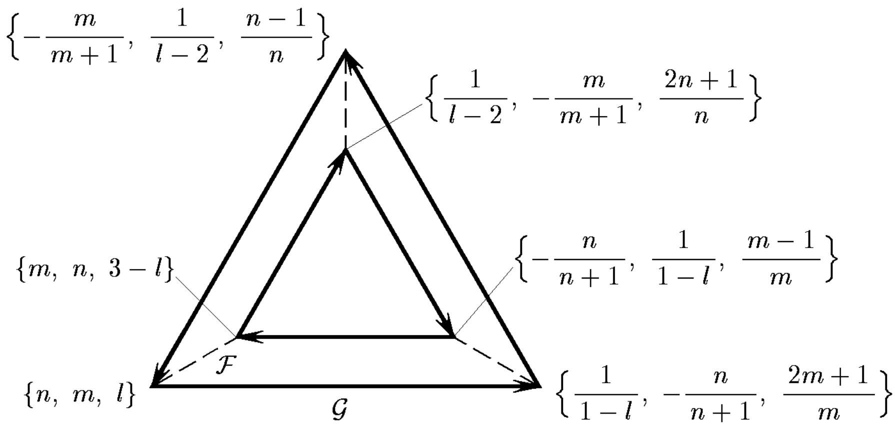

FIGURE 1

Denote this transformation as $\mathcal{G}$ and represent it as follows:

$$\{n,\ m,\ 0\} \quad \longleftrightarrow \quad \{-n-m-3,\ m,\ 0\} \qquad \textit{transformation } \mathcal{H}.$$

With $l = 0$, different compositions of transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate twelve different generalized Emden—Fowler equations whose parameters are shown in Figure 2.

With $l = 0$ and $n = 1$, different compositions of transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate twenty four different generalized Emden—Fowler equations whose parameters are presented in Figure 3.

5. The substitution

$$z = \frac{x}{y} y'_x, \qquad v = A x^{n-l+2} y^{m+l-1}$$

reduces the generalized Emden—Fowler equation to the equation

$$(z^l v - z^2 + z) v'_z = [(m+l-1)z + n - l + 2] v.$$

Furthermore, using the substitution

$$\xi = v - z^{2-l} + z^{1-l},$$

we obtain the Abel equation

$$\xi \xi'_z = [(m+2l-3)z + n - 2l + 3] z^{-l} \xi + [(m+l-1)z^2 + (n-m-2l+3)z - n + l - 2] z^{1-2l}.$$

2.6. Equations of the Form $y''_{xx} = A_1 x^{n_1} y^{m_1} (y'_x)^{l_1} + A_2 x^{n_2} y^{m_2} (y'_x)^{l_2}$

2.6.1. Modified Emden—Fowler Equation $y''_{xx} = A_1 x^{-1} y'_x + A_2 x^n y^m$

See Section 2.3 for the case $A_1 = 0$.

For the sake of clearness, below in this subsection we use the convetional notation $x y''_{xx} - k y'_x = A x^{n+1} y^m$ for the modified Emden—Fowler equation.

The classification Table 2.10 represents all solvable equations whose solutions are outlined in Subsection 2.6.1. Equations are arranged in accordance with the growth of parameter m. The number of the equation sought is indicated in the last column in this table.

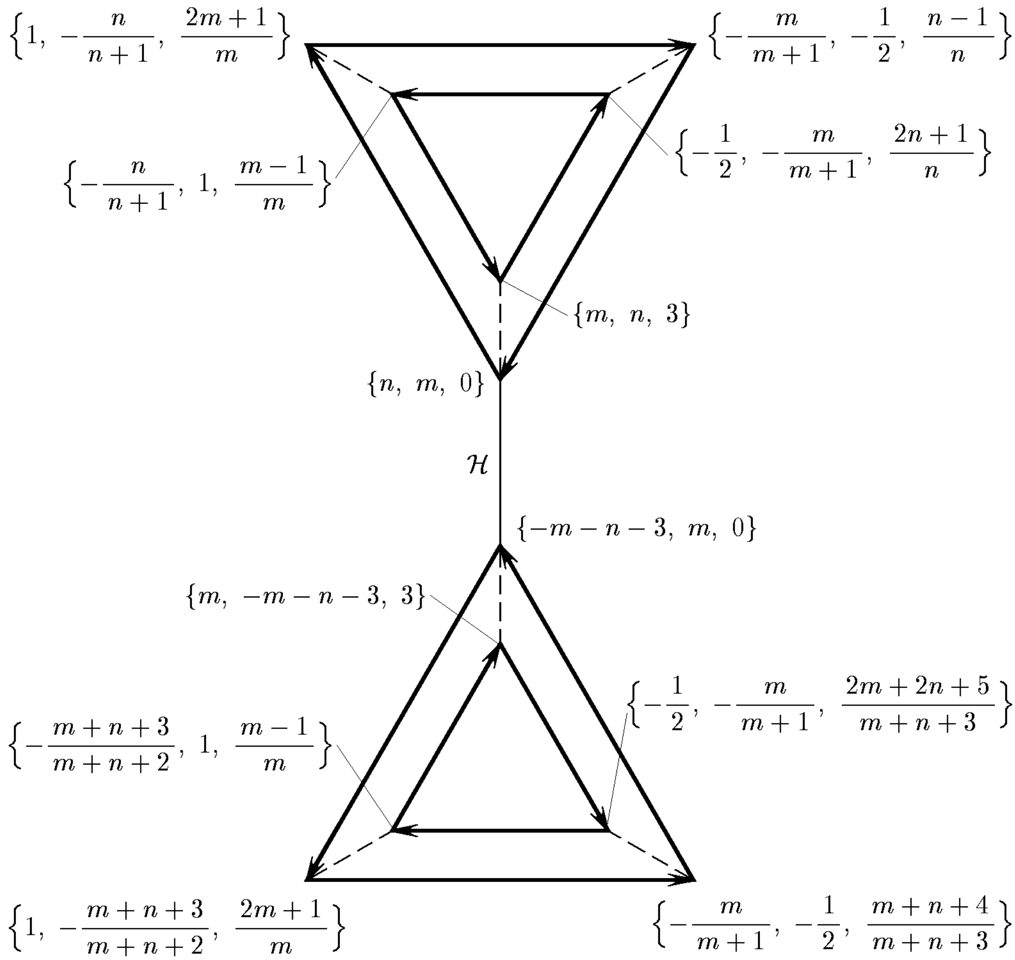

FIGURE 2

1. $\quad xy''_{xx} - \frac{n}{2} y'_x = Ax^{n+1}y^m, \qquad m \neq -1, \quad n \neq -2.$

Solution in the parametric form:

$$x = aC_1^{1-m}\left[\int (1 \pm \tau^{m+1})^{-1/2}\,d\tau + C_2\right]^{\frac{2}{n+2}}, \qquad y = bC_1^{n+2}\tau,$$

where $A = \pm\frac{1}{8}(m+1)(n+2)^2 a^{-n-2} b^{1-m}$.

2. $\quad xy''_{xx} + \frac{n+m+3}{m+1} y'_x = Ax^{n+1}y^m, \qquad m \neq -1, \quad n \neq -2.$

Solution in the parametric form:

$$x = aC_1^{1-m}\left[\int \left(1 \pm \tau^{m+1}\right)^{-1/2} d\tau + C_2\right]^{\frac{m+1}{n+2}},$$

$$y = bC_1^{n+2}\tau\left[\int \left(1 \pm \tau^{m+1}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

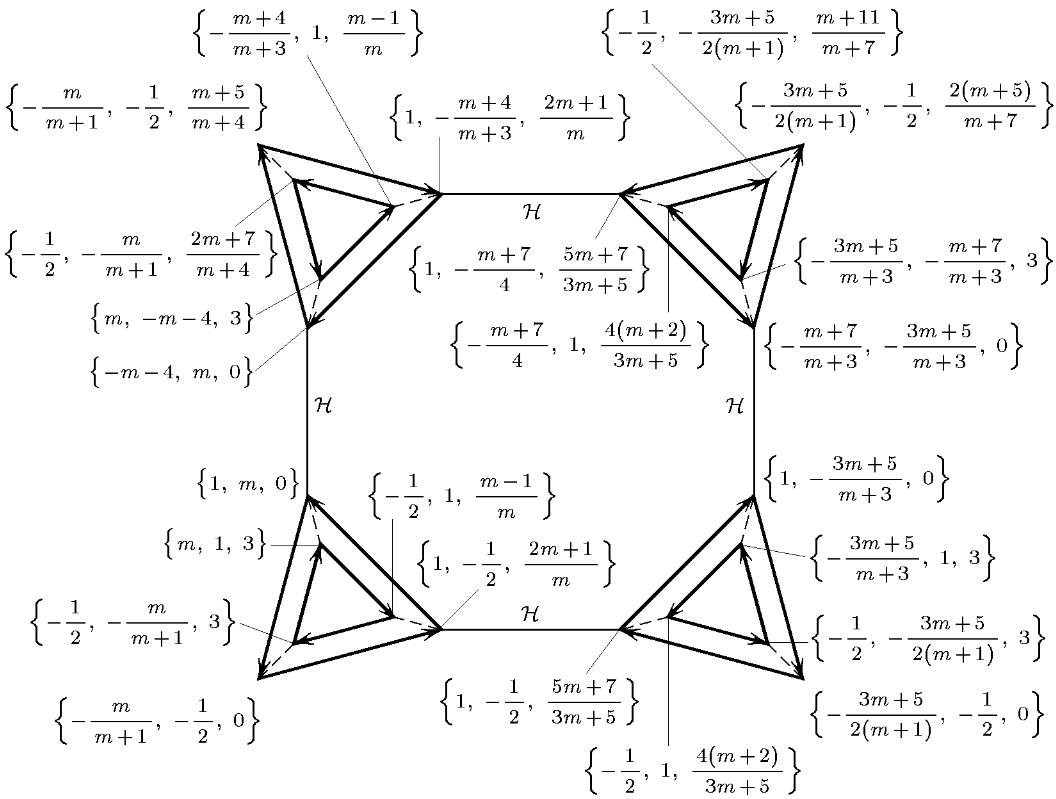

FIGURE 3

where $A = \pm\dfrac{(n+2)^2}{2(m+1)}a^{-n-2}b^{1-m}$.

3. $\boldsymbol{xy''_{xx} + \dfrac{2n+m+3}{m-1}y'_x = Ax^{n+1}y^m, \qquad m \neq -1, \quad n \neq -2.}$

Solution in the parametric form:

$$x = \exp\left[\frac{1-m}{n+2}C_2\int\left(C_1 + \frac{1}{4}\tau^2 + \frac{2B}{m+1}\tau^{m+1}\right)^{-1/2}d\tau\right],$$
$$y = \tau\exp\left[C_2\int\left(C_1 + \frac{1}{4}\tau^2 + \frac{2B}{m+1}\tau^{m+1}\right)^{-1/2}d\tau\right],$$

where $A = \dfrac{4(n+2)^2}{(m-1)^2}B$.

4. $\boldsymbol{xy''_{xx} - \dfrac{n}{2}y'_x = Ax^{n+1}y^{-1}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^2\left[\int\exp(\mp\tau^2)\,d\tau + C_2\right]^{\frac{2}{n+2}}, \qquad y = bC_1^{n+2}\exp(\mp\tau^2),$$

where $A = \mp\frac{1}{2}(n+2)^2a^{-n-2}b^2$.

TABLE 2.10
Solvable cases of the modified Emden—Fowler equations $xy''_{xx} - ky'_x = Ax^{n+1}y^m$

No	m	n	k	Equation
1	arbitrary ($m \neq -1$)	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.1
2	arbitrary ($m \neq -1$)	arbitrary ($n \neq -2$)	$-\dfrac{n+m+3}{m+1}$	2.6.1.2
3	arbitrary ($m \neq -1$)	arbitrary ($n \neq -2$)	$\dfrac{2n+m+3}{1-m}$	2.6.1.3
4	arbitrary ($m \neq -1$)	-2	-1	2.6.1.6
5	-7	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.45
6	-7	arbitrary ($n \neq -2$)	$\frac{1}{5}(n-3)$	2.6.1.46
7	-4	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.40
8	-4	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.42
9	-4	-2	-1	2.6.1.41
10	-2	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.28
11	-2	-2	arbitrary ($k \neq -1$)	2.6.1.29
12	$-\frac{5}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.35
13	$-\frac{5}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(2n+1)$	2.6.1.37
14	$-\frac{5}{2}$	-2	-1	2.6.1.36
15	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$-3n-7$	2.6.1.14
16	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.8
17	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{2}(3n+4)$	2.6.1.9
18	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.13
19	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(2n+1)$	2.6.1.38
20	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{4}(n-2)$	2.6.1.18

TABLE 2.10 *Continued*
Solvable cases of the modified Emden—Fowler equation
$$xy''_{xx} - ky'_x = Ax^{n+1}y^m$$

No	m	n	k	Equation
21	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$-\frac{1}{4}(3n+10)$	2.6.1.19
22	$-\frac{5}{3}$	arbitrary ($n \neq -2$)	$\frac{1}{7}(6n+5)$	2.6.1.39
23	$-\frac{5}{3}$	-2	-1	2.6.1.22
24	$-\frac{7}{5}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.24
25	$-\frac{7}{5}$	arbitrary ($n \neq -2$)	$-\frac{1}{3}(5n+13)$	2.6.1.25
26	-1	arbitrary ($n \neq -2$)	$n+1$	2.6.1.5
27	-1	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.4
28	-1	-2	arbitrary ($k \neq -1$)	2.6.1.7
29	-1	-2	-1	2.6.1.20
30	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$-2n-5$	2.6.1.12
31	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.11
32	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{2}(3n+4)$	2.6.1.43
33	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(n-1)$	2.6.1.16
34	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{3}(2n+1)$	2.6.1.26
35	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$-\frac{1}{3}(2n+7)$	2.6.1.17
36	$-\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{5}(6n+7)$	2.6.1.44
37	$-\frac{1}{2}$	-2	arbitrary ($k \neq -1$)	2.6.1.27
38	$-\frac{1}{2}$	-2	-1	2.6.1.21
39	$\frac{1}{2}$	arbitrary ($n \neq -2$)	$\frac{1}{2}n$	2.6.1.15
40	$\frac{1}{2}$	arbitrary ($n \neq -2$)	$-\frac{1}{3}(2n+7)$	2.6.1.10
41	$\frac{1}{2}$	-2	-1	2.6.1.23

No	m	n	k	Equation
42	2	arbitrary $(n \neq -2)$	$-7n-15$	2.6.1.33
43	2	arbitrary $(n \neq -2)$	$\frac{1}{2}n$	2.6.1.30
44	2	arbitrary $(n \neq -2)$	$-\frac{1}{3}(n+5)$	2.6.1.32
45	2	arbitrary $(n \neq -2)$	$-\frac{1}{6}(7n+20)$	2.6.1.34
46	2	-2	-1	2.6.1.31

5. $\boldsymbol{xy''_{xx} - (n+1)y'_x = Ax^{n+1}y^{-1}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = \exp\left\{\frac{2C_2}{n+2}\int\left[C_1 + \frac{1}{4}\tau^2 + \frac{2A}{(n+2)^2}\ln|\tau|\right]^{-1/2} d\tau\right\},$$
$$y = \tau\exp\left\{C_2\int\left[C_1 + \frac{1}{4}\tau^2 + \frac{2A}{(n+2)^2}\ln|\tau|\right]^{-1/2} d\tau\right\}.$$

6. $\boldsymbol{xy''_{xx} + y'_x = Ax^{-1}y^m, \qquad m \neq -1}$.

Solution in the parametric form:

$$x = C_2\exp\left[\int(C_1 \pm \tau^{m+1})^{-1/2}\,d\tau\right], \qquad y = b\tau,$$

where $A = \pm\frac{1}{2}b^{1-m}(m+1)$.

7. $\boldsymbol{xy''_{xx} - ky'_x = Ax^{-1}y^{-1}, \qquad k \neq -1}$.

Solution in the parametric form:

$$x = \left\{\int\left[\frac{2A}{(k+1)^2}\ln\tau + C_1\right]^{-1/2} d\tau + C_2\right\}^{-\frac{1}{k+1}},$$
$$y = \left\{\int\left[\frac{2A}{(k+1)^2}\ln\tau + C_1\right]^{-1/2} d\tau + C_2\right\}^{-1}.$$

8. $\boldsymbol{xy''_{xx} - \frac{n}{2}y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^8(\tau^3 \pm 3\tau + C_2)^{\frac{2}{n+2}}, \qquad y = bC_1^{3n+6}(\tau^2 \pm 1)^{3/2},$$

where $A = \pm\frac{1}{12}a^{-n-2}b^{8/3}(n+2)^2$.

9. $\boldsymbol{xy''_{xx} - \dfrac{3n+4}{2}y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^8(\tau^3 \pm 3\tau + C_2)^{-\frac{2}{3n+6}}, \qquad y = bC_1^{3n+6}(\tau^2 \pm 1)^{3/2}(\tau^3 \pm 3\tau + C_2)^{-1},$$

where $A = \pm\frac{3}{4}a^{-n-2}b^{8/3}(n+2)^2$.

10. $\boldsymbol{xy''_{xx} + \dfrac{2n+7}{3}y'_x = Ax^{n+1}y^{1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1\left[\int \frac{\tau\, d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_2\right]^{\frac{3}{2n+4}}, \qquad y = bC_1^{2n+4}\tau^2\left[\int \frac{\tau\, d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_2\right]^{-1},$$

where $A = \pm\frac{16}{3}a^{-n-2}b^{1/2}(n+2)^2$.

11. $\boldsymbol{xy''_{xx} - \dfrac{n}{2}y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2)^{\frac{2}{n+2}}, \qquad y = bC_1^{2n+4}(\tau^2 - 1)^2,$$

where $A = \pm\frac{1}{9}(n+2)^2a^{-n-2}b^{3/2}$.

12. $\boldsymbol{xy''_{xx} + (2n+5)y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^3(\tau^3 - 3\tau + C_2)^{\frac{1}{2n+4}}, \qquad y = bC_1^{2n+4}(\tau^2 - 1)^2(\tau^3 - 3\tau + C_2)^{-1},$$

where $A = \pm\frac{16}{9}(n+2)^2a^{-n-2}b^{3/2}$.

13. $\boldsymbol{xy''_{xx} - \dfrac{n-1}{3}y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^8\left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\right]^{\frac{3}{n+2}}, \qquad y = bC_1^{3n+6}(\tau^3 - 3\tau + C_2)^{3/2},$$

where $A = \pm\frac{1}{64}(n+2)^2a^{-n-2}b^{8/3}$.

14. $\boldsymbol{xy''_{xx} + (3n+7)y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^8\left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\right]^{\frac{1}{3n+6}},$$
$$y = bC_1^{3n+6}(\tau^3 - 3\tau + C_2)^{3/2}\left[\pm(\tau^4 - 6\tau^2 + 4C_2\tau - 3)\right]^{-1},$$

where $A = \pm\frac{81}{64}(n+2)^2a^{-n-2}b^{8/3}$.

15. $\boldsymbol{xy''_{xx} - \frac{n}{2} y'_x = Ax^{n+1} y^{1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1 \left[\int \frac{\tau\, d\tau}{\sqrt{\pm(4\tau^3 - 1)}} + C_2 \right]^{\frac{2}{n+2}}, \qquad y = bC_1^{2n+4} \tau^2,$$

where $A = \pm 3a^{-n-2} b^{1/2} (n+2)^2$.

16. $\boldsymbol{xy''_{xx} - \frac{n-1}{3} y'_x = Ax^{n+1} y^{-1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = \left[C_1 e^{2s\tau} + C_2 e^{-s\tau} \sin\left(\sqrt{3}\, s\tau\right) \right]^{\frac{3}{n+2}},$$
$$y = \left\{ 2C_1 s e^{2s\tau} + C_2 s e^{-s\tau} \left[\sqrt{3} \cos\left(\sqrt{3}\, s\tau\right) - \sin\left(\sqrt{3}\, s\tau\right) \right] \right\}^2,$$

where $A = \frac{16}{9} s^3 (n+2)^2$.

17. $\boldsymbol{xy''_{xx} + \frac{2n+7}{3} y'_x = Ax^{n+1} y^{-1/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = \left[C_1 e^{2s\tau} + C_2 e^{-s\tau} \sin\left(\sqrt{3}\, s\tau\right) \right]^{\frac{3}{2n+4}},$$
$$y = \frac{\left\{ 2C_1 s e^{2s\tau} + C_2 s e^{-s\tau} \left[\sqrt{3} \cos\left(\sqrt{3}\, s\tau\right) - \sin\left(\sqrt{3}\, s\tau\right) \right] \right\}^2}{C_1 e^{2s\tau} + C_2 e^{-s\tau} \sin\left(\sqrt{3}\, s\tau\right)},$$

where $A = \frac{64}{9} s^3 (n+2)^2$.

18. $\boldsymbol{xy''_{xx} - \frac{n-2}{4} y'_x = Ax^{n+1} y^{-5/3}, \qquad n \neq -2.}$

1°. Solution in the parametric form with $A < 0$:

$$x = aC_1^8 \left[\cosh(\tau + C_2) \cos\tau \right]^{\frac{4}{n+2}} \left[\tanh(\tau + C_2) + \tan\tau \right]^{\frac{4}{n+2}},$$
$$y = bC_1^{3n+6} \left[\cosh(\tau + C_2) \cos\tau \right]^{3/2},$$

where $A = -\frac{3}{256} a^{-n-2} b^{8/3} (n+2)^2$.

2°. Solution in the parametric form with $A > 0$:

$$x = aC_1^8 \left[\sinh\tau + \cos(\tau + C_2) \right]^{\frac{4}{n+2}},$$
$$y = bC_1^{3n+6} \left[\cosh\tau - \sin(\tau + C_2) \right]^{3/2},$$

where $A = \frac{3}{64} a^{-n-2} b^{8/3} (n+2)^2$.

19. $xy''_{xx} + \dfrac{3n+10}{4} y'_x = A x^{n+1} y^{-5/3}, \qquad n \neq -2.$

1°. Solution in the parametric form with $A < 0$:

$$x = aC_1^8 \bigl[\cosh(\tau + C_2)\cos\tau\bigr]^{\frac{4}{3n+6}} \bigl[\tanh(\tau + C_2) + \tan\tau\bigr]^{\frac{4}{3n+6}},$$
$$y = bC_1^{3n+6} \bigl[\cosh(\tau + C_2)\cos\tau\bigr]^{1/2} \bigl[\tanh(\tau + C_2) + \tan\tau\bigr]^{-1},$$

where $A = -\frac{27}{256} a^{-n-2} b^{8/3} (n+2)^2$.

2°. Solution in the parametric form with $A > 0$:

$$x = aC_1^8 \bigl[\sinh\tau + \cos(\tau + C_2)\bigr]^{\frac{4}{3n+6}},$$
$$y = bC_1^{3n+6} \bigl[\cosh\tau - \sin(\tau + C_2)\bigr]^{3/2} \bigl[\sinh\tau + \cos(\tau + C_2)\bigr]^{-1},$$

where $A = \frac{27}{64} a^{-n-2} b^{8/3} (n+2)^2$.

20. $xy''_{xx} + y'_x = A x^{-1} y^{-1}.$

Solution in the parametric form:

$$x = C_2 \exp\biggl[\int (2A\ln|\tau| + C_1)^{-1/2}\,d\tau\biggr], \qquad y = \tau.$$

21. $xy''_{xx} + y'_x = A x^{-1} y^{-1/2}.$

Solution in the parametric form:

$$x = \exp(\pm\tau^3 - 3C_1\tau + C_2), \quad y = b\bigl(\pm\tau^2 - C_1\bigr)^2, \qquad \text{where} \quad A = \pm\tfrac{4}{9} b^{3/2}.$$

22. $xy''_{xx} + y'_x = A x^{-1} y^{-5/3}.$

Solution in the parametric form:

$$x = \exp(C_1\tau^3 \pm 3\tau + C_2), \qquad y = (\pm 3A/C_1)^{3/8} \bigl(C_1\tau^2 \pm 1\bigr)^{3/2}.$$

23. $xy''_{xx} + y'_x = A x^{-1} y^{1/2}.$

Solution in the parametric form:

$$x = C_1 \exp\biggl[\int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - C_1)}}\biggr], \qquad y = b\tau^2,$$

where $A = \pm 12 b^{1/2}$.

► *In the solutions of equations 24–25, the following notation is used:*

$$S_1 = C_1 e^{2s\tau} + C_2 e^{-s\tau} \sin\bigl(\sqrt{3}\,s\tau\bigr),$$
$$S_2 = 2C_1 s e^{2s\tau} + C_2 s e^{-s\tau} \bigl[\sqrt{3}\cos\bigl(\sqrt{3}\,s\tau\bigr) - \sin\bigl(\sqrt{3}\,s\tau\bigr)\bigr],$$
$$S_3 = S_2^2 - 2S_1 (S_2)'_\tau.$$

24. $\boldsymbol{xy''_{xx} - \dfrac{n-1}{3}y'_x = Ax^{n+1}y^{-7/5}}, \qquad \boldsymbol{n \neq -2}.$

Solution in the parametric form:

$$x = aS_3^{\frac{3}{n+2}}, \quad y = bS_1^{5/2}, \qquad \text{where} \quad A = -\tfrac{5}{9216}a^{-n-2}b^{12/5}s^{-6}(n+2)^2.$$

25. $\boldsymbol{xy''_{xx} + \dfrac{5n+13}{3}y'_x = Ax^{n+1}y^{-7/5}}, \qquad \boldsymbol{n \neq -2}.$

Solution in the parametric form:

$$x = aS_3^{\frac{3}{5n+10}}, \quad y = bS_1^{5/2}S_3^{-1}, \qquad \text{where} \quad A = -\tfrac{125}{9216}a^{-n-2}b^{12/5}s^{-6}(n+2)^2.$$

► *In the solutions of equations 26–29, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions.

26. $\boldsymbol{xy''_{xx} - \dfrac{2n+1}{3}y'_x = Ax^{n+1}y^{-1/2}}, \qquad \boldsymbol{n \neq -2}.$

Solution in the parametric form:

$$x = aC_1^3\tau^{\frac{1}{n+2}}Z^{\frac{3}{n+2}}, \qquad y = bC_1^{2n+4}\tau^{-2/3}(\tau Z'_\tau + \tfrac{1}{3}Z)^2,$$

where $A = \mp\frac{4}{27}a^{-n-2}b^{3/2}(n+2)^2$.

27. $\boldsymbol{xy''_{xx} - ky'_x = Ax^{-1}y^{-1/2}}, \qquad \boldsymbol{k \neq -1}.$

Solution in the parametric form:

$$x = C_1(\tau^{1/3}Z)^{-\frac{2}{k+1}}, \qquad y = b\tau^{-4/3}Z^{-2}(\tau Z'_\tau + \tfrac{1}{3}Z)^2,$$

where $A = \mp\frac{1}{3}b^{3/2}(k+1)^2$.

28. $\boldsymbol{xy''_{xx} - \dfrac{n-1}{3}y'_x = Ax^{n+1}y^{-2}}, \qquad \boldsymbol{n \neq -2}.$

Solution in the parametric form:

$$x = aC_1^3\tau^{-\frac{2}{n+2}}\left[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\right]^{\frac{3}{n+2}}, \qquad y = bC_1^{n+2}\tau^{2/3}Z^2,$$

where $A = -\frac{1}{2}a^{-n-2}b^3(n+2)^2$.

29. $\boldsymbol{xy''_{xx} - ky'_x = Ax^{-1}y^{-2}}, \qquad \boldsymbol{k \neq -1}.$

Solution in the parametric form:

$$x = C_1\tau^{\frac{2}{3k+3}}\left[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\right]^{-\frac{1}{k+1}}, \qquad y = b\tau^{4/3}Z^2\left[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\right]^{-1},$$

where $A = -\frac{9}{2}b^3(k+1)^2$.

► *In the solutions of equations 30–39, function $\wp$ is defined implicitly:*

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3 - 1)}} - C_2, \qquad f = \sqrt{\pm(4\wp^3 - 1)}.$$

The upper sign in the formulae corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$. The solutions outlined below are written in the parametric form—one can assume both τ as the parameter, hence $\wp = \wp(\tau)$, and $\wp$, hence $\tau = \tau(\wp)$.

30. $\boldsymbol{xy''_{xx} - \frac{n}{2} y'_x = Ax^{n+1}y^2, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{\frac{2}{n+2}}, \quad y = bC_1^{n+2}\wp, \qquad \text{where} \quad A = \pm\tfrac{3}{2}a^{-n-2}b^{-1}(n+2)^2.$$

31. $\boldsymbol{xy''_{xx} + y'_x = Ax^{-1}y^2.}$

Solution in the parametric form:

$$x = C_2 e^{\tau}, \qquad y = b\wp(\tau,\ 0,\ C_1),$$

where $A = \pm 6b^{-1}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\ 0,\ C_1)$ is defined implicitly by the integral $\tau = \int_{\infty}^{\wp} (4z^3 - C_1)^{-1/2}\, dz$.

32. $\boldsymbol{xy''_{xx} + \frac{n+5}{3} y'_x = Ax^{n+1}y^2, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}\tau^{-1}\wp, \qquad \text{where} \quad A = \pm\tfrac{2}{3}a^{-n-2}b^{-1}(n+2)^2.$$

33. $\boldsymbol{xy''_{xx} + (7n+15)y'_x = Ax^{n+1}y^2, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{-\frac{1}{n+2}}, \quad y = bC_1^{n+2}\tau(\tau^2\wp \mp 1), \qquad \text{where} \quad A = \pm 6a^{-n-2}b^{-1}(n+2)^2.$$

34. $\boldsymbol{xy''_{xx} + \frac{7n+20}{6} y'_x = Ax^{n+1}y^2, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{\frac{6}{7(n+2)}}, \quad y = bC_1^{n+2}\tau^{-6}(\tau^2\wp \mp 1),$$

where $A = \pm\frac{1}{6}a^{-n-2}b^{-1}(n+2)^2$.

35. $\boldsymbol{xy''_{xx} - \frac{n}{2} y'_x = Ax^{n+1}y^{-5/2}, \qquad n \neq -2.}$

Solution in the parametric form:

$$x = aC_1^7\wp^{-\frac{4}{n+2}}(f \pm 2\tau\wp^2)^{\frac{2}{n+2}}, \quad y = bC_1^{2n+4}\wp^{-2},$$

where $A = \mp\frac{3}{4}a^{-n-2}b^{7/2}(n+2)^2$.

36. $xy''_{xx} + y'_x = Ax^{-1}y^{-5/2}$.

Solution in the parametric form:

$$x = C_2 \exp[\wp^{-2}(f \pm 2\tau\wp^2)], \qquad y = b\wp^{-2},$$

where $A = \mp 3b^{7/2}$, and the elliptic Weierstrass function $\wp = \wp(\tau,\ 0,\ C_1)$ is defined implicitly by the integral $\tau = \displaystyle\int_\infty^\wp (4z^3 - C_1)^{-1/2}\, dz$.

37. $xy''_{xx} - \dfrac{2n+1}{3} y'_x = Ax^{n+1}y^{-5/2}, \qquad n \neq -2$.

Solution in the parametric form:

$$x = aC_1^7 \wp^{\frac{3}{n+2}} (f \pm 2\tau\wp^2)^{-\frac{3}{2n+4}}, \quad y = bC_1^{2n+4}(f \pm 2\tau\wp^2)^{-1},$$

where $A = \mp\frac{4}{3}a^{-n-2}b^{7/2}(n+2)^2$.

38. $xy''_{xx} - \dfrac{2n+1}{3} y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2$.

Solution in the parametric form:

$$x = aC_1^8(\tau f + 2\wp)^{\frac{3}{n+2}}, \quad y = bC_1^{3n+6}(f \pm 2\tau\wp^2)^{3/2},$$

where $A = -\frac{2}{27}a^{-n-2}b^{8/3}(n+2)^2$.

39. $xy''_{xx} - \dfrac{6n+5}{7} y'_x = Ax^{n+1}y^{-5/3}, \qquad n \neq -2$.

Solution in the parametric form:

$$x = aC_1^8(\tau f + 2\wp)^{-\frac{7}{3n+6}}, \quad y = bC_1^{3n+6}(f \pm 2\tau\wp^2)^{3/2}(\tau f + 2\wp)^{-2},$$

where $A = -\frac{6}{49}a^{-n-2}b^{8/3}(n+2)^2$.

► *In the solutions of equations 40–46, the following notation is used:*

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad F_1 = 2\tau I(\tau) + C_2\tau \mp R, \quad F_2 = \tau^{-1}(RF_1 - 1),$$

where $I(\tau) = \int \tau R^{-1}\, d\tau$ *is the incomplete elliptic integral of the second kind in the Weierstrass form.*

40. $xy''_{xx} - \dfrac{n}{2} y'_x = Ax^{n+1}y^{-4}, \qquad n \neq -2$.

Solution in the parametric form:

$$x = aC_1^5(\tau^{-1}F_1)^{\frac{2}{n+2}}, \quad y = bC_1^{n+2}\tau^{-1}, \qquad \text{where} \quad A = \mp\tfrac{3}{2}a^{-n-2}b^5(n+2)^2.$$

41. $\boldsymbol{xy''_{xx} + y'_x = Ax^{-1}y^{-4}}$.

Solution in the parametric form:

$$x = C_2^2 \exp\left[2\int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - C_1)}} + C_2 \mp \frac{1}{\tau}\sqrt{\pm(4\tau^3 - C_1)}\right], \quad y = \mp(AC_1^2/6)^{1/5}\tau^{-1}.$$

42. $\boldsymbol{xy''_{xx} - \frac{n-1}{3}y'_x = Ax^{n+1}y^{-4}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^5(\tau^{-1}F_1)^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}F_1^{-1}, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-n-2}b^5(n+2)^2.$$

43. $\boldsymbol{xy''_{xx} - \frac{3n+4}{2}y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^3F_1^{\frac{2}{n+2}}, \quad y = bC_1^{2n+4}F_2^2, \qquad \text{where} \quad A = \pm 3a^{-n-2}b^{3/2}(n+2)^2.$$

44. $\boldsymbol{xy''_{xx} - \frac{6n+7}{5}y'_x = Ax^{n+1}y^{-1/2}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^3F_1^{-\frac{5}{2n+4}}, \quad y = bC_1^{2n+4}F_1^{-3}F_2^2, \qquad \text{where} \quad A = \pm\tfrac{48}{25}a^{-n-2}b^{3/2}(n+2)^2.$$

45. $\boldsymbol{xy''_{xx} - \frac{n-1}{3}y'_x = Ax^{n+1}y^{-7}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^8(4\tau F_1^2 \mp F_2^2)^{\frac{3}{n+2}}, \quad y = bC_1^{n+2}F_1^{1/2}, \qquad \text{where} \quad A = \pm\tfrac{1}{192}a^{-n-2}b^8(n+2)^2.$$

46. $\boldsymbol{xy''_{xx} - \frac{n-3}{5}y'_x = Ax^{n+1}y^{-7}, \qquad n \neq -2}$.

Solution in the parametric form:

$$x = aC_1^8(4\tau F_1^2 \mp F_2^2)^{-\frac{5}{n+2}}, \quad y = bC_1^{n+2}F_1^{1/2}(4\tau F_1^2 \mp F_2^2)^{-1},$$

where $A = \pm\frac{3}{1600}a^{-n-2}b^8(n+2)^2$.

2.6.2. Equations of the Form $y''_{xx} = (A_1x^{n_1}y^{m_1} + A_2x^{n_2}y^{m_2})(y'_x)^l$

See Section 2.4 for the case $l = 0$.

Table 2.11 represents all solvable equations whose solutions are outlined in Subsection 2.6.2. Equations are arranged in accordance with the growth of l, the growth of m_1 (for identical l), the growth of m_2 (for identical l and m_1, $m_1 \geq m_2$), the growth of n_1 (for identical l, m_1, and m_2), and the growth of n_2 (for identical l, m_1, m_2, and n_1). The number of the equation sought is indicated in the last column in this table.

TABLE 2.11
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
Any ($l \neq 2$)	Any ($m_1 \neq -1$)	Any ($m_2 \neq -1$)	0	0	Any	Any	2.6.2.1
$\dfrac{m_1+2n_1+3}{m_1+n_1+2}$	Any	Any	Any	$\dfrac{m_2(n_1+1)-m_1+n_1}{m_1+1}$	Any	Any	2.6.2.98
Any ($l \neq 1$)	0	0	Any ($n_1 \neq -1$)	Any ($n_2 \neq -1$)	Any	Any	2.6.2.5
Any ($l \neq 2$)	Any ($m_1 \neq -1$)	-1	0	0	Any	Any	2.6.2.2
Any ($l \neq 2$)	0	0	Any ($n_1 \neq -1$)	-1	Any	Any	2.6.2.6
$\dfrac{3m_1+5}{2m_1+3}$	Any	$-m_1-2$	1	0	Any	Any	2.6.2.21
$\dfrac{m_1+5}{m_1+3}$	Any	$\dfrac{m_1-1}{2}$	1	0	Any	Any	2.6.2.94
$\dfrac{3n_1+4}{n_1+1}$	1	0	Any	$-n_1-2$	Any	Any	2.6.2.22
$\dfrac{2(n_1+2)}{n_1+3}$	1	0	Any	$\dfrac{n_1-1}{2}$	Any	Any	2.6.2.95
Any ($l \neq 1, 2$)	1	0	0	1	Any	Any	2.6.2.20
$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{7}{4}$	0	1	Any	Any	2.6.2.71
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{7}{4}$	Any	Any	2.6.2.81
$\frac{1}{2}$	1	0	$-\frac{15}{8}$	$-\frac{13}{8}$	Any	Any	2.6.2.66
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{15}{13}$	Any	Any	2.6.2.68
$\frac{1}{2}$	1	0	$-\frac{20}{13}$	$-\frac{14}{13}$	Any	Any	2.6.2.84
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{3}{4}$	Any	Any	2.6.2.64
$\frac{1}{2}$	1	0	$-\frac{5}{4}$	$-\frac{1}{2}$	Any	Any	2.6.2.78
$\frac{1}{2}$	1	0	0	1	Any	Any	2.6.2.62
$\frac{1}{2}$	1	0	0	2	Any	Any	2.6.2.75
1	0	0	Any ($n_1 \neq -1$)	Any ($n_2 \neq -1$)	Any	Any	2.6.2.7
1	0	0	Any ($n_1 \neq -1$)	-1	Any	Any	2.6.2.8
1	0	-2	0	1	Any	Any	2.6.2.25
1	1	0	0	1	Any	Any	2.6.2.23
$\frac{3}{2}$	Any	Any	m_1	m_2	Any	Any	2.6.2.97

TABLE 2.11 *Continued*
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
$\frac{3}{2}$	0	-2	0	1	Any	Any	2.6.2.107
$\frac{3}{2}$	0	$-\frac{1}{2}$	0	1	Any	Any	2.6.2.105
$\frac{3}{2}$	1	0	-2	0	Any	Any	2.6.2.108
$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	Any	Any	2.6.2.106
2	Any ($m_1 \neq -1$)	Any ($m_2 \neq -1$)	0	0	Any	Any	2.6.2.3
2	Any ($m_1 \neq -1$)	-1	0	0	Any	Any	2.6.2.4
2	1	0	-2	0	Any	Any	2.6.2.26
2	1	0	0	1	Any	Any	2.6.2.24
$\frac{5}{2}$	$-\frac{7}{4}$	$-\frac{15}{8}$	0	1	Any	Any	2.6.2.80
$\frac{5}{2}$	$-\frac{13}{8}$	$-\frac{15}{8}$	0	1	Any	Any	2.6.2.65
$\frac{5}{2}$	$-\frac{15}{13}$	$-\frac{20}{13}$	0	1	Any	Any	2.6.2.67
$\frac{5}{2}$	$-\frac{14}{13}$	$-\frac{20}{13}$	0	1	Any	Any	2.6.2.83
$\frac{5}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$	0	1	Any	Any	2.6.2.63
$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{5}{4}$	0	1	Any	Any	2.6.2.77
$\frac{5}{2}$	1	0	$-\frac{7}{4}$	$-\frac{1}{4}$	Any	Any	2.6.2.72
$\frac{5}{2}$	1	0	0	1	Any	Any	2.6.2.61
$\frac{5}{2}$	2	0	0	1	Any	Any	2.6.2.74
3	Any	Any	$-m_1-3$	$-m_2-3$	Any	Any	2.6.2.9
3	Any	Any	$-2m_1-3$	$-2m_2-3$	Any	Any	2.6.2.93
3	Any ($m_1 \neq -2$)	Any	1	0	Any	Any	2.6.2.49
3	Any	-3	$-m_1-3$	0	Any	Any	2.6.2.19
3	Any ($m_1 \neq -2$)	0	1	-3	Any	Any	2.6.2.51
3	-5	-6	1	3	Any	Any	2.6.2.100
3	-4	-5	0	2	Any	Any	2.6.2.76
3	-3	-5	0	1	Any	Any	2.6.2.44
3	-3	-5	0	2	Any	Any	2.6.2.58
3	-3	$-\frac{7}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.28
3	$-\frac{14}{5}$	$-\frac{18}{5}$	2	3	Any	Any	2.6.2.112

TABLE 2.11 *Continued*

Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	$-\frac{8}{3}$	$-\frac{10}{3}$	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.38
3	$-\frac{5}{2}$	-4	$-\frac{1}{2}$	0	Any	Any	2.6.2.86
3	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	Any	Any	2.6.2.13
3	$-\frac{5}{2}$	-3	$-\frac{1}{2}$	0	Any	Any	2.6.2.32
3	$-\frac{12}{5}$	$-\frac{13}{5}$	$-\frac{3}{5}$	$-\frac{7}{5}$	Any	Any	2.6.2.30
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.34
3	$-\frac{7}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	2.6.2.88
3	$-\frac{7}{3}$	-3	$-\frac{2}{3}$	0	Any	Any	2.6.2.70
3	$-\frac{7}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.42
3	$-\frac{11}{5}$	$-\frac{12}{5}$	2	3	Any	Any	2.6.2.113
3	-2	$-n_2-1$	1	Any	$\dfrac{2(n_2+1)}{n_2+3}$	Any	2.6.2.132
3	-2	-3	-2	0	Any	Any	2.6.2.104
3	-2	-3	1	2	$\frac{6}{25}$	Any	2.6.2.145
3	-2	-3	1	2	$-\frac{6}{25}$	Any	2.6.2.144
3	-2	-3	-1	0	Any	Any	2.6.2.12
3	-2	-3	$-\frac{1}{2}$	0	Any	Any	2.6.2.102
3	-2	-2	1	Any	$\dfrac{2(n_2+1)}{n_2+3}$	Any	2.6.2.116
3	-2	-2	1	-7	$-\frac{15}{4}$	Any	2.6.2.117
3	-2	-2	1	-4	-6	Any	2.6.2.118
3	-2	-2	1	$-\frac{5}{2}$	-12	Any	2.6.2.119
3	-2	-2	1	-2	-2	Any	2.6.2.120
3	-2	-2	1	$-\frac{5}{3}$	$-\frac{63}{4}$	Any	2.6.2.124
3	-2	-2	1	$-\frac{5}{3}$	$-\frac{3}{4}$	Any	2.6.2.123
3	-2	-2	1	$-\frac{5}{3}$	$\frac{3}{16}$	Any	2.6.2.121
3	-2	-2	1	$-\frac{5}{3}$	$\frac{9}{100}$	Any	2.6.2.122
3	-2	-2	1	$-\frac{7}{5}$	$\frac{5}{36}$	Any	2.6.2.125
3	-2	-2	1	$-\frac{1}{2}$	-20	Any	2.6.2.128
3	-2	-2	1	$-\frac{1}{2}$	$\frac{4}{25}$	Any	2.6.2.127
3	-2	-2	1	$-\frac{1}{2}$	$\frac{2}{9}$	Any	2.6.2.126

TABLE 2.11 *Continued*

Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	-2	-2	1	$\frac{1}{2}$	$\frac{12}{49}$	Any	2.6.2.129
3	-2	-2	2	1	Any	$-\frac{6}{25}$	2.6.2.131
3	-2	-2	2	1	Any	$\frac{6}{25}$	2.6.2.130
3	$-\frac{13}{7}$	$-\frac{20}{7}$	0	2	Any	Any	2.6.2.82
3	$-\frac{12}{7}$	$-\frac{20}{7}$	0	2	Any	Any	2.6.2.60
3	$-\frac{5}{3}$	$-\frac{7}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.92
3	$-\frac{8}{5}$	$-\frac{13}{5}$	$-\frac{7}{5}$	$-\frac{7}{5}$	Any	Any	2.6.2.109
3	$-\frac{3}{2}$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.90
3	$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2	Any	Any	2.6.2.48
3	$-\frac{3}{2}$	-2	0	$-\frac{1}{2}$	Any	Any	2.6.2.46
3	$-\frac{3}{2}$	-2	$\frac{1}{2}$	1	Any	$\frac{12}{49}$	2.6.2.143
3	$-\frac{4}{3}$	$-\frac{10}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.36
3	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.40
3	$-\frac{4}{3}$	$-\frac{7}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.14
3	$-\frac{4}{3}$	$-\frac{4}{3}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.15
3	$-\frac{9}{7}$	$-\frac{15}{7}$	0	2	Any	Any	2.6.2.59
3	$-\frac{7}{6}$	$-\frac{5}{3}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.16
3	$-\frac{8}{7}$	$-\frac{15}{7}$	0	2	Any	Any	2.6.2.79
3	-1	-2	-2	-2	Any	Any	2.6.2.110
3	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.115
3	$-\frac{1}{2}$	-3	$-\frac{1}{2}$	0	Any	Any	2.6.2.53
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	-20	2.6.2.141
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	$\frac{4}{25}$	2.6.2.134
3	$-\frac{1}{2}$	-2	$-\frac{1}{2}$	1	Any	$\frac{2}{9}$	2.6.2.137
3	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.45
3	0	-5	-3	1	Any	Any	2.6.2.52
3	0	-2	-3	-2	Any	Any	2.6.2.56
3	0	-2	0	$-\frac{1}{2}$	Any	Any	2.6.2.54
3	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	Any	Any	2.6.2.89
3	0	$-\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.114

TABLE 2.11 *Continued*
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	Any	Any	2.6.2.101
3	0	0	$-\frac{1}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.39
3	0	0	0	-1	Any	Any	2.6.2.11
3	0	0	0	$-\frac{2}{3}$	Any	Any	2.6.2.69
3	0	0	0	$-\frac{1}{2}$	Any	Any	2.6.2.31
3	0	0	2	0	Any	Any	2.6.2.57
3	$\frac{2}{5}$	-2	$-\frac{7}{5}$	1	Any	$\frac{5}{36}$	2.6.2.139
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{63}{4}$	2.6.2.147
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$-\frac{3}{4}$	2.6.2.136
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{9}{100}$	2.6.2.135
3	$\frac{2}{3}$	-2	$-\frac{5}{3}$	1	Any	$\frac{3}{16}$	2.6.2.138
3	1	-2	-2	1	Any	-2	2.6.2.133
3	1	0	-7	-3	Any	Any	2.6.2.17
3	1	0	-4	-3	Any	Any	2.6.2.10
3	1	0	-2	-3	Any	Any	2.6.2.55
3	1	0	-2	$-\frac{3}{2}$	Any	Any	2.6.2.47
3	1	0	-2	0	Any	Any	2.6.2.103
3	1	0	$-\frac{5}{3}$	$-\frac{4}{3}$	Any	Any	2.6.2.91
3	1	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.41
3	1	0	$-\frac{5}{3}$	$\frac{1}{3}$	Any	Any	2.6.2.87
3	1	0	$-\frac{7}{5}$	$-\frac{3}{5}$	Any	Any	2.6.2.29
3	1	0	$-\frac{1}{2}$	0	Any	Any	2.6.2.27
3	1	0	0	$-\frac{1}{2}$	Any	Any	2.6.2.85
3	1	0	0	2	Any	Any	2.6.2.73
3	1	0	1	-3	Any	Any	2.6.2.50
3	1	0	1	0	Any	Any	2.6.2.43
3	1	0	1	3	Any	Any	2.6.2.99
3	$\frac{3}{2}$	-2	$-\frac{5}{2}$	1	Any	-12	2.6.2.146
3	2	0	-5	-5	Any	Any	2.6.2.96
3	2	0	$-\frac{5}{3}$	$-\frac{5}{3}$	Any	Any	2.6.2.35

TABLE 2.11 *Continued*
Solvable cases of the equation $y''_{xx} = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y'_x)^l$

l	m_1	m_2	n_1	n_2	A_1	A_2	Equation
3	2	0	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.37
3	2	1	$-\frac{5}{3}$	$-\frac{1}{3}$	Any	Any	2.6.2.33
3	3	−2	−4	1	Any	−6	2.6.2.140
3	3	0	−7	−3	Any	Any	2.6.2.18
3	4	3	−7	−7	Any	Any	2.6.2.111
3	6	−2	−7	1	Any	Any	2.6.2.142

1. $y''_{xx} = (A_1 y^{m_1} + A_2 y^{m_2})(y'_x)^l, \qquad l \neq 2, \quad m_1 \neq -1, \quad m_2 \neq -1.$

1°. Solution in the parametric form:

$$x = a \int \left(C_1 + \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{\frac{1}{l-2}} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \dfrac{m_1+1}{2-l} a^{l-2} b^{1-m_1-l}$, $A_2 = \pm\dfrac{m_2+1}{2-l} a^{l-2} b^{1-m_2-l}$.

2°. Solution in the parametric form:

$$x = a \int \left(C_1 - \tau^{m_1+1} \pm \tau^{m_2+1}\right)^{\frac{1}{l-2}} d\tau + C_2, \qquad y = b\tau,$$

where $A_1 = \dfrac{m_1+1}{l-2} a^{l-2} b^{1-m_1-l}$, $A_2 = \pm\dfrac{m_2+1}{2-l} a^{l-2} b^{1-m_2-l}$.

2. $y''_{xx} = (A_1 y^{m} + A_2 y^{-1})(y'_x)^l, \qquad l \neq 2, \quad m \neq -1.$

Solution:

$$x = \int \left[C_1 + \frac{A_1(2-l)}{m+1} y^{m+1} + (2-l) A_2 \ln y\right]^{\frac{1}{l-2}} dy + C_2.$$

3. $y''_{xx} = (A_1 y^{m_1} + A_2 y^{m_2})(y'_x)^2, \qquad m_1 \neq -1, \quad m_2 \neq -1.$

Solution:

$$x = C_1 \int \exp\left(-\frac{A_1}{m_1+1} y^{m_1+1} - \frac{A_2}{m_2+1} y^{m_2+1}\right) dy + C_2.$$

4. $y''_{xx} = (A_1 y^{m} + A_2 y^{-1})(y'_x)^2, \qquad m \neq -1.$

Solution:

$$x = C_1 \int y^{-A_2} \exp\left(-\frac{A_1}{m+1} y^{m+1}\right) dy + C_2.$$

5. $y''_{xx} = (A_1 x^{n_1} + A_2 x^{n_2})(y'_x)^l, \qquad l \neq 1, \quad n_1 \neq -1, \quad n_2 \neq -1.$

1°. Solution in the parametric form:

$$x = a\tau, \qquad y = b \int \left(C_1 + \tau^{n_1+1} \pm \tau^{n_2+1}\right)^{\frac{1}{1-l}} d\tau + C_2,$$

where $A_1 = \dfrac{n_1+1}{1-l} a^{l-n_1-2} b^{1-l}$, $A_2 = \pm \dfrac{n_2+1}{1-l} a^{l-n_2-2} b^{1-l}$.

2°. Solution in the parametric form:

$$x = a\tau, \qquad y = b \int \left(C_1 - \tau^{n_1+1} \pm \tau^{n_2+1}\right)^{\frac{1}{1-l}} d\tau + C_2,$$

where $A_1 = \dfrac{n_1+1}{l-1} a^{l-n_1-2} b^{1-l}$, $A_2 = \pm \dfrac{n_2+1}{1-l} a^{l-n_2-2} b^{1-l}$.

6. $y''_{xx} = (A_1 x^n + A_2 x^{-1})(y'_x)^l, \qquad l \neq 1, \quad n \neq -1.$

Solution:

$$y = \int \left[C_1 + \frac{A_1(1-l)}{n+1} x^{n+1} + (1-l) A_2 \ln x\right]^{\frac{1}{1-l}} dx + C_2.$$

7. $y''_{xx} = (A_1 x^{n_1} + A_2 x^{n_2}) y'_x, \qquad n_1 \neq -1, \quad n_2 \neq -1.$

Solution:

$$y = C_1 \int \exp\left(\frac{A_1}{n_1+1} x^{n_1+1} + \frac{A_2}{n_2+1} x^{n_2+1}\right) dx + C_2.$$

8. $y''_{xx} = (A_1 x^n + A_2 x^{-1}) y'_x, \qquad n \neq -1.$

Solution:

$$y = C_1 \int x^{A_2} \exp\left(\frac{A_1}{n+1} x^{n+1}\right) dx + C_2.$$

9. $y''_{xx} = (A_1 x^{-m_1-3} y^{m_1} + A_2 x^{-m_2-3} y^{m_2})(y'_x)^3, \qquad m_1 \neq -2, \quad m_2 \neq -2.$

1°. Solution in the parametric form:

$$x = a\tau \left[\int \left(C_1 + \tau^{-m_1-2} \pm \tau^{-m_2-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b \left[\int \left(C_1 + \tau^{-m_1-2} \pm \tau^{-m_2-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \frac{1}{2} a^{m_1+4} b^{-m_1-2}(m_1+2)$, $A_2 = \pm \frac{1}{2} a^{m_2+4} b^{-m_2-2}(m_2+2)$.

2°. Solution in the parametric form:

$$x = a\tau \left[\int \left(C_1 - \tau^{-m_1-2} \pm \tau^{-m_2-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b \left[\int \left(C_1 - \tau^{-m_1-2} \pm \tau^{-m_2-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = -\frac{1}{2} a^{m_1+4} b^{-m_1-2}(m_1+2)$, $A_2 = \pm \frac{1}{2} a^{m_2+4} b^{-m_2-2}(m_2+2)$.

10. $\boldsymbol{y''_{xx} = (A_1 x^{-4} y + A_2 x^{-3})(y'_x)^3.}$

1°. Solution in the parametric form:

$$x = a\tau\left[\int\left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\left[\int\left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = \frac{3}{2}a^5 b^{-3}$, $A_2 = \pm a^4 b^{-2}$.

2°. Solution in the parametric form:

$$x = a\tau\left[\int\left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$
$$y = b\left[\int\left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{-1/2} d\tau + C_2\right]^{-1},$$

where $A_1 = -\frac{3}{2}a^5 b^{-3}$, $A_2 = \pm a^4 b^{-2}$.

11. $\boldsymbol{y''_{xx} = (A_1 + A_2 x^{-1})(y'_x)^3.}$

Solution:

$$y = \int (C_1 - 2A_1 x - 2A_2 \ln x)^{-1/2}\, dx + C_2.$$

12. $\boldsymbol{y''_{xx} = (A_1 x^{-1} y^{-2} + A_2 y^{-3})(y'_x)^3.}$

Solution in the parametric form:

$$x = \tau\left[\int (C_1 - 2A_1 \ln\tau - 2A_2\tau)^{-1/2}\, d\tau + C_2\right]^{-1},$$
$$y = \left[\int (C_1 - 2A_1 \ln\tau - 2A_2\tau)^{-1/2}\, d\tau + C_2\right]^{-1}.$$

13. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 x^{-1/2} y^{-7/2})(y'_x)^3.}$

Solution in the parametric form:

$$x = \frac{k^2}{F}\left\{2C_1 e^{2k\tau} + C_2 e^{-k\tau}\left[\sqrt{3}\cos(\omega\tau) - \sin(\omega\tau)\right]\right\}^2, \qquad y = \frac{1}{F},$$

where $F = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin(\omega\tau) - \dfrac{A_1}{A_2}$, $A_2 = -16k^3$, $\omega = k\sqrt{3}$.

14. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-5/3} y^{-7/3})(y'_x)^3.}$

Solution in the parametric form:

$$x = \left(\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau^2 + C_3\tau\right)^{-1}\left(\tfrac{1}{9}A_2\tau^3 + 3C_1\tau^2 + 2C_2\tau + C_3\right)^{3/2},$$
$$y = \left(\tfrac{1}{36}A_2\tau^4 + C_1\tau^3 + C_2\tau^2 + C_3\tau\right)^{-1},$$

where $A_1 = 9C_1C_3 - 3C_2^2$.

▶ *In the solutions of equations 15–18, the following notation is used:*

$$R_1 = \left(C_1 + \tau^{-3} \pm \tau^{-2}\right)^{1/2}, \qquad R_2 = \left(C_1 - \tau^{-3} \pm \tau^{-2}\right)^{1/2},$$
$$E_1 = \int R_1^{-1}\,d\tau + C_2, \qquad E_2 = \int R_2^{-1}\,d\tau + C_2,$$
$$F_1 = \tau - R_1E_1, \qquad F_2 = \tau - R_2E_2,$$
$$H_1 = 3\tau^3F_1^2 + 3(1\pm\tau)E_1^2, \qquad H_2 = 3\tau^3F_2^2 + 3(-1\pm\tau)E_2^2.$$

15. $\boldsymbol{y''_{xx} = (A_1y^{-4/3} + A_2x^{-1/2}y^{-4/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = aF_k^2, \qquad y = b\tau^{-3}E_k^3,$$

where $A_1 = \pm\frac{2}{9}ab^{-2/3}$, $A_2 = \frac{1}{3}a^{3/2}b^{-2/3}(-1)^{k+1}$; $k = 1$ and $k = 2$.

16. $\boldsymbol{y''_{xx} = (A_1x^{-1/2}y^{-7/6} + A_2y^{-5/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^3E_k^{-3}F_k^2, \qquad y = b\tau^3E_k^{-3},$$

where $A_1 = \frac{1}{3}a^{3/2}b^{-5/6}(-1)^k$, $A_2 = \mp\frac{2}{9}ab^{-1/3}$; $k = 1$ and $k = 2$.

17. $\boldsymbol{y''_{xx} = (A_1x^{-7}y + A_2x^{-3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-1/2}E_k^{1/2}, \qquad y = b\tau^{-3}H_k,$$

where $A_1 = \frac{1}{36}a^8b^{-3}$, $A_2 = \pm\frac{1}{36}a^4b^{-2}$; $k = 1$ and $k = 2$.

18. $\boldsymbol{y''_{xx} = (A_1x^{-7}y^3 + A_2x^{-3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{5/2}E_k^{1/2}H_k^{-1}, \qquad y = b\tau^3H_k^{-1},$$

where $A_1 = \frac{1}{36}a^8b^{-5}$, $A_2 = \mp\frac{1}{36}a^4b^{-2}$; $k = 1$ and $k = 2$.

▶ *In the solutions of equations 19–22, the following notation is used:*

$$R_1 = \left(C_1 \pm \tau^{\gamma+1} + \tau\right)^{1/2}, \qquad R_2 = \left(C_1 \pm \tau^{\gamma+1} - \tau\right)^{1/2},$$
$$E_1 = \int R_1^{-1}\,d\tau + C_2, \qquad E_2 = \int R_2^{-1}\,d\tau + C_2,$$
$$G_1 = 2R_1 - E_1, \qquad G_2 = 2R_2 + E_2,$$
$$H_1 = 4(\tau - R_1F_1) + E_1^2, \qquad H_2 = 4(\tau - R_2F_2) - E_2^2.$$

19. $y''_{xx} = (A_1 x^{-m-3} y^m + A_2 y^{-3})(y'_x)^3, \qquad m \neq -2.$

Solution in the parametric form:

$$x = a\tau E_k^{-1}, \quad y = bE_k^{-1}, \qquad \gamma = -m-3,$$

where $A_1 = \pm\frac{1}{2}a^{m+4}b^{-m-2}(m+2)$, $A_2 = \frac{1}{2}ab(-1)^k$; $k=1$ and $k=2$.

20. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^l, \qquad l \neq 1, \quad l \neq 2.$

Solution in the parametric form:

$$x = a\tau G_k, \quad y = bE_k, \qquad \gamma = \frac{1}{l-2},$$

where $A_1 = ab^{-1}A_2(-1)^{k+1}$, $A_2 = -\dfrac{\gamma}{2ab}\left[\pm\dfrac{(\gamma+1)a}{b}\right]^{1/\gamma}$; $k=1$ and $k=2$.

21. $y''_{xx} = (A_1 x y^m + A_2 y^{-m-2})(y'_x)^{\frac{3m+5}{2m+3}}.$

Solution in the parametric form:

$$x = aH_k, \quad y = bE_k^{\gamma+2}, \qquad \gamma = -\frac{2m+3}{m+1},$$

where $A_1 = \dfrac{\gamma}{4(\gamma+2)}a^{-1}b^{\frac{1}{\gamma+2}}\left[\mp\dfrac{2(\gamma+1)a}{(\gamma+2)b}\right]^{\frac{1}{\gamma}}$, $A_2 = ab^{-\frac{2}{\gamma+2}}A_1(-1)^k$; $k=1$ and $k=2$.

22. $y''_{xx} = (A_1 x^n y + A_2 x^{-n-2})(y'_x)^{\frac{3n+4}{n+1}}.$

Solution in the parametric form:

$$x = aE_k^{\gamma+2}, \quad y = bH_k, \qquad \gamma = -\frac{2n+3}{n+1},$$

where $A_1 = -\dfrac{\gamma}{4(\gamma+2)}a^{\frac{1}{\gamma+2}}b^{-1}\left[\mp\dfrac{2(\gamma+1)b}{(\gamma+2)a}\right]^{\frac{1}{\gamma}}$, $A_2 = a^{-\frac{2}{\gamma+2}}bA_1(-1)^k$; $k=1$ and $k=2$.

► *In the solutions of equations 23–26, the following notation is used:*

$$R_1 = (C_1 + \tau \pm \ln\tau)^{1/2}, \qquad R_2 = (C_1 - \tau \pm \ln\tau)^{1/2},$$
$$E_1 = \int R_1^{-1}\,d\tau + C_2, \qquad E_2 = \int R_2^{-1}\,d\tau + C_2,$$
$$G_1 = 2R_1 - E_1, \qquad G_2 = 2R_2 + E_2,$$
$$H_1 = 4(\tau - R_1F_1) + E_1^2, \qquad H_2 = 4(\tau - R_2F_2) - E_2^2,$$

23. $y''_{xx} = (A_1 y + A_2 x) y'_x$.

Solution in the parametric form:

$$x = aG_k, \qquad y = bE_k,$$

where $A_1 = ab^{-1}A_2(-1)^k$, $A_2 = \pm\frac{1}{2}a^{-2}$; $k = 1$ and $k = 2$.

24. $y''_{xx} = (A_1 y + A_2 x)(y'_x)^2$.

Solution in the parametric form:

$$x = aE_k, \qquad y = bG_k,$$

where $A_1 = \mp\frac{1}{2}b^{-2}$, $A_2 = a^{-1}bA_1(-1)^k$; $k = 1$ and $k = 2$.

25. $y''_{xx} = (A_1 + A_2 x y^{-2}) y'_x$.

Solution in the parametric form:

$$x = aH_k, \qquad y = bE_k,$$

where $A_1 = ab^{-2}A_2(-1)^k$, $A_2 = \pm\frac{1}{8}a^{-2}b^{-2}$; $k = 1$ and $k = 2$.

26. $y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^2$.

Solution in the parametric form:

$$x = aE_k, \qquad y = bH_k,$$

where $A_1 = \mp\frac{1}{8}a^2b^{-2}$, $A_2 = a^{-2}bA_1(-1)^k$; $k = 1$ and $k = 2$.

► *In the solutions of equations 27–30, the following notation is used:*

$$\begin{aligned}
R_1 &= C_1\tau^{k_1} + C_2\tau^{k_2} + C_3\tau^{k_3},\\
R_2 &= (C_1 + C_2\tau)e^{k\tau} + C_3 e^{\omega\tau},\\
R_3 &= C_1 e^{k\tau} + e^{s\tau}(C_2\sin\omega\tau + C_3\cos\omega\tau),\\
Q_1 &= C_1k_1\tau^{k_1} + C_2k_2\tau^{k_2} + C_3k_3\tau^{k_3},\\
Q_2 &= (kC_1 + C_2 + kC_2\tau)e^{k\tau} + \omega C_3 e^{\omega\tau},\\
Q_3 &= kC_1e^{k\tau} + e^{s\tau}[(sC_2 - \omega C_3)\sin\omega\tau + (sC_3 + \omega C_2)\cos\omega\tau],\\
S_1 &= \tau(Q_1)'_\tau, \quad S_2 = (Q_2)'_\tau, \quad S_3 = (Q_3)'_\tau,
\end{aligned}$$

where k_1, k_2, and k_3 (real numbers) or k and $s \pm i\omega$ (one real and two complex numbers) are the roots of the cubic equation $\lambda^3 - \frac{1}{2}B_2\lambda - \frac{1}{2}B_1 = 0$. Subscripts of functions R_m, Q_m, and S_m ($m = 1,\ 2,\ 3$) are selected depending on the sign of the following expression:

$$2B_2^3 - 27B_1^2 \quad \begin{cases} > 0 & \textit{subscript 1,}\\ = 0 & \textit{subscript 2,}\\ < 0 & \textit{subscript 3;} \end{cases}$$

if $2B_2^3 = 27B_1^2$ (subscript 2), then

$$\begin{aligned}
k &= (\tfrac{1}{6}B_2)^{1/2}, \quad \omega = -2(\tfrac{1}{6}B_2)^{1/2} \qquad \textit{if} \quad B_1 < 0,\\
k &= -(\tfrac{1}{6}B_2)^{1/2}, \quad \omega = 2(\tfrac{1}{6}B_2)^{1/2} \qquad \textit{if} \quad B_1 > 0.
\end{aligned}$$

Remark. *The expressions for R_m, and Q_m contain three constants C_1, C_2, and C_3. One of them may be arbitrarily fixed to let it be any nonzero number (for instance, we may set $C_3 = \pm1$), while the other constants remain arbitrary.*

27. $y''_{xx} = (A_1 x^{-1/2} y + A_2)(y'_x)^3.$

Solution in the parametric form:

$$x = Q_m^2, \quad y = R_m, \qquad A_1 = -B_1, \quad A_2 = -B_2.$$

28. $y''_{xx} = (A_1 y^{-3} + A_2 x^{-1/2} y^{-7/2})(y'_x)^3.$

Solution in the parametric form:

$$x = R_m^{-1} Q_m^2, \quad y = R_m^{-1}, \qquad A_1 = -B_2, \quad A_2 = -B_1.$$

29. $y''_{xx} = (A_1 x^{-7/5} y + A_2 x^{-3/5})(y'_x)^3.$

Solution in the parametric form:

$$x = aR_m^{5/2}, \qquad y = b(2Q_m^2 - 4R_m S_m + B_2 R_m^2),$$

where $A_1 = \frac{5}{32} a^{12/5} b^{-3} B_1^{-1}$, $A_2 = -a^{-4/5} b A_1 B_2$.

30. $y''_{xx} = (A_1 x^{-3/5} y^{-12/5} + A_2 x^{-7/5} y^{-13/5})(y'_x)^3.$

Solution in the parametric form:

$$x = aR_m^{5/2}\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-1}, \quad y = b\bigl(2Q_m^2 - 4R_m S_m + B_2 R_m^2\bigr)^{-2},$$

where $A_1 = -\frac{5}{32} a^{8/5} b^{2/5} B_1^{-2} B_2$, $A_2 = \frac{5}{32} a^{12/5} b^{3/5} B_1^{-2}$.

► *In the solutions of equations 31–32, the following notation is used:*

$$f_1 = \begin{cases} C_1 e^{k\tau} + C_2 e^{-k\tau} - \dfrac{B_1}{B_2}\tau & \textit{if } B_2 > 0, \\[2ex] C_1 \sin(k\tau) + C_2 \cos(k\tau) - \dfrac{B_1}{B_2}\tau & \textit{if } B_2 < 0, \end{cases}$$

$$f_2 = \begin{cases} k(C_1 e^{k\tau} - C_2 e^{-k\tau}) - \dfrac{B_1}{B_2} & \textit{if } B_2 > 0, \\[2ex] k[C_1 \cos(k\tau) - C_2 \sin(k\tau)] - \dfrac{B_1}{B_2} & \textit{if } B_2 < 0, \end{cases}$$

where $k = \sqrt{\frac{1}{2}|B_2|}$.

31. $y''_{xx} = (A_1 + A_2 x^{-1/2})(y'_x)^3.$

Solution in the parametric form:

$$x = f_2^2, \quad y = f_1, \qquad A_1 = -B_2, \quad A_2 = -B_1.$$

32. $y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 y^{-3})(y'_x)^3.$

Solution in the parametric form:

$$x = f_1^{-1} f_2^2, \quad y = f_1^{-1}, \qquad A_1 = -B_1, \quad A_2 = -B_2.$$

► *In the solutions of equations 33–36, the following notation is used:*

For $B_1 > 0$*:*

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin(k\tau), \quad k = \left(\tfrac{4}{3}B_1\right)^{1/4},$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + kC_3 \cos(k\tau);$$

For $B_1 < 0$*:*

$$T_1 = e^{s\tau}[C_1 \sin(s\tau) + C_2 \cos(s\tau)] + C_3 e^{-s\tau} \sin(s\tau), \quad s = \left(-\tfrac{1}{3}B_1\right)^{1/4},$$
$$T_2 = se^{s\tau}[(C_1 - C_2)\sin(s\tau) + (C_1 + C_2)\cos(s\tau)] - sC_3 e^{-s\tau}[\sin(s\tau) - \cos(s\tau)].$$

33. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^2 + A_2 x^{-5/3} y)(y'_x)^3}$.

Solution in the parametric form:

$$x = T_2^{3/2}, \qquad y = T_1 - \frac{A_2}{2A_1},$$

where $B_1 = -A_1$, $B_2 = -A_2$; arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &C_1C_3 = \tfrac{1}{16}A_1^{-2}A_2^2 && \text{if } A_1 > 0,\\ &4C_1C_2 + C_3^2 = \tfrac{1}{4}A_1^{-2}A_2^2 && \text{if } A_1 < 0. \end{aligned}$$

34. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{-5/3} y^{-10/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1} T_2^{3/2}, \qquad y = \left(T_1 - \frac{A_1}{2A_2}\right)^{-1},$$

where $B_1 = -A_2$, $B_2 = -A_1$; arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &C_1C_3 = \tfrac{1}{16}A_1^{2}A_2^{-2} && \text{if } A_2 > 0,\\ &4C_1C_2 + C_3^2 = \tfrac{1}{4}A_1^{2}A_2^{-2} && \text{if } A_2 < 0. \end{aligned}$$

35. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^2 + A_2 x^{-5/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_2^{3/2}, \qquad y = T_1,$$

where $B_1 = -A_1$, $B_2 = -A_2$; arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &C_1C_3 = -\tfrac{1}{4}A_1^{-1}A_2 && \text{if } A_1 > 0,\\ &4C_1C_2 + C_3^2 = -\tfrac{1}{2}A_1^{-1}A_2 && \text{if } A_1 < 0. \end{aligned}$$

36. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-5/3} y^{-10/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_1^{-1} T_2^{3/2}, \qquad y = T_1^{-1},$$

where $B_1 = -A_2$, $B_2 = -A_1$; arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &C_1C_3 = -\tfrac{1}{4}A_1A_2^{-1} && \text{if } A_2 > 0,\\ &4C_1C_2 + C_3^2 = -\tfrac{1}{2}A_1A_2^{-1} && \text{if } A_2 < 0. \end{aligned}$$

► *In the solutions of equations 37–38, the following notation is used:*

1. *For* $B_1 > 0,\ B_2 \neq 0,$

$$T_1 = C_1 e^{k\tau} + C_2 e^{-k\tau} + C_3 \sin\omega\tau,$$
$$T_2 = k(C_1 e^{k\tau} - C_2 e^{-k\tau}) + \omega C_3 \cos\omega\tau,$$

where

$$k = \left\{\tfrac{2}{3}[(B_2^2 + 3B_1)^{1/2} + B_2]\right\}^{1/2}, \quad \omega = \left\{\tfrac{2}{3}[(B_2^2 + 3B_1)^{1/2} - B_2]\right\}^{1/2},$$
$$4k^2 C_1 C_2 + \omega^2 C_3^2 = 0.$$

2. *For* $-B_2^2 < 3B_1 < 0,\ B_2 > 0,$

$$T_1 = C_1 \tau^{k_1} + C_2 \tau^{-k_1} + C_3 \tau^{k_2} + C_4 \tau^{-k_2},$$
$$T_2 = k_1(C_1 \tau^{k_1} - C_2 \tau^{-k_1}) + k_2(C_3 \tau^{k_2} - C_4 \tau^{-k_2}),$$

where

$$k_1 = \left\{\tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2}, \quad k_2 = \left\{\tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2},$$
$$(C_1C_2 + C_3C_4)(B_2^2 + 3B_1)^{1/2} + (C_1C_2 - C_3C_4)B_2 = 0.$$

3. *For* $-B_2^2 < 3B_1 < 0,\ B_2 < 0,$

$$T_1 = C_1 \sin\omega_1\tau + C_2 \cos\omega_1\tau + C_3 \sin\omega_2\tau,$$
$$T_2 = \omega_1(C_1 \cos\omega_1\tau - C_2 \sin\omega_1\tau) + \omega_2 C_3 \cos\omega_2\tau,$$

where

$$\omega_1 = \left\{-\tfrac{2}{3}[B_2 + (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2}, \quad \omega_2 = \left\{-\tfrac{2}{3}[B_2 - (B_2^2 + 3B_1)^{1/2}]\right\}^{1/2},$$
$$\omega_1^2(C_1^2 + C_2^2) - \omega_2^2 C_3^2 = 0.$$

4. *For* $B_2^2 + 3B_1 = 0,\ B_2 > 0,$

$$T_1 = (C_1 + C_2\tau)e^{k\tau} + (C_3 + C_4\tau)e^{-k\tau},$$
$$T_2 = (kC_1 + C_2 + kC_2\tau)e^{k\tau} - (kC_3 - C_4 + kC_4\tau)e^{-k\tau},$$

where

$$k = \left(\tfrac{2}{3}B_2\right)^{1/2}, \quad (C_1C_4 - C_2C_3)\left(\tfrac{2}{3}B_2\right)^{1/2} + 2C_2C_4 = 0.$$

5. *For* $B_2^2 + 3B_1 = 0,\ B_2 < 0,$

$$T_1 = (C_1 + C_2\tau)\sin\omega\tau + C_3\tau\cos\omega\tau,$$
$$T_2 = (\omega C_1 + C_3 + \omega C_2\tau)\cos\omega\tau + (C_2 - \omega C_3\tau)\sin\omega\tau,$$

where

$$\omega = \left(-\tfrac{2}{3}B_2\right)^{1/2}, \quad C_1C_3\left(-\tfrac{2}{3}B_2\right)^{1/2} + C_2^2 + C_3^2 = 0.$$

6. For $3B_1 < -B_2^2$,

$$T_1 = e^{k\tau}(C_1 \sin\omega\tau + C_2\cos\omega\tau) + C_3 e^{-k\tau}\sin\omega\tau,$$
$$T_2 = e^{k\tau}[(kC_2 + \omega C_1)\cos\omega\tau + (kC_1 - \omega C_2)\sin\omega\tau)]$$
$$+ C_3 e^{-k\tau}(\omega\cos\omega\tau - k\sin\omega\tau),$$

where

$$k = \left\{\tfrac{1}{3}[B_2 + (-3B_1)^{1/2}]\right\}^{1/2}, \quad \omega = \left\{\tfrac{1}{3}[-B_2 + (-3B_1)^{1/2}]\right\}^{1/2},$$
$$C_2 B_2 + C_1(-B_2^2 - 3B_1)^{1/2} = 0.$$

37. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^2 + A_2 x^{-1/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_2^{3/2}, \quad y = T_1, \qquad B_1 = -A_1, \quad B_2 = -A_2.$$

38. $\boldsymbol{y''_{xx} = (A_1 x^{-1/3} y^{-8/3} + A_2 x^{-5/3} y^{-10/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_1^{-1} T_2^{3/2}, \quad y = T_1^{-1}, \qquad B_1 = -A_2, \quad B_2 = -A_1.$$

► *In the solutions of equations 39–42, the following notation is used:*

$$T_1 = \begin{cases} C_1 e^{\omega\tau} + C_2 e^{-\omega\tau} + C_3\tau & \text{if } B > 0, \\ C_1 \sin\omega\tau + C_2\cos\omega\tau + C_3\tau & \text{if } B < 0; \end{cases}$$

$$T_2 = \begin{cases} \omega(C_1 e^{\omega\tau} - C_2 e^{-\omega\tau}) + C_3 & \text{if } B > 0, \\ \omega(C_1 \cos\omega\tau - C_2\sin\omega\tau) + C_3 & \text{if } B < 0; \end{cases}$$

where $\omega = |\tfrac{4}{3}B|^{1/2}$.

39. $\boldsymbol{y''_{xx} = (A_1 x^{-1/3} + A_2 x^{-5/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_2^{3/2}, \quad y = T_1,$$

where $B = -A_1$, arbitrary constants C_1, C_2, and C_3 are related by

$$3(A_1 C_3^2 + A_2) - 4A_1^2(C_1^2 + C_2^2) = 0 \quad \text{if } A_1 > 0,$$
$$3(A_1 C_3^2 + A_2) - 16A_1^2 C_1 C_2 = 0 \quad \text{if } A_1 < 0.$$

40. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^{-4/3} + A_2 x^{-1/3} y^{-8/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = T_1^{-1} T_2^{3/2}, \quad y = T_1^{-1},$$

where $B = -A_2$, arbitrary constants C_1, C_2, and C_3 are related by

$$3(A_1 + A_2 C_3^2) - 4A_2^2(C_1^2 + C_2^2) = 0 \quad \text{if } A_2 > 0,$$
$$3(A_1 + A_2 C_3^2) - 16A_2^2 C_1 C_2 = 0 \quad \text{if } A_2 < 0.$$

41. $y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{-1/3})(y'_x)^3.$

Solution in the parametric form:

$$x = \Big(T_2 - \frac{A_1}{2A_2}\tau\Big)^{3/2}, \quad y = T_1 - \frac{A_1}{4A_2}\tau^2,$$

where $B = -A_2$, arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3A_2C_3^2 - 4A_2^2(C_1^2 + C_2^2) - \tfrac{9}{16}A_1^2A_2^{-2} = 0 \quad \text{if} \quad A_2 > 0,\\ &3A_2C_3^2 - 16A_2^2C_1C_2 - \tfrac{9}{16}A_1^2A_2^{-2} = 0 \qquad \text{if} \quad A_2 < 0. \end{aligned}$$

42. $y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{-1/3} y^{-8/3})(y'_x)^3.$

Solution in the parametric form:

$$x = \Big(T_1 - \frac{A_1}{4A_2}\tau^2\Big)^{-1}\Big(T_2 - \frac{A_1}{2A_2}\tau\Big)^{3/2}, \quad y = \Big(T_1 - \frac{A_1}{4A_2}\tau^2\Big)^{-1},$$

where $B = -A_2$, arbitrary constants C_1, C_2, and C_3 are related by

$$\begin{aligned} &3A_2C_3^2 - 4A_2^2(C_1^2 + C_2^2) - \tfrac{9}{16}A_1^2A_2^{-2} = 0 \quad \text{if} \quad A_2 > 0,\\ &3A_2C_3^2 - 16A_2^2C_1C_2 - \tfrac{9}{16}A_1^2A_2^{-2} = 0 \qquad \text{if} \quad A_2 < 0. \end{aligned}$$

▶ *In the solutions of equations 43–48, the following notation is used:*

$$f = \begin{cases} J_{1/3}(\tau) & \textit{for the upper sign (Bessel function),}\\ I_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} Y_{1/3}(\tau) & \textit{for the upper sign (Bessel function),}\\ K_{1/3}(\tau) & \textit{for the lower sign (modified Bessel function),} \end{cases}$$

$$H = C_1 f + C_2 g + \beta\omega\Big(g\int f\,d\tau - f\int g\,d\tau\Big), \qquad \omega = \begin{cases} \tfrac{1}{2}\pi & \textit{for the upper sign,}\\ -1 & \textit{for the lower sign.} \end{cases}$$

43. $y''_{xx} = (A_1 xy + A_2)(y'_x)^3.$

Solution in the parametric form:

$$x = \tau^{1/3}H, \qquad y = b\tau^{2/3},$$

where $A_1 = \pm\frac{9}{4}b^{-3}$, $A_2 = -\frac{9}{4}b^{-2}\beta$.

44. $y''_{xx} = (A_1 y^{-3} + A_2 xy^{-5})(y'_x)^3.$

Solution in the parametric form:

$$x = \tau^{-1/3}H, \qquad y = b\tau^{-2/3},$$

where $A_1 = -\frac{9}{4}b\beta$, $A_2 = \pm\frac{9}{4}b^3$.

45. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-1/2} + A_2 y^{-3/2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-2/3}(\tau H'_\tau + \tfrac{1}{3}H)^2, \qquad y = b\tau^{2/3}H^2,$$

where $A_1 = \pm\frac{1}{3}a^{3/2}b^{-3/2}$, $A_2 = \frac{1}{2}ab^{-1/2}\beta$.

46. $\boldsymbol{y''_{xx} = (A_1 y^{-3/2} + A_2 x^{-1/2} y^{-2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-4/3}H^{-2}(\tau H'_\tau + \tfrac{1}{3}H)^2, \qquad y = b\tau^{-2/3}H^{-2},$$

where $A_1 = \frac{1}{2}ab^{-1/2}\beta$, $A_2 = \pm\frac{1}{3}a^{3/2}$.

47. $\boldsymbol{y''_{xx} = (A_1 x^{-2} y + A_2 x^{-3/2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{2/3}H^2, \qquad y = b\tau^{-2/3}\bigl[\mp\tau^2H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr],$$

where $A_1 = -\frac{9}{2}a^3b^{-3}$, $A_2 = -a^{-1/2}b\beta A_1$.

48. $\boldsymbol{y''_{xx} = (A_1 x^{-3/2} y^{-3/2} + A_2 x^{-2} y^{-2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{4/3}H^2\bigl[\mp\tau^2H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr]^{-1},$$
$$y = b\tau^{2/3}\bigl[\mp\tau^2H^2 + 2\beta\tau H - (\tau H'_\tau + \tfrac{1}{3}H)^2\bigr]^{-1},$$

where $A_1 = \frac{9}{2}a^{5/2}b^{-1/2}\beta$, $A_2 = -\frac{9}{2}a^3$.

49. $\boldsymbol{y''_{xx} = (A_1 x y^{m_1} + A_2 y^{m_2})(y'_x)^3, \qquad m_1 \neq -2}$.

Solution in the parametric form:

$$x = \tau^\nu H, \quad y = b\tau^{2\nu}, \qquad \nu = \frac{1}{m_1+2},$$

where

$$H = C_1 f + C_2 g + \frac{4b^2\beta\omega}{(m_1+2)^2}\Bigl(g\int\tau^k f\,d\tau - f\int\tau^k g\,d\tau\Bigr), \qquad k = \frac{2m_2 - m_1 - 1}{m_1+2};$$

$$f = \begin{cases} J_\nu(\tau) & \text{for the upper sign (Bessel function),} \\ I_\nu(\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$g = \begin{cases} Y_\nu(\tau) & \text{for the upper sign (Bessel function),} \\ K_\nu(\tau) & \text{for the lower sign (modified Bessel function),} \end{cases}$$

$$A_1 = \pm\tfrac{1}{4}(m_1+2)^2b^{-m_1-2}, \quad A_2 = -b^{-m_2}\beta, \quad \omega = \begin{cases} \frac{1}{2}\pi, & \text{for the upper sign,} \\ -1 & \text{for the lower sign.} \end{cases}$$

▶ *In the solutions of equations 50–56, the following notation is used:*

$$U_\nu = \begin{cases} C_1 J_\nu(\tau) & \textit{for the upper sign (Bessel functions),} \\ C_1 I_\nu(\tau) & \textit{for the lower sign (modified Bessel functions),} \end{cases}$$

$$V_\nu = \begin{cases} C_2 Y_\nu(\tau) & \textit{for the upper sign (Bessel functions),} \\ C_2 K_\nu(\tau) & \textit{for the lower sign (modified Bessel functions),} \end{cases}$$

$$Z_\nu = \alpha_1 U_\nu + \alpha_2 V_\nu, \quad X_\nu = \beta_1 U_\nu + \beta_2 V_\nu, \quad F_\nu = \tau Z'_\nu + \nu Z_\nu, \quad G_\nu = \tau X'_\nu + \nu X_\nu,$$

$$N = \begin{cases} Z_\nu X_\nu & \textit{for } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2; \\ \alpha U_\nu^2 + \beta U_\nu V_\nu + \gamma V_\nu^2 & \textit{for } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$N_1 = \begin{cases} Z_\nu G_\nu + X_\nu F_\nu & \textit{for } \Delta = -(\alpha_1\beta_2 - \alpha_2\beta_1)^2, \\ \tau N' + 2\nu N & \textit{for } \Delta = 4\alpha\gamma - \beta^2, \end{cases}$$

$$N_2 = N_1^2 \pm 4\tau^2 N^2 + \omega^2 \Delta, \qquad \omega = \begin{cases} \frac{\pi}{2} & \textit{for the upper sign,} \\ -1 & \textit{for the lower sign.} \end{cases}$$

where prime stands for differentiation with respect to τ.

50. $\boldsymbol{y''_{xx} = (A_1 xy + A_2 x^{-3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{1/3} N^{1/2}, \qquad y = b\tau^{2/3},$$

where $\nu = \frac{1}{3}$, $A_1 = \pm\frac{9}{4} b^{-3}$, $A_2 = -\frac{9}{16} a^4 b^{-2} \omega^2 \Delta$.

51. $\boldsymbol{y''_{xx} = (A_1 x y^m + A_2 x^{-3})(y'_x)^3, \qquad m \neq -2}$.

Solution in the parametric form:

$$x = a\tau^{\nu} N^{1/2}, \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{m+2}$, $A_1 = \pm\frac{1}{4} b^{-m-2}(m+2)^2$, $A_2 = -\frac{1}{16} a^4 b^{-2} \omega^2 \Delta (m+2)^2$.

52. $\boldsymbol{y''_{xx} = (A_1 x^{-3} + A_2 x y^{-5})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-1/3} N^{1/2}, \qquad y = b\tau^{-2/3},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{16} a^4 b^{-2} \omega^2 \Delta$, $A_2 = \pm\frac{9}{4} b^3$.

53. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-1/2} + A_2 y^{-3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-2/3} N^{-1} N_1^2, \qquad y = b\tau^{2/3} N,$$

where $\nu = \frac{1}{3}$, $A_1 = \pm\frac{8}{3} a^{3/2} b^{-3/2}$, $A_2 = 2ab\omega^2\Delta$.

54. $\boldsymbol{y''_{xx} = (A_1 + A_2 x^{-1/2} y^{-2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} N^{-2} N_1^2, \qquad y = b\tau^{-2/3} N^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = 2ab^{-2}\omega^2\Delta$, $A_2 = \pm\frac{8}{3}a^{3/2}$.

55. $\boldsymbol{y''_{xx} = (A_1 x^{-2} y + A_2 x^{-3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{2/3} N, \qquad y = b\tau^{-2/3} N^{-1} N_2,$$

where $\nu = \frac{1}{3}$, $A_1 = \frac{9}{128}a^3 b^{-3}$, $A_2 = -\frac{9}{64}a^4 b^{-2}\omega^2\Delta$.

56. $\boldsymbol{y''_{xx} = (A_1 x^{-3} + A_2 x^{-2} y^{-2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{4/3} N^2 N_2^{-1}, \qquad y = b\tau^{2/3} N N_2^{-1},$$

where $\nu = \frac{1}{3}$, $A_1 = -\frac{9}{64}a^4 b^{-2}\omega^2\Delta$, $A_2 = \frac{9}{128}a^3$.

► *In the solutions of equations 57–72, the following notation is used:*

$$f_1 = \sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}, \qquad \tau = \int \frac{d\wp_1}{\sqrt{\pm 4\wp_1^3 - 2\wp_1 - C_2}} - C_1$$

and

$$f_2 = \sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}, \qquad \tau = \int \frac{d\wp_2}{\sqrt{\pm 4\wp_2^3 + 2\wp_2 - C_2}} - C_1,$$

where functions $\wp_1 = \wp_1(\tau)$ and $\wp_2 = \wp_2(\tau)$ are defined implicitly by the above elliptic integrals. For the upper signs, they are the classical elliptic Weierstrass functions $\wp_1 = \wp(\tau + {}_1,\ 2,\ C_2)$ and $\wp_2 = \wp(\tau + {}_1,\ -2,\ C_2)$.

57. $\boldsymbol{y''_{xx} = (A_1 x^2 + A_2)(y'_x)^3}$.

Solution in the parametric form:

$$x = a\wp_k, \qquad y = b\tau,$$

where $A_1 = \mp 6a^{-1}b^{-2}$, $A_2 = ab^{-2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

58. $\boldsymbol{y''_{xx} = (A_1 y^{-3} + A_2 x^2 y^{-5})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-1}\wp_k, \qquad y = b\tau^{-1},$$

where $A_1 = ab(-1)^{k+1}$, $A_2 = \mp 6a^{-1}b^3$; $k = 1$ and $k = 2$.

59. $\boldsymbol{y''_{xx} = (A_1 y^{-9/7} + A_2 x^2 y^{-15/7})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau(\tau^2 \wp_k \mp 1), \qquad y = b\tau^7,$$

where $A_1 = \frac{1}{49} a b^{-5/7} (-1)^{k+1}$, $A_2 = \mp \frac{6}{49} a^{-1} b^{1/7}$; $k = 1$ and $k = 2$.

60. $\boldsymbol{y''_{xx} = (A_1 y^{-12/7} + A_2 x^2 y^{-20/7})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-6}(\tau^2 \wp_k \mp 1), \qquad y = b\tau^{-7},$$

where $A_1 = \frac{1}{49} a b^{-2/7} (-1)^{k+1}$, $A_2 = \mp \frac{6}{49} a^{-1} b^{6/7}$; $k = 1$ and $k = 2$.

61. $\boldsymbol{y''_{xx} = (A_1 y + A_2 x)(y'_x)^{5/2}}$.

Solution in the parametric form:

$$x = a[f_k - (-1)^k \tau], \qquad y = b\tau,$$

where $A_1 = a b^{-1} A_2 (-1)^k$, $A_2 = -2a^{-1} b^{-1} \left(\pm \dfrac{6a}{b}\right)^{1/2}$; $k = 1$ and $k = 2$.

62. $\boldsymbol{y''_{xx} = (A_1 y + A_2 x)(y'_x)^{1/2}}$.

Solution in the parametric form:

$$x = a\tau, \qquad y = b[f_k - (-1)^k \tau],$$

where $A_1 = 2a^{-1} b^{-1} \left(\pm \dfrac{6b}{a}\right)^{1/2}$, $A_2 = a^{-1} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

63. $\boldsymbol{y''_{xx} = (A_1 y^{-3/4} + A_2 x y^{-5/4})(y'_x)^{5/2}}$.

Solution in the parametric form:

$$x = a[2\tau f_k - 2\wp_k + (-1)^{k+1} \tau^2], \qquad y = b\tau^4,$$

where $A_1 = a b^{-1/2} A_2 (-1)^k$, $A_2 = -\dfrac{1}{4} a^{-1} b^{1/4} \left(\pm \dfrac{3a}{b}\right)^{1/2}$; $k = 1$ and $k = 2$.

64. $\boldsymbol{y''_{xx} = (A_1 x^{-5/4} y + A_2 x^{-3/4})(y'_x)^{1/2}}$.

Solution in the parametric form:

$$x = a\tau^4, \qquad y = b[2\tau f_k - 2\wp_k + (-1)^{k+1} \tau^2],$$

where $A_1 = \dfrac{1}{4} a^{1/4} b^{-1} \left(\pm \dfrac{3b}{a}\right)^{1/2}$, $A_2 = a^{-1/2} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

65. $y''_{xx} = (A_1 y^{-13/8} + A_2 x y^{-15/8})(y'_x)^{5/2}.$

Solution in the parametric form:

$$x = a\tau^{-6}[2\tau^3 f_k + 6\tau^2 \wp_k \mp 2 + (-1)^k \tau^4], \qquad y = b\tau^{-8},$$

where $A_1 = ab^{-1/4} A_2 (-1)^k$, $A_2 = \dfrac{1}{8} a^{-1} b^{7/8} \left(\mp \dfrac{3a}{2b}\right)^{1/2}$; $k = 1$ and $k = 2$.

66. $y''_{xx} = (A_1 x^{-15/8} y + A_2 x^{-13/8})(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = a\tau^{-8}, \qquad y = b\tau^{-6}[2\tau^3 f_k + 6\tau^2 \wp_k \mp 2 + (-1)^k \tau^4],$$

where $A_1 = -\dfrac{1}{8} a^{7/8} b^{-1} \left(\mp \dfrac{3b}{2a}\right)^{1/2}$, $A_2 = a^{-1/4} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

67. $y''_{xx} = (A_1 y^{-15/13} + A_2 x y^{-20/13})(y'_x)^{5/2}.$

Solution in the parametric form:

$$x = a\tau[5\tau^3 f_k - 20\tau^2 \wp_k \pm 30 - (-1)^k \tau^4], \qquad y = b\tau^{13},$$

where $A_1 = ab^{-5/13} A_2 (-1)^k$, $A_2 = -\dfrac{2}{65} a^{-1} b^{7/13} \left(\pm \dfrac{30a}{13b}\right)^{1/2}$; $k = 1$ and $k = 2$.

68. $y''_{xx} = (A_1 x^{-20/13} y + A_2 x^{-15/13})(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = a\tau^{13}, \qquad y = b\tau[5\tau^3 f_k - 20\tau^2 \wp_k \pm 30 - (-1)^k \tau^4],$$

where $A_1 = \dfrac{2}{65} a^{7/13} b^{-1} \left(\pm \dfrac{30b}{13a}\right)^{1/2}$, $A_2 = a^{-5/13} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

69. $y''_{xx} = (A_1 + A_2 x^{-2/3})(y'_x)^3.$

Solution in the parametric form:

$$x = a\wp_k^3, \qquad y = b[f_k - (-1)^k \tau],$$

where $A_1 = \mp\frac{1}{2} ab^{-2}$, $A_2 = \frac{1}{12} a^{5/3} b^{-2} (-1)^{k+1}$; $k = 1$ and $k = 2$.

70. $y''_{xx} = (A_1 x^{-2/3} y^{-7/3} + A_2 y^{-3})(y'_x)^3.$

Solution in the parametric form:

$$x = a\wp_k^3 \bigl[f_k - (-1)^k \tau\bigr]^{-1}, \qquad y = b\bigl[f_k - (-1)^k \tau\bigr]^{-1},$$

where $A_1 = \frac{1}{12} a^{5/3} b^{1/3} (-1)^{k+1}$, $A_2 = \mp\frac{1}{2} ab$; $k = 1$ and $k = 2$.

71. $\boldsymbol{y''_{xx} = (A_1 y^{-1/4} + A_2 x y^{-7/4})(y'_x)^{1/2}}$.

Solution in the parametric form:

$$x = a\bigl(f_k^2 - \tau^2 \mp 4\wp_k^3\bigr), \qquad y = b\bigl[f_k - (-1)^k \tau\bigr]^{4/3},$$

$$\text{where}\quad A_1 = ab^{-3/2}A_2, \quad A_2 = \begin{cases} \pm a^{-3} b^{11/4} \Bigl(\mp \dfrac{a}{b}\Bigr)^{1/2} & \text{if } k = 1, \\ \pm a^{-3} b^{11/4} \Bigl(\pm \dfrac{a}{b}\Bigr)^{1/2} & \text{if } k = 2. \end{cases}$$

72. $\boldsymbol{y''_{xx} = (A_1 x^{-7/4} y + A_2 x^{-1/4})(y'_x)^{5/2}}$.

Solution in the parametric form:

$$x = a[f_k - (-1)^k \tau]^{4/3}, \qquad y = b(f_k^2 - \tau^2 \mp 4\wp_k^3),$$

$$\text{where}\quad A_1 = \begin{cases} \mp a^{11/4} b^{-3} \Bigl(\mp \dfrac{b}{a}\Bigr)^{1/2} & \text{if } k = 1, \\ \mp a^{11/4} b^{-3} \Bigl(\pm \dfrac{b}{a}\Bigr)^{1/2} & \text{if } k = 2, \end{cases} \qquad A_2 = a^{-3/2} b A_1.$$

► *In the solutions of equations 73–92, the following notation is used: functions P_1 and P_2 are the general solutions of four modifications of the first Painlevé equation*

$$P_1'' = \pm 6P_1^2 + \tau, \qquad P_2'' = \pm 6P_2^2 - \tau$$

(in the case of the upper sign, the equation for P_1 is the canonical form of the first Painlevé equation; see Subsection 2.8.2). In addition,

$$\begin{aligned}
Q_1 &= \pm 6P_1^2 + \tau, & Q_2 &= \pm 6P_2^2 - \tau, \\
R_1 &= 2P_1' - \tau^2, & R_2 &= 2P_2' + \tau^2, \\
S_1 &= 3\tau P_1' - 3P_1 - \tau^3, & S_2 &= 3\tau P_2' - 3P_2 + \tau^3, \\
T_1 &= \tau^2 P_1 \mp 1, & T_2 &= \tau^2 P_2 \mp 1, \\
U_1 &= (P_1')^2 - 2P_1 Q_1 \pm 8P_1^3, & U_2 &= (P_2')^2 - 2P_2 Q_2 \pm 8P_2^3, \\
V_1 &= P_1' Q_1' + P_1' - Q_1^2, & V_2 &= P_2' Q_2' - P_2' - Q_2^2, \\
W_1 &= \tau^3 P_1' + 3\tau^2 P_1 \mp 1 + \tau^5, & W_2 &= \tau^3 P_2' + 3\tau^2 P_2 \mp 1 - \tau^5, \\
Z_1 &= 6(\tau^3 P_1' - 4\tau^2 P_1 \pm 6) - \tau^5, & Z_2 &= 6(\tau^3 P_2' - 4\tau^2 P_2 \pm 6) + \tau^5,
\end{aligned}$$

where primes stand for differentiation with respect to τ.

73. $\boldsymbol{y''_{xx} = (A_1 y + A_2 x^2)(y'_x)^3}$.

Solution in the parametric form:

$$x = aP_k, \qquad y = b\tau,$$

where $A_1 = ab^{-3}(-1)^k$, $A_2 = \mp 6a^{-1}b^{-2}$; $k = 1$ and $k = 2$.

74. $y''_{xx} = (A_1 y^2 + A_2 x)(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aR_k, \qquad y = b\tau,$$

where $A_1 = ab^{-2}A_2(-1)^{k+1}$, $A_2 = -2a^{-1}b^{-1}\left(\pm\dfrac{3a}{b}\right)^{1/2}$; $k = 1$ and $k = 2$.

75. $y''_{xx} = (A_1 y + A_2 x^2)(y'_x)^{1/2}$.

Solution in the parametric form:

$$x = a\tau, \qquad y = bR_k,$$

where $A_1 = 2a^{-1}b^{-1}\left(\pm\dfrac{3b}{a}\right)^{1/2}$, $A_2 = a^{-2}bA_1(-1)^{k+1}$; $k = 1$ and $k = 2$.

76. $y''_{xx} = (A_1 y^{-4} + A_2 x^2 y^{-5})(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^{-1}P_k, \qquad y = b\tau^{-1},$$

where $A_1 = ab^2(-1)^k$, $A_2 = \mp 6a^{-1}b^3$; $k = 1$ and $k = 2$.

77. $y''_{xx} = (A_1 y^{-1/2} + A_2 x y^{-5/4})(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = aS_k, \qquad y = b\tau^4,$$

where $A_1 = ab^{-3/4}A_2(-1)^{k+1}$, $A_2 = -\dfrac{1}{2}a^{-1}b^{-1/4}\left(\pm\dfrac{a}{2b}\right)^{1/2}$; $k = 1$ and $k = 2$.

78. $y''_{xx} = (A_1 x^{-5/4} y + A_2 x^{-1/2})(y'_x)^{1/2}$.

Solution in the parametric form:

$$x = a\tau^4, \qquad y = bS_k,$$

where $A_1 = \dfrac{1}{2}a^{-1/4}b^{-1}\left(\pm\dfrac{b}{2a}\right)^{1/2}$, $A_2 = a^{-3/4}bA_1(-1)^{k+1}$; $k = 1$ and $k = 2$.

79. $y''_{xx} = (A_1 y^{-8/7} + A_2 x^2 y^{-15/7})(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau T_k, \qquad y = b\tau^7,$$

where $A_1 = \frac{1}{49}ab^{-6/7}(-1)^k$, $A_2 = \mp\frac{6}{49}a^{-1}b^{1/7}$; $k = 1$ and $k = 2$.

80. $y''_{xx} = (A_1 y^{-7/4} + A_2 x y^{-15/8})(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = a\tau^{-6} W_k, \qquad y = b\tau^{-8},$$

where $A_1 = ab^{-1/8} A_2 (-1)^k$, $A_2 = \dfrac{1}{8} a^{-1} b^{7/8} \Big(\mp \dfrac{3a}{b}\Big)^{1/2}$; $k = 1$ and $k = 2$.

81. $y''_{xx} = (A_1 x^{-15/8} y + A_2 x^{-7/4})(y'_x)^{1/2}$.

Solution in the parametric form:

$$x = a\tau^{-8}, \qquad y = b\tau^{-6} W_k,$$

where $A_1 = -\dfrac{1}{8} a^{7/8} b^{-1} \Big(\mp \dfrac{3b}{a}\Big)^{1/2}$, $A_2 = a^{-1/8} b A_1 (-1)^k$; $k = 1$ and $k = 2$.

82. $y''_{xx} = (A_1 y^{-13/7} + A_2 x^2 y^{-20/7})(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^{-6} T_k, \qquad y = b\tau^{-7},$$

where $A_1 = \frac{1}{49} ab^{-1/7}(-1)^k$, $A_2 = \mp \frac{6}{49} a^{-1} b^{6/7}$; $k = 1$ and $k = 2$.

83. $y''_{xx} = (A_1 y^{-14/13} + A_2 x y^{-20/13})(y'_x)^{5/2}$.

Solution in the parametric form:

$$x = a\tau Z_k, \qquad y = b\tau^{13},$$

where $A_1 = ab^{-6/13} A_2 (-1)^{k+1}$, $A_2 = -\dfrac{2}{13} a^{-1} b^{7/13} \Big(\pm \dfrac{a}{13b}\Big)^{1/2}$; $k = 1$ and $k = 2$.

84. $y''_{xx} = (A_1 x^{-20/13} y + A_2 x^{-14/13})(y'_x)^{1/2}$.

Solution in the parametric form:

$$x = a\tau^{13}, \qquad y = b\tau Z_k,$$

where $A_1 = \dfrac{2}{13} a^{7/13} b^{-1} \Big(\pm \dfrac{b}{13a}\Big)^{1/2}$, $A_2 = a^{-6/13} b A_1 (-1)^{k+1}$; $k = 1$ and $k = 2$.

85. $y''_{xx} = (A_1 y + A_2 x^{-1/2})(y'_x)^3$.

Solution in the parametric form:

$$x = a(P'_k)^2, \qquad y = bP_k,$$

where $A_1 = \mp 24ab^{-3}$, $A_2 = 2a^{3/2} b^{-2} (-1)^k$; $k = 1$ and $k = 2$.

86. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-5/2} + A_2 y^{-4})(y'_x)^3}$.

Solution in the parametric form:

$$x = aP_k^{-1}(P'_k)^2, \qquad y = bP_k^{-1},$$

where $A_1 = 2a^{3/2}b^{1/2}(-1)^k$, $A_2 = \mp 24ab^2$; $k = 1$ and $k = 2$.

87. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{1/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = aP_k^{3/2}, \qquad y = bU_k,$$

where $A_1 = \frac{3}{16}a^{8/3}b^{-3}$, $A_2 = \mp 8a^{-2}bA_1$; $k = 1$ and $k = 2$.

88. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y^{-7/3} + A_2 x^{1/3} y^{-10/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = aP_k^{3/2}U_k^{-1}, \qquad y = bU_k^{-1},$$

where $A_1 = \frac{3}{16}a^{8/3}b^{1/3}$, $A_2 = \mp 8a^{-2}bA_1$; $k = 1$ and $k = 2$.

89. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} + A_2 y^{-3/2})(y'_x)^3}$.

Solution in the parametric form:

$$x = aQ_k^2, \qquad y = b(P'_k)^2,$$

where $A_1 = \mp 6a^{3/2}b^{-2}$, $A_2 = \frac{1}{2}ab^{-1/2}(-1)^{k+1}$; $k = 1$ and $k = 2$.

90. $\boldsymbol{y''_{xx} = (A_1 y^{-3/2} + A_2 x^{-1/2} y^{-5/2})(y'_x)^3}$.

Solution in the parametric form:

$$x = a(P'_k)^{-2}Q_k^2, \qquad y = b(P'_k)^{-2},$$

where $A_1 = \frac{1}{2}ab^{-1/2}(-1)^{k+1}$, $A_2 = \mp 6a^{3/2}b^{1/2}$; $k = 1$ and $k = 2$.

91. $\boldsymbol{y''_{xx} = (A_1 x^{-5/3} y + A_2 x^{-4/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a(P'_k)^3, \qquad y = bV_k,$$

where $A_1 = -\frac{1}{36}a^{8/3}b^{-3}$, $A_2 = a^{-1/3}bA_1(-1)^k$; $k = 1$ and $k = 2$.

92. $\boldsymbol{y''_{xx} = (A_1 x^{-4/3} y^{-5/3} + A_2 x^{-5/3} y^{-7/3})(y'_x)^3}$.

Solution in the parametric form:

$$x = a(P'_k)^3V_k^{-1}, \qquad y = bV_k^{-1},$$

where $A_1 = \frac{1}{36}a^{7/3}b^{-1/3}(-1)^{k+1}$, $A_2 = -\frac{1}{36}a^{8/3}b^{1/3}$; $k = 1$ and $k = 2$.

► *In the solutions of equations 93–96, the following notation is used:*

$$r = \begin{cases} C_1 + \dfrac{1}{4}\tau^2 + \dfrac{2B_1}{k_1+1}\tau^{k_1+1} + \dfrac{2B_2}{k_2+1}\tau^{k_2+1} & \textit{if} \;\; k_1 \neq -1,\ k_2 \neq -1; \\ C_1 + \dfrac{1}{4}\tau^2 + \dfrac{2B_1}{k+1}\tau^{k+1} + 2B_2 \ln|\tau| & \textit{if} \;\; k = k_1 \neq -1,\ k_2 = -1; \end{cases}$$

$$F = C_2 \exp\biggl(\int \frac{d\tau}{\sqrt{r}}\biggr), \qquad G = \tau + 2\sqrt{r} + 4B_2,$$
$$H = \int \Bigl(C_1 - \frac{1}{2A_1A_2}\tau^{-4} - \tau^2\Bigr)^{-1/2} d\tau + C_2.$$

93. $\boldsymbol{y''_{xx} = (A_1 x^{-2m_1-3} y^{m_1} + A_2 x^{-2m_2-3} y^{m_2})(y'_x)^3.}$

Solution in the parametric form:

$$x = \tau F^{1/2}, \qquad y = F,$$

where $k_1 = -2m_1 - 3$, $k_2 = -2m_2 - 3$, $A_1 = -B_1$, $A_2 = -B_2$.

94. $\boldsymbol{y''_{xx} = \Bigl(A_1 x y^m + A_2 y^{\frac{m-1}{2}}\Bigr)(y'_x)^{\frac{m+5}{m+3}}.}$

Solution in the parametric form:

$$x = aF^{-1/2}G, \qquad y = bF^{\frac{1}{m+1}},$$

where $k_1 = k = -\dfrac{m+3}{m+1}$, $k_2 = 0$, $A_1 = \dfrac{k}{k+1}a^{-1}b^{k+1}\biggl[-\dfrac{4aB_1}{(k+1)b}\biggr]^{1/k}$, $A_2 = -4ab^{-\frac{1}{k+1}}A_1B_2$.

95. $\boldsymbol{y''_{xx} = \Bigl(A_1 x^n y + A_2 x^{\frac{n-1}{2}}\Bigr)(y'_x)^{\frac{2n+4}{n+3}}.}$

Solution in the parametric form:

$$x = aF^{\frac{1}{n+1}}, \qquad y = F^{-1/2}G,$$

where $k_1 = k = -\dfrac{n+3}{n+1}$, $k_2 = 0$, $A_1 = -\dfrac{k}{k+1}a^{\frac{2}{k+1}}b^{-1}\biggl[-\dfrac{4bB_1}{(k+1)a}\biggr]^{1/k}$, $A_2 = -4a^{-\frac{1}{k+1}}bA_1B_2$.

96. $\boldsymbol{y''_{xx} = (A_1 x^{-5} y^2 + A_2 x^{-5})(y'_x)^3.}$

Solution in the parametric form:

$$x = \frac{\tau\sqrt{-A_2}}{\cos H}, \qquad y = \sqrt{\frac{A_2}{A_1}}\tan H.$$

97. $y''_{xx} = (A_1 x^{m_1} y^{m_1} + A_2 x^{m_2} y^{m_2})(y'_x)^{3/2}$.

Solution in the parametric form:

$$x = C_1 \tau^{1/2} \exp\left(-\frac{1}{2}\int \frac{f\,d\tau}{\tau\sqrt{f^2+4}}\right), \qquad y = C_1^{-1}\tau^{1/2}\exp\left(\frac{1}{2}\int \frac{f\,d\tau}{\tau\sqrt{f^2+4}}\right),$$

where

$$f = \begin{cases} \tau^{-1/2}\left[C_2 + \dfrac{A_1}{2(m_1+1)}\tau^{m_1+1} + \dfrac{A_2}{2(m_2+1)}\tau^{m_2+1}\right] & \text{if } m_1 \neq -1,\ m_2 \neq -1, \\ \tau^{-1/2}\left[C_2 + \dfrac{A_1}{2(m+1)}\tau^{m+1} + \dfrac{1}{2}A_2 \ln\tau\right] & \text{if } m_1 = m \neq -1,\ m_2 = -1. \end{cases}$$

98. $y''_{xx} = \left(A_1 x^n y^{m_1} + A_2 x^{\frac{m_2(n+1)-m_1+n}{m_1+1}} y^{m_2}\right)(y'_x)^{\frac{m_1+2n+3}{m_1+n+2}}$.

Solution in the parametric form:

$$x = C_1 \exp\left(\int \frac{d\tau}{\tau z}\right), \qquad y = C_1^{-\frac{n+1}{m_1+1}}\tau \exp\left(-\frac{n+1}{m_1+1}\int \frac{d\tau}{\tau z}\right),$$

where $z = z(\tau)$ is the solution of the algebraic equation

$$\left(z - \frac{m_1+n+2}{m_1+1}\right)\left(z - \frac{n+1}{m_1+1}\right)^{\frac{n+1}{m_1+n+2}} = \tau^{-\frac{m_1+1}{m_1+n+2}}\left(C_2 + \frac{A_1}{m_1+n+2}\tau^{m_1+1} + F\right),$$

$$F = \begin{cases} \dfrac{A_2(m_1+1)}{(m_1+n+2)(m_2+1)}\tau^{m_2+1} & \text{if } m_2 \neq -1, \\ \dfrac{A_2}{m_1+n+2}\ln|\tau| & \text{if } m_2 = -1. \end{cases}$$

► *In the solutions of equations 99–108, the following notation is used: functions P_1 and P_2 are the general solutions of four modifications of the second Painlevé equation (with parameter $a = 0$)*

$$P_1'' = \tau P_1 \pm 2P_1^3, \qquad P_2'' = -\tau P_2 \pm 2P_2^3.$$

In the case of the upper sign, the equation for P_1 is the canonical form of the second Painlevé equation (with parameter $a = 0$; see Subsection 2.8.2);

$$\begin{aligned} Q_1 &= \tau P_1^2 \pm P_1^4 - (P_1')^2, & Q_2 &= \tau P_2^2 \pm P_2^4 - (P_2')^2, \\ R_1 &= P_1' \mp P_1 Q_1, & R_2 &= P_2' \pm P_2 Q_2, \\ S_1 &= 2P_1' Q_1 - P_1^3 \mp P_1 Q_1^2, & S_2 &= 2P_2' Q_2 + P_2^3 \pm P_2 Q_2^2, \end{aligned}$$

where primes stand for differentiation with respect to τ.

99. $y''_{xx} = (A_1 xy + A_2 x^3)(y'_x)^3$.

Solution in the parametric form:

$$x = aP_k, \qquad y = b\tau,$$

where $A_1 = b^3(-1)^k$, $A_2 = \mp 2a^{-2}b^{-2}$; $k = 1$ and $k = 2$.

100. $\boldsymbol{y''_{xx} = (A_1 x y^{-5} + A_2 x^3 y^{-6})(y'_x)^3}$.

Solution in the parametric form:

$$x = a\tau^{-1}P_k, \qquad y = b\tau^{-1},$$

where $A_1 = b^3(-1)^k$, $A_2 = \mp 2a^{-2}b^4$; $k = 1$ and $k = 2$.

101. $\boldsymbol{y''_{xx} = (A_1 + A_2 x^{-1/2} y^{-1/2})(y'_x)^3}$.

Solutions in the parametric form:

$$x = a(P'_k)^2, \quad y = bP_k^2, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \mp 2ab^{-2}$, $A_2 = \frac{1}{2}a^{3/2}b^{-3/2}(-1)^k$; $k = 1$ and $k = 2$.

102. $\boldsymbol{y''_{xx} = (A_1 x^{-1/2} y^{-2} + A_2 y^{-3})(y'_x)^3}$.

Solutions in the parametric form:

$$x = aP_k^{-2}(P'_k)^2, \quad y = bP_k^{-2}, \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = \frac{1}{2}a^{3/2}(-1)^k$, $A_2 = \mp 2ab$; $k = 1$ and $k = 2$.

103. $\boldsymbol{y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^2}$.

Solutions in the parametric form:

$$x = aP_k^2, \quad y = b\big[\tau P_k^2 \pm P_k^4 - (P'_k)^2\big], \qquad P'_k = (P_k)'_\tau,$$

where $A_1 = 2a^3b^{-3}(-1)^k$, $A_2 = \pm 2ab^{-2}(-1)^k$; $k = 1$ and $k = 2$.

104. $\boldsymbol{y''_{xx} = (A_1 x^{-2} y^{-2} + A_2 y^{-3})(y'_x)^3}$.

Solutions in the parametric form:

$$x = aP_k^2\big[\tau P_k^2 \pm P_k^4 - (P'_k)^2\big]^{-1}, \quad y = b\big[\tau P_k^2 \pm P_k^4 - (P'_k)^2\big]^{-1},$$

where $A_1 = -2a^3$, $A_2 = \mp 2ab$; $k = 1$ and $k = 2$.

105. $\boldsymbol{y''_{xx} = (A_1 + A_2 x y^{-1/2})(y'_x)^{3/2}}$.

Solutions in the parametric form:

$$x = aP_k^{-1}R_k, \quad y = bQ_k^2,$$

$$\text{where} \quad A_1 = \mp ab^{-1/2}A_2(-1)^k, \quad A_2 = \begin{cases} 4a^{-2}b^{1/2}\Big(\dfrac{a}{2b}\Big)^{1/2} & \text{if } k = 1, \\ -4a^{-2}b^{1/2}\Big(-\dfrac{a}{2b}\Big)^{1/2} & \text{if } k = 2, \end{cases}.$$

106. $y''_{xx} = (A_1 x^{-1/2} y + A_2)(y'_x)^{3/2}$.

Solutions in the parametric form:

$$x = aQ_k^2, \quad y = bP_k^{-1}R_k,$$

$$\text{where} \quad A_1 = \begin{cases} -4a^{1/2}b^{-2}\Big(\dfrac{b}{2a}\Big)^{1/2} & \text{if } k=1, \\ 4a^{1/2}b^{-2}\Big(-\dfrac{b}{2a}\Big)^{1/2} & \text{if } k=2, \end{cases} \qquad A_2 = \mp a^{-1/2} b A_1 (-1)^k.$$

107. $y''_{xx} = (A_1 + A_2 x y^{-2})(y'_x)^{3/2}$.

Solutions in the parametric form:

$$x = aS_k, \quad y = bQ_k,$$

$$\text{where} \quad A_1 = \mp a b^{-2} A_2 (-1)^k, \quad A_2 = \begin{cases} a^{-2}b^2\Big(\dfrac{2a}{b}\Big)^{1/2} & \text{if } k=1, \\ a^{-2}b^2\Big(-\dfrac{2a}{b}\Big)^{1/2} & \text{if } k=2. \end{cases}$$

108. $y''_{xx} = (A_1 x^{-2} y + A_2)(y'_x)^{3/2}$.

Solutions in the parametric form:

$$x = aQ_k, \quad y = bS_k,$$

$$\text{where} \quad A_1 = \begin{cases} -a^2b^{-2}\Big(\dfrac{2b}{a}\Big)^{1/2} & \text{if } k=1, \\ -a^2b^{-2}\Big(-\dfrac{2b}{a}\Big)^{1/2} & \text{if } k=2, \end{cases} \qquad A_2 = \mp a^{-2} b A_1 (-1)^k.$$

109. $y''_{xx} = (A_1 x^{-7/5} y^{-8/5} + A_2 x^{-7/5} y^{-13/5})(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^5 S^{5/2}\Big(bC_1^4 F - \frac{A_1}{A_2}\Big)^{-1}, \qquad y = \Big(bC_1^4 F - \frac{A_1}{A_2}\Big)^{-1},$$

where $S = C_1 e^{2k\tau} + C_2 e^{-k\tau}\sin\big(\sqrt{3}\,k\tau\big)$, $F = (S'_\tau)^2 - 2SS''_{\tau\tau}$, $A_2 = \frac{5}{1024}a^{12/5}b^{-3}k^{-6}$.

110. $y''_{xx} = (A_1 x^{-2} y^{-1} + A_2 x^{-2} y^{-2})(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1\tau^{2/3}Z^2\Big\{bC_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big] - \frac{A_1}{A_2}\Big\}^{-1},$$
$$y = \Big\{bC_1\tau^{-2/3}\big[(\tau Z'_\tau + \tfrac{1}{3}Z)^2 \pm \tau^2 Z^2\big] - \frac{A_1}{A_2}\Big\}^{-1},$$

where

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign}, \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign}, \end{cases}$$

$J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions; $A_2 = \frac{9}{2}a^3b^{-3}$.

111. $\boldsymbol{y''_{xx} = (A_1x^{-7}y^4 + A_2x^{-7}y^3)(y'_x)^3}$.

Solution in the parametric form:

$$x = aC_1^3F^{1/2}\Big(bC_1^8G - \frac{A_1}{A_2}\Big)^{-1}, \qquad y = \Big(bC_1^8G - \frac{A_1}{A_2}\Big)^{-1},$$

where $R = \sqrt{\pm(4\tau^3 - 1)}$, $F = 2\tau\int \tau R^{-1}\,d\tau + C_2\tau \mp R$, $G = 4\tau F^2 \mp \tau^{-2}(RF - 1)^2$, $A_2 = \mp\frac{3}{64}a^8b^{-3}$.

► *In the solutions of equations 112–113, the following notation is used:*

$$E = \int (1 \pm \tau^4)^{-1/2}\,d\tau + C_2, \qquad k^2 = \pm 1;$$

function E can be expressed in terms of the elliptic integrals or the lemniscate functions.

112. $\boldsymbol{y''_{xx} = (A_1x^2y^{-14/5} + A_2x^3y^{-18/5})(y'_x)^3}$.

Solutions in the parametric form:

$$x = aC_1^4E^{-4}(\tau E - k), \qquad y = bC_1^5E^{-5},$$

where $A_1 = \mp\frac{6}{25}a^{-1}b^{4/5}k$, $A_2 = \mp\frac{2}{25}a^{-2}b^{8/5}$.

113. $\boldsymbol{y''_{xx} = (A_1x^2y^{-11/5} + A_2x^3y^{-12/5})(y'_x)^3}$.

Solutions in the parametric form:

$$x = aC_1E(\tau E - k), \qquad y = bC_1^5E^5,$$

where $A_1 = \mp\frac{6}{25}a^{-1}b^{1/5}k$, $A_2 = \mp\frac{2}{25}a^{-2}b^{2/5}$.

► *In the solutions of equations 114–115, the following notation is used:*

$$\Delta = C_2^2 - 2C_1, \qquad R = (36\Delta + 54B\tau - 2\tau^3)^{1/2}, \qquad z = 3\int \tau^{-1}R^{-1}\,d\tau,$$

$$W(z) = \begin{cases} \dfrac{\sqrt{-\Delta}}{C_1}\tan\big(\pm\sqrt{-\Delta}\,z\big) + \dfrac{C_2}{C_1} & \textit{if } \Delta < 0; \\ \dfrac{\sqrt{\Delta}}{C_1}\tanh\big(\mp\sqrt{\Delta}\,z\big) + \dfrac{C_2}{C_1} & \textit{if } \Delta > 0; \\ \mp\dfrac{1}{C_1z} - \sqrt{\dfrac{2}{|C_1|}} & \textit{if } \Delta = 0,\ C_2 < 0; \\ \mp\dfrac{1}{C_1z} + \sqrt{\dfrac{2}{|C_1|}} & \textit{if } \Delta = 0,\ C_2 > 0. \end{cases}$$

114. $y''_{xx} = (A_1 x^{-5/3} + A_2 x^{-5/3} y^{-2/3})(y'_x)^3.$

Solutions in the parametric form:

$$x = a\tau^{-9/4}(C_1W^2 - 2C_2W + 2)^{3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$
$$y = b\tau^{-3/2}(C_1W^2 - 2C_2W + 2)^{3/2},$$

where $A_1 = -24a^{8/3}b^{-2}C_1$, $A_2 = 36a^{8/3}b^{-4/3}B$.

115. $y''_{xx} = (A_1 x^{-5/3} y^{-2/3} + A_2 x^{-5/3} y^{-4/3})(y'_x)^3.$

Solutions in the parametric form:

$$x = a\tau^{-3/4}(C_1W^2 - 2C_2W + 2)^{-3/4}(6C_1W - 6C_2 \mp R)^{3/2},$$
$$y = b\tau^{3/2}(C_1W^2 - 2C_2W + 2)^{-3/2},$$

where $A_1 = 36a^{8/3}b^{-4/3}B$, $A_2 = -24a^{8/3}b^{-2/3}C_1$.

116. $y''_{xx} = \left[\dfrac{2(m+1)}{(m+3)^2}x + Ax^n\right]y^{-2}(y'_x)^3, \qquad n \neq -3, \quad n \neq -1.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.4 for function $x = x(y)$:

$$x''_{yy} = y^{-2}\left[-\frac{2(n+1)}{(n+3)^2}x - Ax^n\right].$$

117. $y''_{xx} = (-\tfrac{15}{4}x + Ax^{-7})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.35 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(\tfrac{15}{4}x - Ax^{-7}).$$

118. $y''_{xx} = (-6x + Ax^{-4})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.31 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(6x - Ax^{-4}).$$

119. $y''_{xx} = (-12x + Ax^{-5/2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.64 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(12x - Ax^{-5/2}).$$

120. $y''_{xx} = (-2x + Ax^{-2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.6 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(2x - Ax^{-2}).$$

121. $y''_{xx} = (\frac{3}{16}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.26 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{3}{16}x - Ax^{-5/3}).$$

122. $y''_{xx} = (\frac{9}{100}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.10 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{9}{100}x - Ax^{-5/3}).$$

123. $y''_{xx} = (-\frac{3}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.12 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(\tfrac{3}{4}x - Ax^{-5/3}).$$

124. $y''_{xx} = (-\frac{63}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.66 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(\tfrac{63}{4}x - Ax^{-5/3}).$$

125. $y''_{xx} = (\frac{5}{36}x + Ax^{-7/5})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.29 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{5}{36}x - Ax^{-7/5}).$$

126. $y''_{xx} = (\frac{2}{9}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.14 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{2}{9}x - Ax^{-1/2}).$$

127. $y''_{xx} = (\frac{4}{25}x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.8 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{4}{25}x - Ax^{-1/2}).$$

128. $y''_{xx} = (-20x + Ax^{-1/2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.33 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(20x - Ax^{-1/2}).$$

129. $y''_{xx} = (\frac{12}{49}x + Ax^{1/2})y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.37 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-\tfrac{12}{49}x - Ax^{1/2}).$$

130. $y''_{xx} = (Ax^2 + \frac{6}{25}x)y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.60 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-Ax^2 - \tfrac{6}{25}x).$$

131. $y''_{xx} = (Ax^2 - \frac{6}{25}x)y^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.4.2.62 for function $x = x(y)$:

$$x''_{yy} = y^{-2}(-Ax^2 + \tfrac{6}{25}x).$$

132. $y''_{xx} = \left[\frac{2(m+1)}{(m+3)^2}xy^{-2} + Ax^m y^{-m-1}\right](y'_x)^3, \qquad m \neq -3,\ m \neq -1.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.5 for function $x = x(y)$:

$$x''_{yy} = \left[-\frac{2(m+1)}{(m+3)^2}y^{-2}x - Ay^{-m-1}x^m\right].$$

133. $y''_{xx} = (Ax^{-2}y - 2xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.7 for function $x = x(y)$:

$$x''_{yy} = (2y^{-2}x - Ayx^{-2}).$$

134. $y''_{xx} = (Ax^{-1/2}y^{-1/2} + \frac{4}{25}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.9 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{4}{25}y^{-2}x - Ay^{-1/2}x^{-1/2}).$$

135. $y''_{xx} = (Ax^{-5/3}y^{2/3} + \frac{9}{100}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.11 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{9}{100}y^{-2}x - Ay^{2/3}x^{-5/3}).$$

136. $y''_{xx} = (Ax^{-5/3}y^{2/3} - \frac{3}{4}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.13 for function $x = x(y)$:

$$x''_{yy} = (\tfrac{3}{4}y^{-2}x - Ay^{2/3}x^{-5/3}).$$

137. $y''_{xx} = (Ax^{-1/2}y^{-1/2} + \frac{2}{9}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.15 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{2}{9}y^{-2}x - Ay^{-1/2}x^{-1/2}).$$

138. $y''_{xx} = (Ax^{-5/3}y^{2/3} + \frac{3}{16}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.27 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{3}{16}y^{-2}x - Ay^{2/3}x^{-5/3}).$$

139. $y''_{xx} = (Ax^{-7/5}y^{2/5} + \frac{5}{36}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.30 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{5}{36}y^{-2}x - Ay^{2/5}x^{-7/5}).$$

140. $y''_{xx} = (Ax^{-4}y^3 - 6xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.32 for function $x = x(y)$:

$$x''_{yy} = (6y^{-2}x - Ay^3x^{-4}).$$

141. $y''_{xx} = (Ax^{-1/2}y^{-1/2} - 20xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.34 for function $x = x(y)$:

$$x''_{yy} = (20y^{-2}x - Ay^{-1/2}x^{-1/2}).$$

142. $y''_{xx} = (Ax^{-7}y^6 - \frac{15}{4}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.36 for function $x = x(y)$:

$$x''_{yy} = (\tfrac{15}{4}y^{-2}x - Ay^6x^{-7}).$$

143. $y''_{xx} = (Ax^{1/2}y^{-3/2} + \frac{12}{49}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.38 for function $x = x(y)$:

$$x''_{yy} = (-\tfrac{12}{49}y^{-2}x - Ay^{-3/2}x^{1/2}).$$

144. $y''_{xx} = (\frac{6}{25}xy^{-2} + Ax^2y^{-3})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.61 for function $x = x(y)$:

$$x''_{yy} = (Ay^{-3}x^2 - \tfrac{6}{25}y^{-2}x).$$

145. $y''_{xx} = (-\frac{6}{25}xy^{-2} + Ax^2y^{-3})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.63 for function $x = x(y)$:

$$x''_{yy} = (Ay^{-3}x^2 + \tfrac{6}{25}y^{-2}x).$$

146. $y''_{xx} = (Ax^{-5/2}y^{3/2} - 12xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.65 for function $x = x(y)$:

$$x''_{yy} = (12y^{-2}x - Ay^{3/2}x^{-5/2}).$$

147. $y''_{xx} = (Ax^{-5/3}y^{2/3} - \frac{63}{4}xy^{-2})(y'_x)^3.$

Assuming y as the independent variables, we obtain an equation of the form 2.4.2.67 for function $x = x(y)$:

$$x''_{yy} = (\tfrac{63}{4}y^{-2}x - Ay^{2/3}x^{-5/3}).$$

2.6.3. Equations of the Form $y''_{xx} = \sigma Ax^n y^m (y'_x)^l + Ax^{n-1}y^{m+1}(y'_x)^{l-1}$

Table 2.12 represents all solvable equations whose solutions are outlined in Subsection 2.6.3. The two-parameter families (in the space of parameters n, m, and l), one-parameter families, and isolated points are represented in a consecutive fashion. Equations are arranged in accordance with the growth of l. The number of the equation sought is indicated in the last column in this table.

1. $y''_{xx} = Ax^{-m-2}y^m y'_x - Ax^{-m-3}y^{m+1}, \qquad m \neq -1.$

Solution in the parametric form:

$$x = aC_1^m\left(\int \frac{d\tau}{1 \pm \tau^{m+1}} + C_2\right)^{-1}, \qquad y = bC_1^{m+1}\tau\left(\int \frac{d\tau}{1 \pm \tau^{m+1}} + C_2\right)^{-1},$$

where $A = \mp(m+1)a^{m+1}b^{-m}$.

2. $y''_{xx} = Ax^{-1}y^{-1}y'_x - Ax^{-2}.$

Solution in the parametric form:

$$x = C_1\left[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\right]^{-1},$$
$$y = -\frac{A}{2}\exp(\mp\tau^2)\left[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}.$$

3. $y''_{xx} = Ax^{-m-2}y^m(y'_x)^3 - Ax^{-m-3}y^{m+1}(y'_x)^2, \qquad m \neq -2.$

Solution in the parametric form:

$$x = aC_1^{m+2}\tau\left(\int \frac{d\tau}{1 \pm \tau^{-m-2}} + C_2\right)^{-1}, \qquad y = bC_1^{m+3}\left(\int \frac{d\tau}{1 \pm \tau^{-m-2}} + C_2\right)^{-1},$$

where $A = \pm(m+2)a^{m+3}b^{-m-2}$.

TABLE 2.12
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
arbitrary ($l \neq 2$)	arbitrary ($m \neq -1$)	$-m-1$	-1	2.6.3.75
arbitrary ($l \neq 2$)	$1-l$	$l-2$	-1	2.6.3.76
arbitrary ($l \neq 3$)	-2	1	-1	2.6.3.79
arbitrary ($l \neq 1$)	0	-1	-1	2.6.3.80
$\dfrac{m+3}{m+2}$	arbitrary ($m \neq -1, -2$)	1	$m+1$	2.6.3.74
$\dfrac{3n+2}{n+1}$	0	arbitrary ($n \neq 0, -1$)	$\dfrac{1}{n}$	2.6.3.73
1	arbitrary ($m \neq -1$)	$-m-2$	-1	2.6.3.1
1	0	arbitrary ($n \neq -1$)	$\frac{1}{n}$	2.6.3.23
1	1	arbitrary ($n \neq 0, -2$)	$\frac{2}{n}$	2.6.3.37
$\dfrac{3}{2}$	0	arbitrary ($n \neq 0, -1$)	$\frac{1}{n}$	2.6.3.41
2	arbitrary ($m \neq -1$)	arbitrary ($n \neq 0$)	-1	2.6.3.85
2	arbitrary ($m \neq -1$)	$m+1$	-1	2.6.3.82
2	arbitrary ($m \neq -1$)	$-2m-2$	-1	2.6.3.83
2	arbitrary ($m \neq -1$)	$-\dfrac{m+1}{2}$	-1	2.6.3.84
2	arbitrary ($m \neq -1$)	0	arbitrary	2.6.3.87
2	-1	arbitrary ($n \neq 0$)	arbitrary	2.6.3.86
$\dfrac{5}{2}$	arbitrary ($m \neq -1, -2$)	1	$m+1$	2.6.3.42
3	arbitrary ($m \neq -2$)	$-m-2$	-1	2.6.3.3
3	arbitrary ($m \neq -2$)	1	$m+1$	2.6.3.24
3	arbitrary ($m \neq -1, -3$)	2	$\dfrac{m+1}{2}$	2.6.3.38

TABLE 2.12 *Continued*
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
0	-3	-1	2	2.6.3.65
0	-3	$\frac{1}{2}$	-4	2.6.3.61
0	-3	2	-1	2.6.3.35
0	0	-2	-1	2.6.3.48
0	0	-1	-2	2.6.3.50
0	0	-1	-1	2.6.3.33
0	0	$-\frac{2}{5}$	$-\frac{5}{2}$	2.6.3.59
0	0	2	$\frac{1}{2}$	2.6.3.63
1	-3	1	-1	2.6.3.15
1	-2	-2	$\frac{1}{2}$	2.6.3.51
1	-2	-1	arbitrary	2.6.3.71
1	-2	-1	-1	2.6.3.7
1	-2	-1	1	2.6.3.5
1	-2	$-\frac{1}{2}$	2	2.6.3.55
1	-2	$\frac{1}{2}$	-1	2.6.3.13
1	-2	1	arbitrary	2.6.3.69
1	-2	1	-2	2.6.3.11
1	-2	1	-1	2.6.3.29
1	-1	-1	-1	2.6.3.2
1	$-\frac{1}{2}$	-2	$-\frac{1}{4}$	2.6.3.53
1	$-\frac{1}{2}$	-1	-1	2.6.3.45
1	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	2.6.3.31
1	$-\frac{1}{2}$	1	$\frac{1}{2}$	2.6.3.57
1	0	-1	-1	2.6.3.77
1	0	1	arbitrary	2.6.3.67
1	0	1	-1	2.6.3.9
1	1	-4	$-\frac{1}{2}$	2.6.3.21
1	1	-1	-2	2.6.3.39
1	1	-2	-1	2.6.3.25
1	1	1	1	2.6.3.17

TABLE 2.12 *Continued*
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
$\frac{3}{2}$	0	-1	-1	2.6.3.27
$\frac{3}{2}$	0	$-\frac{1}{2}$	-2	2.6.3.43
$\frac{3}{2}$	0	1	1	2.6.3.19
2	-1	0	arbitrary	2.6.3.81
$\frac{5}{2}$	-2	1	-1	2.6.3.28
$\frac{5}{2}$	$-\frac{3}{2}$	1	$-\frac{1}{2}$	2.6.3.44
$\frac{5}{2}$	0	1	1	2.6.3.20
3	-5	2	-2	2.6.3.22
3	-3	-1	2	2.6.3.52
3	-3	$\frac{1}{2}$	-4	2.6.3.54
3	-3	2	-1	2.6.3.26
3	-2	-1	arbitrary	2.6.3.72
3	-2	-1	-1	2.6.3.8
3	-2	-1	1	2.6.3.6
3	-2	0	-1	2.6.3.4
3	-2	$\frac{1}{2}$	-1	2.6.3.46
3	-2	1	-1	2.6.3.78
3	-2	2	$-\frac{1}{2}$	2.6.3.40
3	$-\frac{3}{2}$	-1	$\frac{1}{2}$	2.6.3.56
3	$-\frac{3}{2}$	$\frac{1}{2}$	-1	2.6.3.32
3	$-\frac{1}{2}$	-1	-1	2.6.3.14
3	0	-2	-1	2.6.3.16
3	0	-1	arbitrary	2.6.3.70
3	0	-1	-1	2.6.3.30
3	0	-1	$-\frac{1}{2}$	2.6.3.12
3	0	$\frac{1}{2}$	2	2.6.3.58
3	0	1	arbitrary	2.6.3.68
3	0	1	-1	2.6.3.10
3	0	2	1	2.6.3.18
4	-3	1	-1	2.6.3.47

TABLE 2.12 *Continued*
Solvable cases of the equation $y''_{xx} = \sigma A x^n y^m (y'_x)^l + A x^{n-1} y^{m+1} (y'_x)^{l-1}$

l	m	n	σ	Equation
4	-2	-2	$\frac{1}{2}$	2.6.3.66
4	-2	1	-1	2.6.3.34
4	-2	1	$-\frac{1}{2}$	2.6.3.49
4	$-\frac{1}{2}$	-2	$-\frac{1}{4}$	2.6.3.62
4	$-\frac{2}{5}$	1	$-\frac{2}{5}$	2.6.3.60
4	1	-2	-1	2.6.3.36
4	1	1	2	2.6.3.64

4. $\boldsymbol{y''_{xx} = Ay^{-2}(y'_x)^3 - Ax^{-1}y^{-2}(y'_x)^2}$.

Solution in the parametric form:

$$x = -\frac{A}{2}\exp(\mp\tau^2)\left[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\right]^{-1},$$
$$y = C_1\left[\int \tau^{-1}\exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}.$$

► *In the solutions of equations 5–12, the following notation is used:*

$$f = \int \exp(\mp\tau^2)\,d\tau + C_2.$$

5. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-2}y'_x + Ax^{-2}y^{-1}}$.

Solution in the parametric form:

$$x = C_1\exp(\mp\tau^2)f^{-1}, \qquad y = b[2\tau \pm \exp(\mp\tau^2)f^{-1}],$$

where $A = \pm 2b^2$.

6. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-2}(y'_x)^3 + Ax^{-2}y^{-1}(y'_x)^2}$.

Solution in the parametric form:

$$x = a[2\tau \pm \exp(\mp\tau^2)f^{-1}], \qquad y = C_1\exp(\mp\tau^2)f^{-1},$$

where $A = \mp 2a^2$.

7. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-2}y'_x - Ax^{-2}y^{-1}}$.

Solution in the parametric form:

$$x = C_1\left[2\tau f \pm \exp(\mp\tau^2)\right]^{-1}, \qquad y = bf\left[2\tau f \pm \exp(\mp\tau^2)\right]^{-1},$$

where $A = \pm\frac{1}{2}b^2$.

8. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-2}(y'_x)^3 - Ax^{-2}y^{-1}(y'_x)^2}.$

Solution in the parametric form:

$$x = af\big[2\tau f \pm \exp(\mp\tau^2)\big]^{-1}, \qquad y = C_1\big[2\tau f \pm \exp(\mp\tau^2)\big]^{-1},$$

where $A = \pm\frac{1}{2}a^2$.

9. $\boldsymbol{y''_{xx} = Axy'_x - Ay}.$

Solution in the parametric form:

$$x = a\tau, \qquad y = C_1[2\tau f \pm \exp(\mp\tau^2)],$$

where $A = \mp 2a^{-2}$.

10. $\boldsymbol{y''_{xx} = Ax(y'_x)^3 - Ay(y'_x)^2}.$

Solution in the parametric form:

$$x = C_1[2\tau f \pm \exp(\mp\tau^2)], \qquad y = b\tau,$$

where $A = \mp 2b^{-2}$.

11. $\boldsymbol{y''_{xx} = 2Axy^{-2}y'_x - Ay^{-1}}.$

Solution in the parametric form:

$$x = aC_1[2\tau^2 f \pm \tau\exp(\mp\tau^2) \pm f], \qquad y = bC_1[2\tau f \pm \exp(\mp\tau^2)],$$

where $A = \mp\frac{1}{2}a^{-2}b^2$.

12. $\boldsymbol{y''_{xx} = Ax^{-1}(y'_x)^3 - 2Ax^{-2}y(y'_x)^2}.$

Solution in the parametric form:

$$x = aC_1[2\tau f \pm \exp(\mp\tau^2)], \qquad y = bC_1[2\tau^2 f \pm \tau\exp(\mp\tau^2) \pm f],$$

where $A = \mp\frac{1}{2}a^2b^{-2}$.

► *In the solutions of equations 13–22, the following notation is used:*

$$E = \sqrt{\tau(\tau+1)} - \ln\big(\sqrt{\tau} + \sqrt{\tau+1}\big) + C_2, \qquad F = E\sqrt{\frac{\tau+1}{\tau}} - \tau.$$

13. $\boldsymbol{y''_{xx} = Ax^{1/2}y^{-2}y'_x - Ax^{-1/2}y^{-1}}.$

Solution in the parametric form:

$$x = aC_1^4F^{-2}, \qquad y = bC_1^3\tau^{-1}EF^{-2},$$

where $A = -a^{-3/2}b^2$.

14. $\boldsymbol{y''_{xx} = Ax^{-1}y^{-1/2}(y'_x)^3 - Ax^{-2}y^{1/2}(y'_x)^2}$.

Solution in the parametric form:

$$x = aC_1^3\tau^{-1}EF^{-2}, \qquad y = bC_1^4F^{-2},$$

where $A = -a^2b^{-3/2}$.

15. $\boldsymbol{y''_{xx} = Axy^{-3}y'_x - Ay^{-2}}$.

Solution in the parametric form:

$$x = aC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad y = bC_1^2F^{-1},$$

where $A = -2a^{-2}b^3$.

16. $\boldsymbol{y''_{xx} = Ax^{-2}(y'_x)^3 - Ax^{-3}y(y'_x)^2}$.

Solution in the parametric form:

$$x = aC_1^2F^{-1}, \qquad y = bC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}},$$

where $A = -2a^3b^{-2}$.

17. $\boldsymbol{y''_{xx} = Axyy'_x + Ay^2}$.

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{-1}E^{-1}(\tau F^2 + \tau^2F - E^2), \qquad y = bC_1^2F^{-1},$$

where $A = a^{-2}b^{-1}$.

18. $\boldsymbol{y''_{xx} = Ax^2(y'_x)^3 + Axy(y'_x)^2}$.

Solution in the parametric form:

$$x = aC_1^2F^{-1}, \qquad y = bC_1^{-1}\tau^{-1}E^{-1}(\tau F^2 + \tau^2F - E^2),$$

where $A = -a^{-1}b^{-2}$.

19. $\boldsymbol{y''_{xx} = Ax(y'_x)^{3/2} + Ay(y'_x)^{1/2}}$.

Solution in the parametric form:

$$x = aC_1^{-1}\left(F\sqrt{\frac{\tau+1}{\tau}} - E\tau^{-1}\right), \qquad y = bC_1^3F^{-1}\sqrt{\frac{\tau+1}{\tau}},$$

where $A = 2a^{-2}\left(-\frac{a}{b}\right)^{1/2}$.

20. $y''_{xx} = Ax(y'_x)^{5/2} + Ay(y'_x)^{3/2}$.

Solution in the parametric form:

$$x = aC_1^3 F^{-1}\sqrt{\frac{\tau+1}{\tau}}, \qquad y = bC_1^{-1}\left(F\sqrt{\frac{\tau+1}{\tau}} - E\tau^{-1}\right),$$

where $A = -2b^{-2}\left(-\dfrac{b}{a}\right)^{1/2}$.

21. $y''_{xx} = Ax^{-4}yy'_x - 2Ax^{-5}y^2$.

Solution in the parametric form:

$$x = aC_1\tau E\bigl(\tau F^2 + \tau^2 F - E^2\bigr)^{-1}, \qquad y = bC_1^3\tau EF^{-1}\bigl(\tau F^2 + \tau^2 F - E^2\bigr)^{-1},$$

where $A = -2a^3b^{-1}$.

22. $y''_{xx} = 2Ax^2y^{-5}(y'_x)^3 - Axy^{-4}(y'_x)^2$.

Solution in the parametric form:

$$x = aC_1^3\tau EF^{-1}\bigl(\tau F^2 + \tau^2 F - E^2\bigr)^{-1}, \qquad y = bC_1\tau E\bigl(\tau F^2 + \tau^2 F - E^2\bigr)^{-1},$$

where $A = -2a^{-1}b^3$.

23. $y''_{xx} = Ax^n y'_x + nAx^{n-1}y, \qquad n \neq -1$.

Solution in the parametric form:

$$x = a\tau^{\frac{1}{n+1}}, \qquad y = C_1e^{\beta\tau}\left(\int \tau^{-\frac{n}{n+1}}e^{-\beta\tau}\,d\tau + C_2\right),$$

where $A = (n+1)a^{-n-1}\beta$.

24. $y''_{xx} = Axy^m(y'_x)^3 + \dfrac{A}{m+1}y^{m+1}(y'_x)^2, \qquad m \neq -2$.

Solution in the parametric form:

$$x = C_1e^{\beta\tau}\left(\int \tau^{-\frac{m+1}{m+2}}e^{-\beta\tau}\,d\tau + C_2\right), \qquad y = b\tau^{\frac{1}{m+2}},$$

where $A = -\dfrac{m+2}{m+1}b^{-m-2}\beta$.

► *In the solutions of equations 25–36, the following notation is used:*

$$R = \begin{cases} \tau^\nu + C_2\tau^{-\nu} & \textit{for the upper sign,} \\ \sin(\nu\ln\tau) + C_2\cos(\nu\ln\tau) & \textit{for the lower sign,} \\ \ln\tau + C_2 & \textit{for } \nu = 0, \end{cases}$$

$$Q = \begin{cases} (1+\nu)\tau^\nu + (1-\nu)C_2\tau^{-\nu} & \textit{for the upper sign,} \\ (1-\nu C_2)\sin(\nu\ln\tau) + (C_2+\nu C_1)\cos(\nu\ln\tau) & \textit{for the lower sign,} \\ \ln\tau + 1 + C_2 & \textit{for } \nu = 0. \end{cases}$$

25. $y''_{xx} = Ax^{-2}yy'_x - Ax^{-3}y^2.$

Solution in the parametric form:

$$x = a\tau^{-2}, \qquad y = b\tau^{-2}R^{-1}Q,$$

where $\nu = C_1$, $A = ab^{-1}$; the solution is valid for all three cases of functions R and Q given above.

26. $y''_{xx} = Ax^2y^{-3}(y'_x)^3 - Axy^{-2}(y'_x)^2.$

Solution in the parametric form:

$$x = a\tau^{-2}R^{-1}Q, \qquad y = b\tau^{-2},$$

where $\nu = C_1$, $A = a^{-1}b$; the solution is valid for all three cases of functions R and Q given above.

27. $y''_{xx} = Ax^{-1}(y'_x)^{3/2} - Ax^{-2}y(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = a\tau^{-2}, \qquad y = b\tau^{-2}\left(QR^{-1} - \frac{1 \mp \nu^2}{2}\right),$$

where $\nu = C_1$, $A = \left(\dfrac{2a}{b}\right)^{1/2}$.

28. $y''_{xx} = Axy^{-2}(y'_x)^{5/2} - Ay^{-1}(y'_x)^{3/2}.$

Solution in the parametric form:

$$x = a\tau^{-2}\left(QR^{-1} - \frac{1 \mp \nu^2}{2}\right), \qquad y = b\tau^{-2},$$

where $\nu = C_1$, $A = \left(\dfrac{2b}{a}\right)^{1/2}$.

29. $y''_{xx} = Axy^{-2}y'_x - Ay^{-1}.$

Solution in the parametric form:

$$x = aC_1\tau R, \qquad y = bC_1\tau Q,$$

where $A = a^{-2}b^2(1 \mp \nu^2)$.

30. $y''_{xx} = Ax^{-1}(y'_x)^3 - Ax^{-2}y(y'_x)^2.$

Solution in the parametric form:

$$x = aC_1\tau Q, \qquad y = bC_1\tau R,$$

where $A = a^2b^{-2}(1 \mp \nu^2)$.

31. $y''_{xx} = Ax^{-1/2}y^{-1/2}y'_x - Ax^{-3/2}y^{1/2}$.

Solution in the parametric form:

$$x = a\tau^2 R^2, \qquad y = b\tau^2 Q^2,$$

where $\nu = C_1$, $A = a^{-1/2}b^{1/2}$.

32. $y''_{xx} = Ax^{1/2}y^{-3/2}(y'_x)^3 - Ax^{-1/2}y^{-1/2}(y'_x)^2$.

Solution in the parametric form:

$$x = a\tau^2 Q^2, \qquad y = b\tau^2 R^2,$$

where $\nu = C_1$, $A = a^{1/2}b^{-1/2}$.

33. $y''_{xx} = Ax^{-1} - Ax^{-2}y(y'_x)^{-1}$.

Solution in the parametric form:

$$x = a\tau^2 R^2, \qquad y = b\tau^2[Q^2 + (1 \mp \nu^2)R^2],$$

where $\nu = C_1$, $A = 2a^{-1}b$.

34. $y''_{xx} = Axy^{-2}(y'_x)^4 - Ay^{-1}(y'_x)^3$.

Solution in the parametric form:

$$x = a\tau^2[Q^2 + (1 \mp \nu^2)R^2], \qquad y = b\tau^2 R^2,$$

where $\nu = C_1$, $A = 2ab^{-1}$.

35. $y''_{xx} = Ax^2y^{-3} - Axy^{-2}(y'_x)^{-1}$.

Solution in the parametric form:

$$x = aC_1\tau R, \qquad y = bC_1\tau\big[Q^2 + (1 \mp \nu^2)R^2\big]^{1/2},$$

where $A = 4(1 \mp \nu^2)a^{-4}b^4$.

36. $y''_{xx} = Ax^{-2}y(y'_x)^4 - Ax^{-3}y^2(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1\tau\big[Q^2 + (1 \mp \nu^2)R^2\big]^{1/2}, \qquad y = bC_1\tau R,$$

where $A = 4(1 \mp \nu^2)a^4b^{-4}$.

▶ *In the solutions of equations 37–50, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \text{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \text{for the lower sign,} \end{cases}$$

where J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

37. $y''_{xx} = 2Ax^n yy'_x + nAx^{n-1}y^2, \qquad n \neq 0, \quad n \neq -2.$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{2-2\nu}, \qquad y = bC_1^{n+1}\tau^{-2\nu}Z^{-1}(\tau Z'_\tau + \nu Z),$$

where $\nu = \dfrac{n+1}{n+2}$, $A = -\dfrac{n+2}{2}a^{-n-1}b^{-1}$.

38. $y''_{xx} = (m+1)Ax^2y^m(y'_x)^3 + 2Axy^{m+1}(y'_x)^2, \qquad m \neq -1, \quad m \neq -3.$

Solution in the parametric form:

$$x = aC_1^{-m-2}\tau^{-2\nu}Z^{-1}(\tau Z'_\tau + \nu Z), \qquad y = bC_1\tau^{2-2\nu},$$

where $\nu = \dfrac{m+2}{m+3}$, $A = \dfrac{m+3}{2}a^{-1}b^{-m-2}$.

39. $y''_{xx} = 2Ax^{-1}yy'_x - Ax^{-2}y^2.$

Solution in the parametric form:

$$x = C_1\tau^2, \qquad y = b\tau Z^{-1}Z'_\tau,$$

where $\nu = 0$, $A = -\frac{1}{2}b^{-1}$.

40. $y''_{xx} = Ax^2y^{-2}(y'_x)^3 - 2Axy^{-1}(y'_x)^2.$

Solution in the parametric form:

$$x = a\tau Z^{-1}Z'_\tau, \qquad y = C_1\tau^2,$$

where $\nu = 0$, $A = -\frac{1}{2}a^{-1}$.

41. $y''_{xx} = Ax^n(y'_x)^{3/2} + nAx^{n-1}y(y'_x)^{1/2}, \qquad n \neq 0, \; n \neq -1.$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{4-2\nu}, \qquad y = bC_1^{2n+1}\tau^{-2\nu}\left[Z^{-1}(\tau Z'_\tau + \nu Z) \pm \frac{1}{2(1-\nu)}\tau^2\right],$$

where $\nu = \dfrac{2n+1}{n+1}$, $A = -(n+1)a^{-n-1}\left[-\dfrac{2a}{(n+1)b}\right]^{1/2}$.

42. $y''_{xx} = (m+1)Axy^m(y'_x)^{5/2} + Ay^{m+1}(y'_x)^{3/2}, \qquad m \neq -1, \quad m \neq -2.$

Solution in the parametric form:

$$x = aC_1^{2m-3}\tau^{-2\nu}\left[Z^{-1}(\tau Z'_\tau + \nu Z) \pm \frac{1}{2(1-\nu)}\tau^2\right], \qquad y = bC_1\tau^{4-2\nu},$$

where $\nu = \dfrac{2m+3}{m+2}$, $A = (m+2)b^{-m-2}\left[-\dfrac{2b}{(m+2)a}\right]^{1/2}$.

43. $y''_{xx} = 2Ax^{-1/2}(y'_x)^{3/2} - Ax^{-3/2}y(y'_x)^{1/2}.$

Solution in the parametric form:

$$x = C_1\tau^4, \qquad y = b(\tau Z^{-1}Z'_\tau \pm \tfrac{1}{2}\tau^2),$$

where $\nu = 0$, $A = -\frac{1}{2}(-b)^{-1/2}$.

44. $y''_{xx} = Axy^{-3/2}(y'_x)^{5/2} - 2Ay^{-1/2}(y'_x)^{3/2}.$

Solution in the parametric form:

$$x = a(\tau Z^{-1}Z'_\tau \pm \tfrac{1}{2}\tau^2), \qquad y = C_1\tau^4,$$

where $\nu = 0$, $A = -\frac{1}{2}(-a)^{-1/2}$.

45. $y''_{xx} = Ax^{-1}y^{-1/2}y'_x - Ax^{-2}y^{1/2}.$

Solution in the parametric form:

$$x = C_1Z^{-2}, \qquad y = b\tau^2Z^{-2}(Z'_\tau)^2,$$

where $\nu = 0$, $A = -b^{1/2}$.

46. $y''_{xx} = Ax^{1/2}y^{-2}(y'_x)^3 - Ax^{-1/2}y^{-1}(y'_x)^2.$

Solution in the parametric form:

$$x = a\tau^2Z^{-2}(Z'_\tau)^2, \qquad y = C_1Z^{-2},$$

where $\nu = 0$, $A = -a^{1/2}$.

47. $y''_{xx} = Axy^{-3}(y'_x)^4 - Ay^{-2}(y'_x)^3.$

Solution in the parametric form:

$$x = aZ^{-1}(2\tau Z'_\tau \pm \tau^2 Z), \qquad y = C_1Z^{-1},$$

where $\nu = 0$, $A = 4a$.

48. $y''_{xx} = Ax^{-2} - Ax^{-3}y(y'_x)^{-1}.$

Solution in the parametric form:

$$x = C_1Z^{-1}, \qquad y = bZ^{-1}(2\tau Z'_\tau \pm \tau^2 Z),$$

where $\nu = 0$, $A = 4b$.

49. $y''_{xx} = Axy^{-2}(y'_x)^4 - 2Ay^{-1}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1[\tau^2(Z'_\tau)^2 + 2\tau ZZ'_\tau \pm \tau^2 Z^2], \qquad y = bC_1 Z^2,$$

where $\nu = 0,\ A = \frac{1}{2}ab^{-1}$.

50. $y''_{xx} = 2Ax^{-1} - Ax^{-2}y(y'_x)^{-1}.$

Solution in the parametric form:

$$x = aC_1 Z^2, \qquad y = bC_1[\tau^2(Z'_\tau)^2 + 2\tau ZZ'_\tau \pm \tau^2 Z^2],$$

where $\nu = 0,\ A = \frac{1}{2}a^{-1}b$.

► *In the solutions of equations 51–66, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \textit{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \textit{for the lower sign,} \end{cases}$$

$$U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \quad U_2 = U_1^2 \pm \tau^2 Z^2, \quad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1U_2,$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions.

51. $y''_{xx} = Ax^{-2}y^{-2}y'_x + 2Ax^{-3}y^{-1}.$

Solution in the parametric form:

$$x = aC_1^{-2}\tau^{4/3}Z^2U_2^{-1}, \qquad y = bC_1\tau^{-2/3}Z^{-1}U_2^{-1}U_3,$$

where $A = 2ab^2$.

52. $y''_{xx} = 2Ax^{-1}y^{-3}(y'_x)^3 + Ax^{-2}y^{-2}(y'_x)^2.$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \qquad y = bC_1^2\tau^{4/3}Z^2U_2^{-1},$$

where $A = -2a^2b$.

53. $y''_{xx} = Ax^{-2}y^{-1/2}y'_x - 4Ax^{-3}y^{1/2}.$

Solution in the parametric form:

$$x = aC_1^{-1}\tau^{-4/3}Z^{-2}U_2, \qquad y = bC_1^2\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2,$$

where $A = \mp\frac{2}{3}ab^{1/2}$.

54. $y''_{xx} = 4Ax^{1/2}y^{-3}(y'_x)^3 - Ax^{-1/2}y^{-2}(y'_x)^2.$

Solution in the parametric form:

$$x = aC_1^{-2}\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \qquad y = bC_1\tau^{-4/3}Z^{-2}U_2,$$

where $A = \mp\frac{2}{3}a^{1/2}b$.

55. $y''_{xx} = 2Ax^{-1/2}y^{-2}y'_x + Ax^{-3/2}y^{-1}$.

Solution in the parametric form:

$$x = aC_1^4\tau^{-4/3}Z^{-2}U_1^2, \qquad y = bC_1\tau^{-4/3}Z^{-2}U_2,$$

where $A = \pm\frac{1}{6}a^{-1/2}b^2$.

56. $y''_{xx} = Ax^{-1}y^{-3/2}(y'_x)^3 + 2Ax^{-2}y^{-1/2}(y'_x)^2$.

Solution in the parametric form:

$$x = aC_1\tau^{-4/3}Z^{-2}U_2, \qquad y = bC_1^4\tau^{-4/3}Z^{-2}U_1^2,$$

where $A = \mp\frac{1}{6}a^2b^{-1/2}$.

57. $y''_{xx} = Axy^{-1/2}y'_x + 2Ay^{1/2}$.

Solution in the parametric form:

$$x = aC_1\tau^{-2/3}Z^{-1}U_1, \qquad y = bC_1^4\tau^{-8/3}Z^{-4}U_2^2,$$

where $A = 2a^{-2}b^{1/2}$.

58. $y''_{xx} = 2Ax^{1/2}(y'_x)^3 + Ax^{-1/2}y(y'_x)^2$.

Solution in the parametric form:

$$x = aC_1^4\tau^{-8/3}Z^{-4}U_2^2, \qquad y = bC_1\tau^{-2/3}Z^{-1}U_1,$$

where $A = -2a^{1/2}b^{-2}$.

59. $y''_{xx} = 5Ax^{-2/5} - 2Ax^{-7/5}y(y'_x)^{-1}$.

Solution in the parametric form:

$$x = aC_1^5\tau^{-5/3}Z^{-5/2}U_1^{5/2}, \qquad y = bC_1^8\tau^{-8/3}Z^{-4}(U_2^2 \pm \tfrac{4}{3}\tau^2Z^3U_1),$$

where $A = \frac{32}{125}a^{-8/5}b$.

60. $y''_{xx} = 2Axy^{-2/5}(y'_x)^4 - 5Ay^{-7/5}(y'_x)^3$.

Solution in the parametric form:

$$x = aC_1^8\tau^{-8/3}Z^{-4}(U_2^2 \pm \tfrac{4}{3}\tau^2Z^3U_1), \qquad y = bC_1^5\tau^{-5/3}Z^{-5/2}U_1^{5/2},$$

where $A = \frac{32}{125}ab^{-8/5}$.

61. $y''_{xx} = 4Ax^{1/2}y^{-3} - Ax^{-1/2}y^{-2}(y'_x)^{-1}$.

Solution in the parametric form:

$$x = aC_1^8\tau^{-4/3}Z^{-2}U_1^2, \qquad y = bC_1^5\tau^{-4/3}Z^{-2}(U_2^2 \pm \tfrac{4}{3}\tau^2Z^3U_1)^{1/2},$$

where $A = \pm\frac{1}{3}a^{-5/2}b^4$.

62. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^4 - 4Ax^{-3}y^{1/2}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1^5\tau^{-4/3}Z^{-2}\bigl(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1\bigr)^{1/2}, \qquad y = bC_1^8\tau^{-4/3}Z^{-2}U_1^2,$$

where $A = \pm\frac{1}{3}a^4 b^{-5/2}$.

63. $y''_{xx} = Ax^2 + 2Axy(y'_x)^{-1}.$

Solution in the parametric form:

$$x = aC_1\tau^{2/3}ZU_2^{-1/2}, \qquad y = bC_1^4\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3),$$

where $A = \frac{32}{9}a^{-4}b$.

64. $y''_{xx} = 2Axy(y'_x)^4 + Ay^2(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1^4\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \qquad y = bC_1\tau^{2/3}ZU_2^{-1/2},$$

where $A = -\frac{32}{9}ab^{-4}$.

65. $y''_{xx} = 2Ax^{-1}y^{-3} + Ax^{-2}y^{-2}(y'_x)^{-1}.$

Solution in the parametric form:

$$x = aC_1^4\tau^{4/3}Z^2U_2^{-1}, \qquad y = bC_1\tau^{-2/3}Z^{-1}U_2^{-1}\bigl(U_3^2 - 4U_2^3\bigr)^{1/2},$$

where $A = -\frac{8}{9}a^{-1}b^4$.

66. $y''_{xx} = Ax^{-2}y^{-2}(y'_x)^4 + 2Ax^{-3}y^{-1}(y'_x)^3.$

Solution in the parametric form:

$$x = aC_1\tau^{-2/3}Z^{-1}U_2^{-1}\bigl(U_3^2 - 4U_2^3\bigr)^{1/2}, \qquad y = bC_1^4\tau^{4/3}Z^2U_2^{-1},$$

where $A = \frac{8}{9}a^4b^{-1}$.

► *In the solutions of equations 67–72, the following notation is used:*

$$M = C_1\Phi(\lambda, \tfrac{1}{2}, \pm\tau) + C_2\Psi(\lambda, \tfrac{1}{2}, \pm\tau),$$

where Φ *and* Ψ *are linearly-independent solutions of the degenerate hypergeometric equation*

$$\tau M''_{\tau\tau} + (\tfrac{1}{2} \pm \tau)M'_\tau - \lambda M = 0.$$

The function $\Phi = \Phi(\lambda, \frac{1}{2}, \pm\tau)$ *can be expressed in terms of a degenerate hypergeometric series (see Equation 2.1.2.65).*

67. $\boldsymbol{y''_{xx} = A_1 x y'_x + A_2 y.}$

Solution in the parametric form:

$$x = a\tau^{1/2}, \qquad y = M,$$

where $A_1 = \pm 2a^{-2}$, $A_2 = \pm 4a^{-2}\lambda$.

68. $\boldsymbol{y''_{xx} = A_1 x (y'_x)^3 + A_2 y (y'_x)^2.}$

Solution in the parametric form:

$$x = M, \qquad y = b\tau^{1/2},$$

where $A_1 = \mp 4b^{-2}\lambda$, $A_2 = \mp 2b^{-2}$.

69. $\boldsymbol{y''_{xx} = A_1 x y^{-2} y'_x + A_2 y^{-1}.}$

Solution in the parametric form:

$$x = M, \qquad y = \pm b\tau^{1/2} M'_\tau,$$

where $A_1 = \mp b^2\lambda$, $A_2 = \pm b^2(\lambda + \frac{1}{2})$.

70. $\boldsymbol{y''_{xx} = A_1 x^{-1} (y'_x)^3 + A_2 x^{-2} y (y'_x)^2.}$

Solution in the parametric form:

$$x = \pm a\tau^{1/2} M'_\tau, \qquad y = M,$$

where $A_1 = \mp a^2(\lambda + \frac{1}{2})$, $A_2 = \pm a^2\lambda$.

71. $\boldsymbol{y''_{xx} = A_1 x^{-1} y^{-2} y'_x + A_2 x^{-2} y^{-1}.}$

Solution in the parametric form:

$$x = M^{-1}, \qquad y = \pm b\tau^{1/2} M^{-1} M'_\tau,$$

where $A_1 = \pm b^2\lambda$, $A_2 = \pm \frac{1}{2} b^2$.

72. $\boldsymbol{y''_{xx} = A_1 x^{-1} y^{-2} (y'_x)^3 + A_2 x^{-2} y^{-1} (y'_x)^2.}$

Solution in the parametric form:

$$x = \pm a\tau^{1/2} M^{-1} M'_\tau, \qquad y = M^{-1},$$

where $A_1 = \mp \frac{1}{2} a^2$, $A_2 = \mp a^2\lambda$.

73. $\boldsymbol{y''_{xx} = A x^n (y'_x)^{\frac{3n+2}{n+1}} + nA x^{n-1} y (y'_x)^{\frac{2n+1}{n+1}}, \qquad n \neq 0, \quad n \neq -1.}$

Solution in the parametric form:

$$x = aC_1^{-2n-1}\left(\int \frac{d\tau}{\beta\tau^k + 1} + C_2\right)^{-1/n}, \qquad y = bC_1^{n^2}\left(\tau - \int \frac{d\tau}{\beta\tau^k + 1} - C_2\right),$$

where $k = -\dfrac{n+1}{n}$, $A = \dfrac{n+1}{n^2\beta} a^{1-n} b^{-2} \Big(-\dfrac{nb\beta}{a}\Big)^{\frac{1}{n+1}}$.

74. $y''_{xx} = A(m+1)xy^m(y'_x)^{\frac{m+3}{m+2}} + Ay^{m+1}(y'_x)^{\frac{1}{m+2}}, \qquad m \neq -1, \quad m \neq -2.$

Solution in the parametric form:

$$x = aC_1^{(m+1)^2}\left(\tau - \int \frac{d\tau}{\beta\tau^k + 1} - C_2\right), \qquad y = bC_1^{-2m-3}\left(\int \frac{d\tau}{\beta\tau^k + 1} + C_2\right)^{-\frac{1}{m+1}},$$

where $k = -\dfrac{m+2}{m+1}$, $A = -\dfrac{m+2}{(m+1)^2\beta}a^{-2}b^{-m}\left[-\dfrac{a(m+1)\beta}{b}\right]^{\frac{1}{m+2}}$.

75. $y''_{xx} = Ax^{-m-1}y^m(y'_x)^l - Ax^{-m-2}y^{m+1}(y'_x)^{l-1}, \quad m \neq -1,\ l \neq 2,\ m+l-1 \neq 0.$

Solution in the parametric form:

$$x = aC_1 \exp\left(\frac{l-2}{m+l-1}\int \frac{d\tau}{F}\right), \qquad y = bC_1\tau^{\frac{l-2}{m+l-1}} \exp\left(\frac{l-2}{m+l-1}\int \frac{d\tau}{F}\right),$$

where

$$F = \frac{m+l-1}{l-2}\left(\beta + C_2\tau^k\right)^{\frac{1}{2-l}} - \tau, \quad k = \frac{(m+1)(l-2)}{m+l-1},$$
$$A = -\frac{(m+1)(m+l-1)}{(l-2)^3}a^{m-1}b^{1-m}\beta\left[\frac{(l-2)a}{(m+l-1)b}\right]^l.$$

76. $y''_{xx} = Ax^{l-2}y^{1-l}(y'_x)^l - Ax^{l-3}y^{2-l}(y'_x)^{l-1}, \qquad l \neq 2.$

Solution in the parametric form:

$$x = C_1 \exp\left(\int \frac{d\tau}{F}\right), \qquad y = C_2 \exp\left(\tau + \int \frac{d\tau}{F}\right),$$

where

$$F = (2-l)\left[\beta + e^{(l-2)\tau}\right]^{\frac{1}{2-l}} - 1, \quad A = (2-l)^{2-l}\beta.$$

77. $y''_{xx} = Ax^{-1}y'_x - Ax^{-2}y.$

1°. Solution with $A \neq 1$:

$$y = C_1x^A + \frac{C_2}{1-A}x.$$

2°. Solution with $A = 1$:

$$y = x(C_1 + C_2 \ln x).$$

78. $y''_{xx} = Axy^{-2}(y'_x)^3 - Ay^{-1}(y'_x)^2.$

1°. Solution with $A \neq 1$:

$$x = C_1y^A + \frac{C_2}{1-A}y.$$

2°. Solution with $A = 1$:

$$x = y(C_1 + C_2 \ln y).$$

► *In the solutions of equations 79–80, the following notation is used:*

$$f = \begin{cases} \dfrac{1}{\beta+1}\tau^{\beta+1} + \dfrac{1}{\beta}\tau^{\beta} + C_2 & \text{if } \beta \neq 0, \\ \tau + \ln|\tau| + C_2 & \text{if } \beta = 0. \end{cases}$$

79. $\boldsymbol{y''_{xx} = Axy^{-2}(y'_x)^l - Ay^{-1}(y'_x)^{l-1}, \qquad l \neq 3.}$

Solution in the parametric form:

$$x = aC_1[\tau^{\beta+1} - (\beta+1)f]\exp\left(-\int \tau^{\beta-1}f^{-1}\,d\tau\right), \qquad y = bC_1\exp\left(-\int \tau^{\beta-1}f^{-1}\,d\tau\right),$$

where $\beta = \dfrac{2-l}{l-3}$, $A = -a^{l-3}b^{3-l}$.

80. $\boldsymbol{y''_{xx} = Ax^{-1}(y'_x)^l - Ax^{-2}y(y'_x)^{l-1}, \qquad l \neq 1.}$

Solution in the parametric form:

$$x = aC_1\exp\left(-\int \tau^{\beta-1}f^{-1}\,d\tau\right), \qquad y = bC_1[\tau^{\beta+1} - (\beta+1)f]\exp\left(-\int \tau^{\beta-1}f^{-1}\,d\tau\right),$$

where $\beta = \dfrac{l-2}{1-l}$, $A = -a^{l-1}b^{1-l}$.

81. $\boldsymbol{y''_{xx} = A_1y^{-1}(y'_x)^2 + A_2x^{-1}y'_x.}$

1°. Solution with $A_1 \neq 1$, $A_2 \neq -1$:

$$y = (C_1x^{A_2+1} + C_2)^{A_1-1}.$$

2°. Solution with $A_1 \neq 1$, $A_2 = -1$:

$$y = (C_1\ln x + C_2)^{A_1-1}.$$

3°. Solution with $A_1 = 1$, $A_2 \neq -1$:

$$y = C_2\exp(C_1x^{A_2+1}).$$

4°. Solution with $A_1 = 1$, $A_2 = -1$:

$$y = C_2x^{C_1}.$$

► *In the solutions of equations 82–84, the following notation is used:*

$$U = \exp\left(\int \frac{W\,d\tau}{\tau\sqrt{W^2+4}}\right), \qquad W = C_2\tau^{-1/2}\exp\left[\frac{A}{2(k+1)}\tau^{k+1}\right].$$

82. $y''_{xx} = Ax^{m+1}y^m(y'_x)^2 - Ax^m y^{m+1}y'_x, \qquad m \neq -1.$

Solution in the parametric form:

$$x = C_1\tau^{1/2}U^{-1/2}, \quad y = C_1^{-1}\tau^{1/2}U^{1/2}, \qquad k = m.$$

83. $y''_{xx} = Ax^{-2m-2}y^m(y'_x)^2 - Ax^{-2m-3}y^{m+1}y'_x, \qquad m \neq -1.$

Solution in the parametric form:

$$x = C_1\tau^{-1/2}U^{1/2}, \quad y = C_1^2 U, \qquad k = m.$$

84. $y''_{xx} = Ax^{-\frac{m+1}{2}}y^m(y'_x)^2 - Ax^{-\frac{m+3}{2}}y^{m+1}y'_x, \qquad m \neq -1.$

Solution in the parametric form:

$$x = C_1^2 U, \quad y = C_1\tau^{-1/2}U^{1/2}, \qquad k = -\frac{m+3}{2}.$$

85. $y''_{xx} = Ax^n y^m(y'_x)^2 - Ax^{n-1}y^{m+1}y'_x, \qquad m \neq -1, \quad n \neq 0.$

Solution in the parametric form:

$$x = C_1 \exp\biggl(\int \frac{d\tau}{\tau F}\biggr), \quad y = C_1^k \tau \exp\biggl(k\int \frac{d\tau}{\tau F}\biggr), \qquad k = -\frac{n}{m+1},$$

where $F = F(\tau)$ is the solution of the transcendental equation

$$\frac{(F+k)^k}{(F+k-1)^{k-1}} = C_2\tau^{-1}\exp\Bigl(\frac{A}{m+1}\tau^{m+1}\Bigr).$$

86. $y''_{xx} = A_1 x^n y^{-1}(y'_x)^2 + A_2 x^{n-1}y'_x, \qquad n \neq 0.$

Solution:

$$y = C_1 \exp\biggl[\int \exp\Bigl(\frac{A_2}{n}x^n\Bigr)(F+C_2)^{-1}dx\biggr],$$

where $F = \displaystyle\int (1 - A_1 x^n)\exp\Bigl(\frac{A_2}{n}x^n\Bigr)\,dx.$

87. $y''_{xx} = A_1 y^m(y'_x)^2 + A_2 x^{-1}y^{m+1}y'_x, \qquad m \neq -1.$

Solution:

$$x = C_1 \exp\biggl[\int \exp\Bigl(-\frac{A_1}{m+1}y^{m+1}\Bigr)(F+C_2)^{-1}\,dy\biggr],$$

where $F = \displaystyle\int (1 + A_2 y^{m+1})\exp\Bigl(-\frac{A_1}{m+1}y^{m+1}\Bigr)\,dy.$

TABLE 2.13

Solvable cases of the equation $y''_{xx} = A_1 x^{n_1} y^{m_1} (y'_x)^{l_1} + A_2 x^{n_2} y^{m_2} (y'_x)^{l_2}$, $l_1 \neq l_2$

l_1	l_2	m_1	m_2	n_1	n_2	Equation
arbitrary	arbitrary	arbitrary	m_1	0	0	2.6.4.3
arbitrary	arbitrary	0	0	arbitrary	n_1	2.6.4.4
arbitrary	arbitrary	$1-l_1$	$1-l_2$	l_1-2	l_2-2	2.6.4.9
arbitrary	$3-l_1$	$1-l_1$	l_1-2	l_1-2	$1-l_1$	2.6.4.8
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	2.6.4.1
1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	2.6.4.14
1	0	$-\frac{1}{2}$	0	0	0	2.6.4.16
2	0	-1	1	arbitrary	arbitrary	2.6.4.18
2	1	arbitrary	arbitrary	0	-1	2.6.4.13
2	1	arbitrary ($m_1 \neq -1$)	0	0	arbitrary ($n_2 \neq -1$)	2.6.4.5
2	1	arbitrary ($m_1 \neq -1$)	0	0	-1	2.6.4.7
2	1	-1	0	arbitrary	arbitrary	2.6.4.12
2	1	-1	0	0	arbitrary ($n_2 \neq -1$)	2.6.4.6
$\frac{5}{2}$	$\frac{1}{2}$	arbitrary	m_1+2	m_1+2	m_1	2.6.4.11
3	0	arbitrary* ($m_1 \neq -2$)	m_1+3	$m_1 3$	m_1	2.6.4.10
3	1	arbitrary	arbitrary	1	-1	2.6.4.19
3	2	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	2.6.4.2
3	2	0	0	0	$-\frac{1}{2}$	2.6.4.17
3	2	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	2.6.4.15

* For $m_1 = -2$, see Equation 2.6.4.8 with $l = 3$

2.6.4. Other Equations ($l_1 \neq l_2$)

Table 2.13 represents all solvable equations whose solutions are outlined in Subsection 2.6.4. Equations are arranged in accordance with the growth of l_1. The number of the equation sought is indicated in the last column in this table.

1. $y''_{xx} = A_1 y^{-1/2} y'_x + A_2 y^{-1/2}$.

Solution in the parametric form:

$$x = C_1 \exp(A_1\tau) - \frac{A_2}{4A_1}\tau^2 + C_2, \qquad y = \left[A_1 C_1 \exp(A_1\tau) - \frac{A_2}{2A_1}\tau\right]^2$$

2. $y''_{xx} = A_1 x^{-1/2} (y'_x)^3 + A_2 x^{-1/2} (y'_x)^2.$

Solution in the parametric form:

$$x = \left[A_2 C_1 \exp(-A_2 \tau) + \frac{A_1}{2A_2} \tau \right]^2, \qquad y = C_1 \exp(-A_2 \tau) - \frac{A_1}{4A_2} \tau^2 + C_2.$$

3. $y''_{xx} = A_1 y^m (y'_x)^{l_1} + A_2 y^m (y'_x)^{l_2}.$

1°. Solution in the parametric form with $m \neq -1$:

$$x = C_2 + \int \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} f^{-\frac{m}{m+1}} \, d\tau, \qquad y = f^{\frac{1}{m+1}},$$

where $f = C_1 + (m+1) \displaystyle\int \tau \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} d\tau.$

2°. Solution in the parametric form with $m = -1$:

$$x = C_2 + \int \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} e^f \, d\tau, \qquad y = e^f,$$

where $f = C_1 \displaystyle\int \tau \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} d\tau.$

4. $y''_{xx} = A_1 x^n (y'_x)^{l_1} + A_2 x^n (y'_x)^{l_2}.$

1°. Solution in the parametric form with $n \neq -1$:

$$x = f^{\frac{1}{n+1}}, \qquad y = C_2 + \int \tau \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} f^{-\frac{n}{n+1}} \, d\tau,$$

where $f = C_1 + (n+1) \displaystyle\int \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} d\tau.$

2°. Solution in the parametric form with $n = -1$:

$$x = e^f, \qquad y = C_2 + \int \tau \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} e^f \, d\tau,$$

where $f = C_1 \displaystyle\int \left(A_1 \tau^{l_1} + A_2 \tau^{l_2} \right)^{-1} d\tau.$

5. $y''_{xx} = A_1 y^m (y'_x)^2 + A_2 x^n y'_x, \qquad n \neq -1, \quad m \neq -1.$

Solution:

$$\int \exp\left(-\frac{A_1}{m+1} y^{m+1} \right) dy = C_1 \int \exp\left(\frac{A_2}{n+1} x^{n+1} \right) dx + C_2.$$

6. $y''_{xx} = A_1 y^{-1} (y'_x)^2 + A_2 x^n y'_x, \qquad n \neq -1.$

1°. Solution with $A_1 \neq -1$:

$$y = \left[C_1 \int \exp\left(\frac{A_2}{n+1} x^{n+1} \right) dx + C_2 \right]^{A_1 - 1}.$$

2°. Solution with $A_1 = -1$:

$$y = C_2 \exp\left[C_1 \int \exp\left(\frac{A_2}{n+1} x^{n+1} \right) dx \right].$$

7. $y''_{xx} = A_1 y^m (y'_x)^2 + A_2 x^{-1} y'_x, \qquad m \neq -1.$

1°. Solution with $A_2 \neq -1$:

$$x = \left[C_1 \int \exp\Bigl(-\frac{A_1}{m+1} y^{m+1}\Bigr)\,dy + C_2\right]^{-A_2-1}.$$

2°. Solution with $A_2 = -1$:

$$y = C_2 \exp\left[C_1 \int \exp\Bigl(-\frac{A_1}{m+1} y^{m+1}\Bigr)\,dy\right].$$

8. $y''_{xx} = A_1 x^{l-2} y^{1-l} (y'_x)^l + A_2 x^{1-l} y^{l-2} (y'_x)^{3-l}.$

Solution in the parametric form:

$$x = C_2 \exp\left[\int \bigl(A_1 \tau^l + A_2 \tau^{3-l} - \tau^2 + \tau\bigr)^{-1}\,d\tau\right],$$
$$y = C_1 \exp\left[\int \tau \bigl(A_1 \tau^l + A_2 \tau^{3-l} - \tau^2 + \tau\bigr)^{-1}\,d\tau\right].$$

9. $y''_{xx} = A_1 x^{l_1-2} y^{1-l_1} (y'_x)^{l_1} + A_2 x^{l_2-2} y^{1-l_2} (y'_x)^{l_2}.$

Solution in the parametric form:

$$x = C_2 \exp\left[\int \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2} - \tau^2 + \tau\bigr)^{-1}\,d\tau\right],$$
$$y = C_1 \exp\left[\int \tau \bigl(A_1 \tau^{l_1} + A_2 \tau^{l_2} - \tau^2 + \tau\bigr)^{-1}\,d\tau\right].$$

10. $y''_{xx} = A x^{m+3} y^m (y'_x)^3 - A x^m y^{m+3}, \qquad m \neq -2.$

Solution in the parametric form:

$$x = C_1 \tau^{1/2} \exp\Bigl(-\frac{1}{2}\int \frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr), \qquad y = C_1^{-1} \tau^{1/2} \exp\Bigl(\frac{1}{2}\int \frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr),$$

where $V = \tau^{-1/2} \exp\Bigl(\dfrac{3A}{2m+4}\tau^{m+2}\Bigr)\left[C_2 - A\displaystyle\int \tau^m \exp\Bigl(\frac{3A}{m+2}\tau^{m+2}\Bigr)\,d\tau\right]^{-1/2}.$

11. $y''_{xx} = A x^{m+2} y^m (y'_x)^{5/2} + A x^m y^{m+2} (y'_x)^{1/2}.$

Solution in the parametric form:

$$x = C_1 \tau^{1/2} \exp\Bigl(-\frac{1}{2}\int \frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr), \qquad y = C_1^{-1} \tau^{1/2} \exp\Bigl(\frac{1}{2}\int \frac{V\,d\tau}{\tau\sqrt{V^2+4}}\Bigr),$$

where function $V = V(\tau)$ is defined in the parametric form $\tau = \tau(u)$, $V = V(u)$ as follows:

1) For $m \neq -1$, $m \neq -3/2$,

$$\tau = a u^{\frac{2}{2m+3}}, \qquad V = \frac{b(2m+3)}{2(m+1)} u^{-1} Z^{-1} (\tau Z'_u + \nu Z),$$

where

$$Z = \begin{cases} C_2 J_\nu(u) + Y_\nu(u) & \text{for the upper sign,} \\ C_2 I_\nu(u) + K_\nu(u) & \text{for the lower sign,} \end{cases}$$

J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions,

$$\nu = \frac{m+1}{2m+3}, \quad a = \Bigl(-\frac{2m+2}{Ab}\Bigr)^{-\frac{1}{m+1}}, \quad \pm\frac{(2m+3)^2}{8(m+1)^2} b^{\frac{2m+3}{m+1}} = \Bigl(-\frac{2m+2}{A}\Bigr)^{\frac{1}{m+1}};$$

2) For $m = -1$,

$$\tau = \frac{u^2}{2A^2}, \qquad V = \frac{A}{\sqrt{2}} Z^{-1} Z'_u,$$

where $Z = C_2 J_0(u) + C_3 Y_0(u)$, J_0 and Y_0 are Bessel functions;

3) For $m = -3/2$,

$$\tau = A^2 u^{-4}, \quad V = \begin{cases} \dfrac{1}{2A}\dfrac{(1+k)C_2 u^k + (1-k)u^{-k}}{C_2 u^k + u^{-k}} & \text{if } A^2 < \frac{1}{8}, \\ \dfrac{1}{2A}\dfrac{C_2 \ln u + C_2 + 1}{C_2 \ln u + 1} & \text{if } A^2 = \frac{1}{8}, \\ \dfrac{1}{2A}\dfrac{(C_2 - k)\sin(k\ln u) + (1 + kC_2)\cos(k\ln u)}{C_2 \sin(k\ln u) + \cos(k\ln u)} & \text{if } A^2 > \frac{1}{8}, \end{cases}$$

where $k = \sqrt{|1 - 8A^2|}$.

12. $y''_{xx} = A_1 x^{n_1} y^{-1} (y'_x)^2 + A_2 x^{n_2} y'_x.$

1°. Solution with $n_2 \neq -1$:

$$y = C_1 \exp\Bigl(\int u\,dx\Bigr),$$

where

$$u = \exp\Bigl(\frac{A_2}{n_2+1} x^{n_2+1}\Bigr)\Bigl[C_2 + \int (1 - A_1 x^{n_1}) \exp\Bigl(\frac{A_2}{n_2+1} x^{n_2+1}\Bigr)\,dx\Bigr]^{-1}.$$

2°. Solution with $n_2 = -1$, $A_2 \neq -1$, $A_2 \neq -n_1 - 1$:

$$y = C_1 \exp\Bigl[\int x^{A_2}\Bigl(C_2 + \frac{1}{A_2+1} x^{A_2+1} - \frac{A_1}{n_1 + A_2 + 1} x^{n_1+A_2+1}\Bigr)^{-1} dx\Bigr].$$

3°. Solution with $n_2 = -1$, $A_2 = -1$:

$$y = C_1 \exp\Bigl[\int x^{-1}\Bigl(C_2 + \ln x - \frac{A_1}{n_1} x^{n_1}\Bigr)^{-1} dx\Bigr].$$

4°. Solution with $n_2 = -1$, $A_2 = -n_1 - 1$:

$$y = C_1 \exp\Bigl[\int x^{-n_1-1}\Bigl(C_2 - \frac{1}{n_1} x^{-n_1} - A_1 \ln x\Bigr)^{-1} dx\Bigr].$$

13. $\boldsymbol{y''_{xx} = A_1 y^{m_1} (y'_x)^2 + A_2 x^{-1} y^{m_2} y'_x.}$

1°. Solution with $m_1 \neq -1$:

$$x = C_1 \exp\Bigl(\int u\,dy\Bigr),$$

where

$$u = \exp\Bigl(-\frac{A_1}{m_1+1} y^{m_1+1}\Bigr) \biggl[C_2 + \int (1 + A_2 y^{m_2}) \exp\Bigl(-\frac{A_1}{m_1+1} y^{m_1+1}\Bigr)\,dy\biggr]^{-1}.$$

2°. Solution with $m_1 = -1$, $A_1 \neq 1$, $A_1 \neq m_2 + 1$:

$$x = C_1 \exp\biggl[\int y^{-A_1}\Bigl(C_2 + \frac{1}{1-A_1} y^{1-A_1} + \frac{A_2}{m_2 - A_1 + 1} y^{m_2 - A_1 + 1}\Bigr)^{-1} dy\biggr].$$

3°. Solution with $m_1 = -1$, $A_1 = 1$:

$$x = C_1 \exp\biggl[\int y^{-1}\Bigl(C_2 + \ln y + \frac{A_2}{m_2} y^{m_2}\Bigr)^{-1} dy\biggr].$$

4°. Solution with $m_1 = -1$, $A_1 = m_2 + 1$:

$$x = C_1 \exp\biggl[\int y^{-m_2-1}\Bigl(C_2 - \frac{1}{m_2} y^{-m_2} + A_2 \ln y\Bigr)^{-1} dy\biggr].$$

► *In the solutions of equations 14–15, the following notation is used:*

$$R = \begin{cases} C_1 \tau^{k_1} + C_2 \tau^{k_2} + C_3 \tau^{k_3} & \textit{if } B_2(8B_1^3 + 27B_2) < 0, \\ C_1 \tau e^{k\tau} + C_2 e^{\sigma\tau} & \textit{if } 8B_1^3 + 27B_2 = 0, \\ C_1 e^{k\tau} + C_2 e^{\rho\tau} \cos \omega\tau & \textit{if } B_2(8B_1^3 + 27B_2) > 0, \end{cases}$$

$$Q = \begin{cases} C_1 k_1 \tau^{k_1} + C_2 k_2 \tau^{k_2} + C_3 k_3 \tau^{k_3} & \textit{if } B_2(8B_1^3 + 27B_2) < 0, \\ C_1 (1 + k\tau) \tau e^{k\tau} + C_2 \sigma e^{\sigma\tau} & \textit{if } 8B_1^3 + 27B_2 = 0, \\ C_1 k e^{k\tau} + C_2 e^{\rho\tau} (\rho \cos \omega\tau - \omega \sin \omega\tau) & \textit{if } B_2(8B_1^3 + 27B_2) > 0, \end{cases}$$

where k_1, k_2, and k_3 (real numbers) or k and $\rho \pm i\omega$ (one real and two complex numbers) are the roots of the cubic equation

$$\lambda^3 - B_1 \lambda^2 - \tfrac{1}{2} B_2 = 0.$$

In the special case $8B_1^3 = -27B_2$, we have $k = \frac{2}{3} B_1$ (multiple root) and $\sigma = -\frac{1}{3} B_1$ (simple root).

Remark. *In the expressions for R and Q, constant C_3 may be set to any nonzero number (for example, one may set $C_3 = \pm 1$).*

14. $\boldsymbol{y''_{xx} = A_1 y^{-1/2} y'_x + A_2 x y^{-1/2}.}$

Solution in the parametric form:

$$x = R, \quad y = Q^2; \qquad B_1 = A_1, \quad B_2 = A_2.$$

15. $\boldsymbol{y''_{xx} = A_1 x^{-1/2} y (y'_x)^3 + A_2 x^{-1/2} (y'_x)^2.}$

Solution in the parametric form:

$$x = Q^2, \quad y = R; \qquad B_1 = -A_2, \quad B_2 = -A_1.$$

▶ *In the solutions of equations 16–17, the following notation is used:*

$$R = \begin{cases} \tau^{B_1/2}(C_1\tau^k + C_2\tau^{-k}) + C_3 & \textit{if } B_1^2 + 2B_2 > 0, \\ C_1\tau\exp(\frac{1}{2}B_1\tau) + C_2 & \textit{if } B_1^2 + 2B_2 = 0, \\ C_1\exp(\frac{1}{2}B_1\tau)\cos(\omega\tau) + C_2 & \textit{if } B_1^2 + 2B_2 < 0, \end{cases}$$

$$Q = \begin{cases} \tau^{B_1/2}[(C_1(B_1 + 2k)\tau^k + C_2(B_1 - 2k)\tau^{-k}] & \textit{if } B_1^2 + 2B_2 > 0, \\ C_1(B_1\tau + 2)\exp(\frac{1}{2}B_1\tau) + C_2 & \textit{if } B_1^2 + 2B_2 = 0, \\ C_1\exp(\frac{1}{2}B_1\tau)[B_1\cos(\omega\tau) - 2\omega\sin(\omega\tau)] & \textit{if } B_1^2 + 2B_2 < 0, \end{cases}$$

where $k = \frac{1}{2}\sqrt{B_1^2 + 2B_2}$, $\omega = -\frac{1}{2}\sqrt{-(B_1^2 + 2B_2)}$.

16. $\boldsymbol{y''_{xx} = A_1 y^{-1/2} y'_x + A_2}$.

Solution in the parametric form:

$$x = R, \quad y = \tfrac{1}{4}Q^2; \qquad B_1 = A_1, \quad B_2 = A_2.$$

17. $\boldsymbol{y''_{xx} = A_1(y'_x)^3 + A_2 x^{-1/2}(y'_x)^2}$.

Solution in the parametric form:

$$x = \tfrac{1}{4}Q^2, \quad y = R; \qquad B_1 = -A_2, \quad B_2 = -A_1.$$

18. $\boldsymbol{y''_{xx} = A_1 x^{n_1} y^{-1}(y'_x)^2 + A_2 x^{n_2} y}$.

Solution:

$$y = C_1\exp\biggl[-\int\frac{w'_x\,dx}{(A_1x^{n_1} - 1)w}\biggr],$$

where $w = w(x)$ is the general solution of the second order linear equation

$$(A_1x^{n_1} - 1)w''_{xx} - A_1n_1x^{n_1-1}w'_x + A_1x^{n_2}(A_1x^{n_1} - 1)^2 w = 0.$$

19. $\boldsymbol{y''_{xx} = A_1 x y^{m_1}(y'_x)^3 + A_2 x^{-1} y^{m_2} y'_x}$.

Solution:

$$x = C_1\exp\biggl[\int\frac{w'_y\,dy}{(A_2y^{m_2} + 1)w}\biggr],$$

where $w = w(y)$ is the general solution of the second order linear equation

$$(A_2y^{m_2} + 1)w''_{yy} - A_2m_2y^{m_2-1}w'_y - A_1y^{m_1}(A_2y^{m_2} + 1)^2 w = 0.$$

2.7. Equations of the Form $y''_{xx} = f(x)g(y)h(y'_x)$

See Section 2.3 for the case

$$f(x) = \text{const}\,x^n, \quad g(y) = \text{const}\,y^m, \quad h(w) = \text{const}\,.$$

See Section 2.5 for the case

$$f(x) = \text{const}\,x^n, \quad g(y) = \text{const}\,y^m, \quad h(w) = \text{const}\,w^l.$$

2.7.1. Equations of the Form $y''_{xx} = f(x)g(y)$

1. $y''_{xx} = x^{-2}\left[-\dfrac{2(m+1)}{(m+3)^2}y + Ay^m\right], \qquad m \neq -3, \quad m \neq -1.$

   See equation 2.4.2.4.

2. $y''_{xx} = x^{-2}(\frac{15}{4}y + Ay^{-7}).$

   See equation 2.4.2.35.

3. $y''_{xx} = x^{-2}(6y + Ay^{-4}).$

   See equation 2.4.2.31.

4. $y''_{xx} = x^{-2}(12y + Ay^{-5/2}).$

   See equation 2.4.2.64.

5. $y''_{xx} = x^{-2}(2y + Ay^{-2}).$

   See equation 2.4.2.6.

6. $y''_{xx} = x^{-2}(-\frac{3}{16}y + Ay^{-5/3}).$

   See equation 2.4.2.26.

7. $y''_{xx} = x^{-2}(-\frac{9}{100}y + Ay^{-5/3}).$

   See equation 2.4.2.10.

8. $y''_{xx} = x^{-2}(\frac{3}{4}y + Ay^{-5/3}).$

   See equation 2.4.2.12.

9. $y''_{xx} = x^{-2}(\frac{63}{4}y + Ay^{-5/3}).$

   See equation 2.4.2.66.

10. $y''_{xx} = x^{-2}(-\frac{5}{36}y + Ay^{-7/5}).$

    See equation 2.4.2.29.

11. $y''_{xx} = x^{-2}(-\frac{2}{9}y + Ay^{-1/2}).$

    See equation 2.4.2.14.

12. $y''_{xx} = x^{-2}(-\frac{4}{25}y + Ay^{-1/2}).$

    See equation 2.4.2.8.

13. $y''_{xx} = x^{-2}(20y + Ay^{-1/2}).$

    See equation 2.4.2.33.

14. $y''_{xx} = x^{-2}(-\frac{12}{49}y + Ay^{1/2})$.

See equation 2.4.2.37.

15. $y''_{xx} = x^{-2}(Ay^2 - \frac{6}{25}y)$.

See equation 2.4.2.60.

16. $y''_{xx} = x^{-2}(Ay^2 + \frac{6}{25}y)$.

See equation 2.4.2.62.

17. $y''_{xx} = x^{-4/3}(A + By^{-1/2})$.

See equation 2.4.2.40.

18. $y''_{xx} = (Ax^4 + Bx^3)y^{-7}$.

See equation 2.4.2.39.

19. $y''_{xx} = (Ax^2 + B)y^{-5}$.

See equation 2.4.2.16.

20. $y''_{xx} = (Ax^{-1} + Bx^{-2})y^{-2}$.

See equation 2.4.2.28.

21. $y''_{xx} = (Ax^{-7/3} + Bx^{-10/3})y^{-5/3}$.

See equation 2.4.2.48.

22. $y''_{xx} = (Ax^{-4/3} + Bx^{-10/3})y^{-5/3}$.

See equation 2.4.2.49.

23. $y''_{xx} = (Ax^{-4/3} + Bx^{-7/3})y^{-5/3}$.

See equation 2.4.2.24.

24. $y''_{xx} = (Ax^{-2/3} + Bx^{-4/3})y^{-5/3}$.

See equation 2.4.2.90.

25. $y''_{xx} = (A + Bx^{-2/3})y^{-5/3}$.

See equation 2.4.2.89.

26. $y''_{xx} = (Ax^2 + B)y^{-5/3}$.

See equation 2.4.2.47.

27. $y''_{xx} = (Ax^2 + Bx)y^{-5/3}$.

See equation 2.4.2.46.

28. $y''_{xx} = A(ax^{-2/3} + bx^{-5/3})^2 y^{-5/3}$.

This is a special case of equation 2.7.1.37 with $c = 1$, $d = 0$.

29. $y''_{xx} = (Ax^{-8/5} + Bx^{-13/5}) y^{-7/5}$.

See equation 2.4.2.25.

30. $y''_{xx} = (Ax^{-5/2} + Bx^{-7/2}) y^{-1/2}$.

See equation 2.4.2.23.

31. $y''_{xx} = A(ax^5 + bx^4)^{-1/2} y^{-1/2}$.

This is a special case of equation 2.7.1.38 with $c = 1$, $d = 0$.

32. $y''_{xx} = A(ax^{15/8} + bx^{7/8})^{-4/3} y^{-1/2}$.

This is a special case of equation 2.7.1.39 with $c = 1$, $d = 0$.

33. $y''_{xx} = A(ax^{7/3} + bx^{4/3})^{-15/7} y^2$.

This is a special case of equation 2.7.1.40 with $c = 1$, $d = 0$.

34. $y''_{xx} = (ax^2 + bx + c) y^{-5/3}$.

The transformation $x = x(t)$, $y = (x'_t)^{3/2}$ leads to a third order equation:

$$2x'_t x'''_{ttt} - (x''_{tt})^2 = \tfrac{4}{3}(ax^2 + bx + c).$$

Differentiating the latter equation with respect to t and dividing it by x'_t, we obtain a constant-coefficient fourth-order linear equation: $3x''''_{tttt} = 4ax + 2b$.

35. $y''_{xx} = (ax^{-10/3} + bx^{-7/3} + cx^{-4/3}) y^{-5/3}$.

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 2.7.1.34: $w''_{tt} = (at^2 + bt + c) w^{-5/3}$.

36. $y''_{xx} = (ax^2 + bx + c)^n y^{-2n-3}$.

This is a special case of equation 2.9.1.9 with $f(\xi) = \xi^{-2n}$.

37. $y''_{xx} = A(ax + b)^2 (cx + d)^{-10/3} y^{-5/3}$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden—Fowler equation of the form 2.3.1.9: $w''_{\xi\xi} = A\Delta^{-2}\xi^2 w^{-5/3}$, where $\Delta = ad - bc$.

38. $y''_{xx} = A(ax + b)^{-1/2} (cx + d)^{-2} y^{-1/2}$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden—Fowler equation of the form 2.3.1.25: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-1/2} w^{-1/2}$, where $\Delta = ad - bc$.

39. $\boldsymbol{y''_{xx} = A(ax+b)^{-4/3}(cx+d)^{-7/6}y^{-1/2}}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden—Fowler equation of the form 2.3.1.17: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-4/3}w^{-1/2}$, where $\Delta = ad - bc$.

40. $\boldsymbol{y''_{xx} = A(ax+b)^{-15/7}(cx+d)^{-20/7}y^2}.$

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden—Fowler equation of the form 2.3.1.20: $w''_{\xi\xi} = A\Delta^{-2}\xi^{-15/7}w^2$, where $\Delta = ad - bc$.

41. $\boldsymbol{y''_{xx} = A\exp(ax^2+bx)\exp(ky)}.$

The sbubstitution $kw = ky + ax^2 + bx$ leads to an equation of the form 2.9.1.1: $w''_{xx} = Ae^{kw} + 2ak^{-1}$.

2.7.2. Equations Containing Power Functions ($h \not\equiv$ const)

1. $\boldsymbol{y''_{xx} = \left[\dfrac{2(n+1)}{(n+3)^2}x + Ax^n\right]y^{-2}(y'_x)^3, \qquad n \neq -3, \quad n \neq -1}.$

See equation 2.6.2.116.

2. $\boldsymbol{y''_{xx} = (-\tfrac{15}{4}x + Ax^{-7})y^{-2}(y'_x)^3}.$

See equation 2.6.2.117.

3. $\boldsymbol{y''_{xx} = (-6x + Ax^{-4})y^{-2}(y'_x)^3}.$

See equation 2.6.2.118.

4. $\boldsymbol{y''_{xx} = (-12x + Ax^{-5/2})y^{-2}(y'_x)^3}.$

See equation 2.6.2.119.

5. $\boldsymbol{y''_{xx} = (-2x + Ax^{-2})y^{-2}(y'_x)^3}.$

See equation 2.6.2.120.

6. $\boldsymbol{y''_{xx} = (\tfrac{3}{16}x + Ax^{-5/3})y^{-2}(y'_x)^3}.$

See equation 2.6.2.121.

7. $\boldsymbol{y''_{xx} = (\tfrac{9}{100}x + Ax^{-5/3})y^{-2}(y'_x)^3}.$

See equation 2.6.2.122.

8. $\boldsymbol{y''_{xx} = (-\tfrac{3}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3}.$

See equation 2.6.2.123.

9. $\boldsymbol{y''_{xx} = (-\tfrac{63}{4}x + Ax^{-5/3})y^{-2}(y'_x)^3}.$

See equation 2.6.2.124.

10. $y''_{xx} = (\frac{5}{36}x + Ax^{-7/5})y^{-2}(y'_x)^3$.

See equation 2.6.2.125.

11. $y''_{xx} = (\frac{2}{9}x + Ax^{-1/2})y^{-2}(y'_x)^3$.

See equation 2.6.2.126.

12. $y''_{xx} = (\frac{4}{25}x + Ax^{-1/2})y^{-2}(y'_x)^3$.

See equation 2.6.2.127.

13. $y''_{xx} = (-20x + Ax^{-1/2})y^{-2}(y'_x)^3$.

See equation 2.6.2.128.

14. $y''_{xx} = (\frac{12}{49}x + Ax^{1/2})y^{-2}(y'_x)^3$.

See equation 2.6.2.129.

15. $y''_{xx} = (Ax^2 + \frac{6}{25}x)y^{-2}(y'_x)^3$.

See equation 2.6.2.130.

16. $y''_{xx} = (Ax^2 - \frac{6}{25}x)y^{-2}(y'_x)^3$.

See equation 2.6.2.131.

17. $y''_{xx} = (A + Bx^{-1/2})y^{-4/3}(y'_x)^3$.

See equation 2.6.2.15.

18. $y''_{xx} = x^{-7}(Ay^4 + By^3)(y'_x)^3$.

See equation 2.6.2.111.

19. $y''_{xx} = x^{-5}(Ay^2 + B)(y'_x)^3$.

See equation 2.6.2.96.

20. $y''_{xx} = x^{-2}(Ay^{-1} + By^{-2})(y'_x)^3$.

See equation 2.6.2.110.

21. $y''_{xx} = x^{-5/3}(Ay^{-7/3} + By^{-10/3})(y'_x)^3$.

See equation 2.6.2.34.

22. $y''_{xx} = x^{-5/3}(Ay^{-4/3} + By^{-10/3})(y'_x)^3$.

See equation 2.6.2.36.

23. $y''_{xx} = x^{-5/3}(Ay^{-4/3} + By^{-7/3})(y'_x)^3$.

See equation 2.6.2.14.

24. $y''_{xx} = x^{-5/3}(Ay^{-2/3} + By^{-4/3})(y'_x)^3.$

See equation 2.6.2.115.

25. $y''_{xx} = x^{-5/3}(A + By^{-2/3})(y'_x)^3.$

See equation 2.6.2.114.

26. $y''_{xx} = x^{-5/3}(Ay^2 + B)(y'_x)^3.$

See equation 2.6.2.35.

27. $y''_{xx} = x^{-5/3}(Ay^2 + By)(y'_x)^3.$

See equation 2.6.2.33.

28. $y''_{xx} = Ax^{-5/3}(ay^{-2/3} + by^{-5/3})^2(y'_x)^3.$

This is a special case of equation 2.7.2.37 with $c = 1$, $d = 0$.

29. $y''_{xx} = x^{-7/5}(Ay^{-8/5} + By^{-13/5})(y'_x)^3.$

See equation 2.6.2.109.

30. $y''_{xx} = x^{-1/2}(Ay^{-5/2} + By^{-7/2})(y'_x)^3.$

See equation 2.6.2.13.

31. $y''_{xx} = Ax^{-1/2}(ay^5 + by^4)^{-1/2}(y'_x)^3.$

This is a special case of equation 2.7.2.38 with $c = 1$, $d = 0$.

32. $y''_{xx} = Ax^{-1/2}(ay^{15/8} + by^{7/8})^{-4/3}(y'_x)^3.$

This is a special case of equation 2.7.2.39 with $c = 1$, $d = 0$.

33. $y''_{xx} = Ax^2(ay^{7/3} + by^{4/3})^{-15/7}(y'_x)^3.$

This is a special case of equation 2.7.2.40 with $c = 1$, $d = 0$.

34. $y''_{xx} = x^{-5/3}(ay^2 + by + c)(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.34 for function $x = x(y)$: $x''_{yy} = -(ay^2 + by + c)x^{-5/3}$.

35. $y''_{xx} = x^{-5/3}(ay^{-10/3} + by^{-7/3} + cy^{-4/3})(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.35 for function $x = x(y)$: $x''_{yy} = -(ay^{-10/3} + by^{-4/3} + cy^{-4/3})x^{-5/3}$.

36. $y''_{xx} = x^{-2n-3}(ay^2 + by + c)^n(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.9.1.9 with $f(\xi) = -\xi^{-2n}$ for function $x = x(y)$: $x''_{yy} = -(ay^2 + by + c)^n x^{-2n-3}$.

37. $y''_{xx} = Ax^{-5/3}(ay+b)^2(cy+d)^{-10/3}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.37 for function $x = x(y)$: $x''_{yy} = -A(ay+b)^2(cy+d)^{-10/3}x^{-5/3}$.

38. $y''_{xx} = Ax^{-1/2}(ay+b)^{-1/2}(cy+d)^{-2}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.38 for function $x = x(y)$: $x''_{yy} = -A(ay+b)^{-1/2}(cy+d)^{-2}x^{-1/2}$.

39. $y''_{xx} = Ax^{-1/2}(ay+b)^{-4/3}(cy+d)^{-7/6}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.39 for function $x = x(y)$: $x''_{yy} = -A(ay+b)^{-4/3}(cy+d)^{-7/6}x^{-1/2}$.

40. $y''_{xx} = Ax^2(ay+b)^{-15/7}(cy+d)^{-20/7}(y'_x)^3.$

Assuming y as the independent variable, we obtain an equation of the form 2.7.1.40 for function $x = x(y)$: $x''_{yy} = -A(ay+b)^{-15/7}(cy+d)^{-20/7}x^2$.

41. $y''_{xx} = Ax^{-1/2}y^{-2}\left[(y'_x)^2 + B^2\right]^{1/2}.$

Solution in the parametric form:

$$x = a(u^2-1)^{-1}(\tau u \pm R)^2, \qquad y = b\tau^{-1}(u^2-1)^{-1/2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \mp\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

42. $y''_{xx} = Ax^{-1/2}y^{-2}\left[(y'_x)^2 - B^2\right]^{1/2}.$

Solution in the parametric form:

$$x = a(u^2+1)^{-1}(\tau u \pm R)^2, \qquad y = b\tau^{-1}(u^2+1)^{-1/2},$$

where $R = \sqrt{-\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tan\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

43. $y''_{xx} = Ax^{-1/2}y^{-2}\left[B^2 - (y'_x)^2\right]^{1/2}.$

Solution in the parametric form:

$$x = a(1-u^2)^{-1}(\tau u \mp R)^2, \qquad y = b\tau^{-1}(1-u^2)^{-1/2},$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr)$, $A = -\frac{1}{2}a^{-1/2}b^2$, $B = \frac{1}{2}a^{-1}b$.

44. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2\big[(y'_x)^2 + B^2\big]^{1/2}$.

Solution in the parametric form:

$$x = a\tau^{-1}(u^2-1)^{-1/2}, \qquad y = b(u^2-1)^{-1}(\tau u \pm R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \mp\tanh\Big(C_2 + \int R^{-1}\,d\tau\Big)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

45. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2\big[(y'_x)^2 - B^2\big]^{1/2}$.

Solution in the parametric form:

$$x = a\tau^{-1}(1-u^2)^{-1/2}, \qquad y = b(1-u^2)^{-1}(\tau u \mp R)^2,$$

where $R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tanh\Big(C_2 + \int R^{-1}\,d\tau\Big)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

46. $y''_{xx} = Ax^{-2}y^{-1/2}(y'_x)^2\big[B^2 - (y'_x)^2\big]^{1/2}$.

Solution in the parametric form:

$$x = a\tau^{-1}(1+u^2)^{-1/2}, \qquad y = b(1+u^2)^{-1}(\tau u \pm R)^2,$$

where $R = \sqrt{-\tau^2 - 2\tau^{-1} + C_1}$, $u = \pm\tan\Big(C_2 + \int R^{-1}\,d\tau\Big)$, $A = -\frac{1}{4}a^3b^{-3/2}$, $B = 2a^{-1}b$.

2.7.3. Equations Containing Exponential Functions ($h \not\equiv \text{const}$)

Preliminary Comments.

1. With $l \neq 1-m$, the equation

$$y''_{xx} = Ae^x y^m (y'_x)^l \tag{1}$$

has the particular solution

$$y = Be^{\lambda x}, \qquad \text{where} \quad \lambda = \frac{1}{1-m-l}, \quad B = (A\lambda^{l-2})^{\lambda}.$$

2. With $m \neq 0$ and $l \neq 1$, equation (1) can be reduced, with the aid of the transformation

$$t = (y'_x)^{1-l}, \qquad w = e^x,$$

to the generalized Emden—Fowler equation with respect to $w = w(t)$:

$$w''_{tt} = Bt^{\frac{1}{1-l}} w^{-1} (w'_t)^{\frac{2m+1}{m}}, \tag{2}$$

where $B = -m\big[A(1-l)\big]^{\frac{1}{m}}$. Equations of the form (2) are outlined in Section 2.5.

When obtained the general solution $w = w(t)$ of the Emden—Fowler equation (2), the solution of the original equation (1) can be written in the parametric form with the formulae

$$x = \ln w, \quad y = k(w_t')^{-\frac{1}{m}},$$

where $k = \bigl[A(1-l)\bigr]^{-\frac{1}{m}}$.

3. With $l \neq n+2$, the equation

$$y''_{xx} = Ax^n e^y (y'_x)^l \tag{3}$$

has the particular solution

$$y = \lambda \ln(Bx), \qquad \text{where} \quad \lambda = l - n - 2, \quad B = \left(-\frac{\lambda^{1-l}}{A}\right)^{\frac{1}{\lambda}}.$$

4. Taking y as the independent variable and x as the dependent one, we obtain from equation (3) an equation of the form (1) for $x = x(y)$:

$$x''_{yy} = -Ae^y x^n (x'_y)^{3-l}.$$

5. With $n \neq -1$ and $l \neq 1$, equation (3) can be reduced, with the aid of the transformation

$$t = (y'_x)^{1-l}, \qquad u = x^{n+1},$$

to the generalized Emden—Fowler equation for $u = u(t)$:

$$u''_{tt} = -\frac{1}{n+1} t^{\frac{1}{1-l}} u^{-\frac{n}{n+1}} (u'_t)^2. \tag{4}$$

Equations of this form are outlined in Section 2.5.

When obtained the general solution $u = u(t)$ of the Emden—Fowler equation (4), the solution of the original equation (3) can be written in the parametric form with the formulae

$$x = u^{\frac{1}{n+1}}, \quad y = -\ln(u'_t) + \ln \frac{n+1}{A(1-l)}.$$

1. $\boldsymbol{y''_{xx} = Ae^x (y'_x)^l}$.

1°. Solution in the parametric form with $l \neq 1$:

$$x = \ln\left[\pm \frac{1}{A(1-l)} C_1^{1-l} \tau\right], \qquad y = C_1 \int \frac{1}{\tau} (1 \pm \tau)^{\frac{1}{1-l}}\, d\tau + C_2.$$

2°. Solution in the parametric form with $l = 1$:

$$x = \ln\left(\pm \frac{\tau}{A}\right), \qquad y = C_1 \int \frac{1}{\tau} \exp(\pm\tau)\, d\tau + C_2.$$

2. $y''_{xx} = Ae^x y^m (y'_x)^2.$

1°. Solution in the parametric form with $m \neq -1$:

$$x = \int \frac{d\tau}{f} + C_2, \qquad y = \tau \exp\biggl[-\frac{1}{m+1}\biggl(\int \frac{d\tau}{f} + C_2\biggr)\biggr],$$

where function $f = f(\tau)$ is defined implicitly by the relation

$$\ln\biggl(\frac{f}{\tau} - \frac{1}{m+1}\biggr) - \frac{\tau}{(m+1)f - \tau} = \frac{A}{m+1}\tau^{m+1} - \ln\tau + C_1.$$

2°. Solution with $m = -1$:

$$y = C_2 \exp\biggl(\int \frac{dx}{x + Ae^x + C_1}\biggr).$$

3. $y''_{xx} = Ae^x y.$

1°. Solution with $A > 0$:

$$y = C_1 I_0\bigl(2\sqrt{A}\,e^{x/2}\bigr) + C_2 K_0\bigl(2\sqrt{A}\,e^{x/2}\bigr),$$

where I_0 and K_0 are modified Bessel functions.

2°. Solution with $A < 0$:

$$y = C_1 J_0\bigl(2\sqrt{-A}\,e^{x/2}\bigr) + C_2 Y_0\bigl(2\sqrt{-A}\,e^{x/2}\bigr),$$

where J_0 and Y_0 are Bessel functions.

4. $y''_{xx} = Ae^x y^{-1/2} (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = \tau^2 - \ln(Af), \qquad y = C_1\bigl[2\tau f - \exp(\tau^2)\bigr]^2,$$

where $f = \int \exp(\tau^2)\,d\tau + C_2$.

5. $y''_{xx} = Ae^x y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = -\ln\bigl[AC_1^3\bigl(\sqrt{\tau^2+\tau} - f\bigr)\bigr], \qquad y = 2C_1^2\biggl(1 - f\sqrt{\frac{\tau+1}{\tau}}\biggr),$$

where $f = \ln\bigl(\sqrt{\tau} + \sqrt{\tau+1}\bigr) + C_2$.

6. $y''_{xx} = Ae^y (y'_x)^l.$

1°. Solution in the parametric form with $l \neq 2$:

$$x = C_1 \int \frac{1}{\tau}(1 \pm \tau)^{\frac{1}{l-2}}\,d\tau + C_2, \qquad y = \ln\biggl[\pm\frac{1}{A(2-l)}C_1^{l-2}\tau\biggr].$$

2°. Solution in the parametric form with $l = 2$:

$$x = C_1 \int \frac{1}{\tau}\exp(\mp\tau)\,d\tau + C_2, \qquad y = \ln\Bigl(\pm\frac{\tau}{A}\Bigr).$$

7. $y''_{xx} = Ax^n e^y y'_x.$

1°. Solution in the parametric form with $n \neq -1$:

$$x = \tau \exp\left[-\frac{1}{n+1}\left(\int \frac{d\tau}{f} + C_2\right)\right], \qquad y = \int \frac{d\tau}{f} + C_2,$$

where function $f = f(\tau)$ is defined implicitly by the relation

$$\ln\left(\frac{f}{\tau} - \frac{1}{n+1}\right) - \frac{\tau}{(n+1)f - \tau} = -\frac{A}{n+1}\tau^{n+1} - \ln\tau + C_1.$$

2°. Solution with $n = -1$:

$$x = C_2 \exp\left(\int \frac{dy}{y - Ae^y + C_1}\right).$$

8. $y''_{xx} = Ax^{-1/2} e^y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = C_1\left[2\tau f - \exp(\tau^2)\right]^2, \qquad y = \tau^2 - \ln(-Af),$$

where $f = \int \exp(\tau^2)\, d\tau + C_2$.

9. $y''_{xx} = Axe^y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = 2C_1^2\left(1 - f\sqrt{\frac{\tau+1}{\tau}}\right), \qquad y = -\ln\left[AC_1^3\left(f - \sqrt{\tau^2+\tau}\right)\right],$$

where $f = \ln\left(\sqrt{\tau} + \sqrt{\tau+1}\right) + C_2$.

10. $y''_{xx} = Axe^y (y'_x)^3.$

1°. Solution in the parametric form with $A > 0$:

$$x = C_1 J_0(2\tau) + C_2 Y_0(2\tau), \qquad y = \ln\left(\tau/\sqrt{A}\right),$$

where J_0 and Y_0 are Bessel functions.

2°. Solution in the parametric form with $A < 0$:

$$x = C_1 I_0(2\tau) + C_2 K_0(2\tau), \qquad y = \ln\left(\tau/\sqrt{-A}\right),$$

where I_0 and K_0 are modified Bessel functions.

11. $y''_{xx} = Ae^x e^y (y'_x)^l.$

Solution in the parametric form:

$$x = \int \frac{d\tau}{f\tau^l} + C_2, \qquad y = \ln\left(\frac{f}{A}\right) - \int \frac{d\tau}{f\tau^l} - C_2,$$

$$\text{where } f = \begin{cases} \dfrac{1}{2-l}\tau^{2-l} + \dfrac{1}{1-l}\tau^{1-l} + C_1 & \text{if } l \neq 1, 2; \\ \tau + \ln|\tau| + C_1 & \text{if } l = 1; \\ \ln|\tau| - \dfrac{1}{\tau} + C_1 & \text{if } l = 2. \end{cases}$$

12. $y''_{xx} = A\exp(kx)\exp(ay^2+by)(y'_x)^3.$

Taking y as the independent variable, we obtain an equation of the form 2.7.1.41 for function $x=x(y)$: $x''_{yy} = -A\exp(ay^2+by)\exp(kx)$.

13. $y''_{xx} = Ae^x y^{-1/2}(y'_x)^{3/2}\sqrt{y'_x-2B}.$

Solution in the parametric form:

$$x=\ln[a\tau(\cosh u)^{-1}], \qquad y=B\cosh^2 u\,(\tau\tanh u \pm R)^2,$$

where $a=-A^{-1}B^{-1/2}$, $R=\sqrt{2\ln\tau+\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

14. $y''_{xx} = Ae^x y^{-1/2}(y'_x)^{3/2}\sqrt{2B-y'_x}.$

Solution in the parametric form:

$$x=\ln[a\tau(\cos u)^{-1}], \qquad y=B\cos^2 u\,(\tau\tan u \pm R)^2,$$

where $a=-A^{-1}B^{-1/2}$, $R=\sqrt{2\ln\tau-\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

15. $y''_{xx} = Ax^{-1/2}e^y y'_x\sqrt{y'_x-B}.$

Solution in the parametric form:

$$x=\frac{1}{2B}\cos^2 u\,(\tau\tan u \pm R)^2, \qquad y=\ln[b\tau(\cos u)^{-1}],$$

where $b=A^{-1}\sqrt{2}$, $R=\sqrt{2\ln\tau-\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

16. $y''_{xx} = Ax^{-1/2}e^y y'_x\sqrt{B-y'_x}.$

Solution in the parametric form:

$$x=\frac{1}{2B}\cosh^2 u\,(\tau\tanh u \pm R)^2, \qquad y=\ln[b\tau(\cosh u)^{-1}],$$

where $b=A^{-1}\sqrt{2}$, $R=\sqrt{2\ln\tau+\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

2.7.4. Equations Containing Hyperbolic Functions ($h \not\equiv \text{const}$)

1. $y''_{xx} = Ax[\cosh(\omega y)]^{-2}y'_x.$

Solution in the parametric form:

$$x=a\cosh u\,(\tau\tanh u \pm R), \qquad y=u/\omega,$$

where $A=a^{-2}$, $R=\sqrt{2\ln\tau+\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

2. $y''_{xx} = Ax[\sinh(\omega y)]^{-2}y'_x.$

Solution in the parametric form:

$$x=a\sinh u\,(\tau\coth u \pm R), \qquad y=u/\omega,$$

where $A=a^{-2}$, $R=\sqrt{2\ln\tau+\tau^2+C_1}$, $u=C_2 \mp \int R^{-1}\,d\tau$.

3. $\boldsymbol{y''_{xx} = Ax\cosh(\omega y)(y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = a(u^2+1)^{-1/2}(\tau u \pm R), \qquad y = \omega^{-1}\ln\bigl(u+\sqrt{u^2+1}\,\bigr),$$

where $A = 2a^{-2}\sqrt{a\omega}$, $R = \sqrt{C_1 - \tau^2 - 2\tau^{-1}}$, $u = \pm\tan(C_2 + \int R^{-1}\,d\tau)$.

4. $\boldsymbol{y''_{xx} = Ax\sinh(\omega y)(y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = a(u^2-1)^{-1/2}(\tau u \pm R), \qquad y = \pm\omega^{-1}\ln\bigl(u+\sqrt{u^2-1}\,\bigr),$$

where $A = \pm 2a^{-2}\sqrt{a\omega}$, $R = \sqrt{C_1 + \tau^2 - 2\tau^{-1}}$, $u = \mp\tanh(C_2 + \int R^{-1}\,d\tau)$.

5. $\boldsymbol{y''_{xx} = A\cosh(\omega x)y(y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = \omega^{-1}\ln(u+\sqrt{u^2+1}), \qquad y = b(u^2+1)^{-1/2}(\tau u \pm R),$$

where $A = -2b^{-2}\sqrt{b\omega}$, $R = \sqrt{C_1 - \tau^2 - 2\tau^{-1}}$, $u = \pm\tan(C_2 + \int R^{-1}\,d\tau)$.

6. $\boldsymbol{y''_{xx} = A\sinh(\omega x)y(y'_x)^{3/2}}$.

Solution in the parametric form:

$$x = \pm\omega^{-1}\ln(u+\sqrt{u^2-1}), \qquad y = b(u^2-1)^{-1/2}(\tau u \pm R),$$

where $A = \mp 2b^{-2}\sqrt{b\omega}$, $R = \sqrt{C_1 + \tau^2 - 2\tau^{-1}}$, $u = \mp\tanh(C_2 + \int R^{-1}\,d\tau)$.

7. $\boldsymbol{y''_{xx} = A[\cosh(\omega x)]^{-2}y(y'_x)^2}$.

Solution in the parametric form:

$$x = u/\omega, \qquad y = b\cosh u\,(\tau\tanh u \pm R),$$

where $A = -b^{-2}$, $R = \sqrt{2\ln\tau + \tau^2 + C_1}$, $u = C_2 \mp \int R^{-1}\,d\tau$.

8. $\boldsymbol{y''_{xx} = A[\sinh(\omega x)]^{-2}y(y'_x)^2}$.

Solution in the parametric form:

$$x = u/\omega, \qquad y = b\sinh u\,(\tau\coth u \pm R),$$

where $A = -b^{-2}$, $R = \sqrt{2\ln\tau + \tau^2 + C_1}$, $u = C_2 \mp \int R^{-1}\,d\tau$.

2.7.5. Equations Containing Trigonometric Functions ($h \not\equiv \text{const}$)

► *In the solutions of equations 1–4, the following notation is used:*

$$R = \sqrt{2\ln\tau - \tau^2 + C_1}, \qquad u = C_2 \pm \int R^{-1}\,d\tau.$$

1. $y''_{xx} = Ax[\cos(\omega y)]^{-2} y'_x.$

Solution in the parametric form:

$$x = a\cos u\,(\tau\tan u \pm R), \quad y = u/\omega, \qquad \text{where} \quad A = a^{-2}.$$

2. $y''_{xx} = Ax[\sin(\omega y)]^{-2} y'_x.$

Solution in the parametric form:

$$x = a\sin u\,(\tau\cot u \mp R), \quad y = u/\omega, \qquad \text{where} \quad A = a^{-2}.$$

3. $y''_{xx} = A[\cos(\omega x)]^{-2} y (y'_x)^2.$

Solution in the parametric form:

$$x = \omega^{-1}u, \quad y = b\cos u\,(\tau\tan u \pm R), \qquad \text{where} \quad A = -b^2.$$

4. $y''_{xx} = A[\sin(\omega x)]^{-2} y (y'_x)^2.$

Solution in the parametric form:

$$x = \omega^{-1}u, \quad y = b\sin u\,(\tau\cot u \mp R), \qquad \text{where} \quad A = -b^2.$$

► *In the solutions of equations 5–8, the following notation is used:*

$$R = \sqrt{\tau^2 - 2\tau^{-1} + C_1}, \qquad u = \pm\tanh\Bigl(C_2 + \int R^{-1}\,d\tau\Bigr).$$

5. $y''_{xx} = Ax\cos(\omega y)(y'_x)^{3/2}.$

Solution in the parametric form:

$$x = a\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \quad y = \omega^{-1}\arccos u, \qquad \text{where} \quad A = 2a^{-2}(-a\omega)^{1/2}.$$

6. $y''_{xx} = Ax\sin(\omega y)(y'_x)^{3/2}.$

Solution in the parametric form:

$$x = a\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \quad y = \omega^{-1}\arccos u, \qquad \text{where} \quad A = 2a^{-2}(a\omega)^{1/2}.$$

7. $y''_{xx} = A\cos(\omega x) y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = \omega^{-1}\arccos u, \quad y = b\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \qquad \text{where} \quad A = -2b^{-2}(-b\omega)^{1/2}.$$

8. $y''_{xx} = A\sin(\omega x) y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = \omega^{-1}\arccos u, \quad y = b\bigl(1-u^2\bigr)^{-1/2}(\tau u \mp R), \qquad \text{where} \quad A = -2b^{-2}(b\omega)^{1/2}.$$

2.7.6. Some Transformations

For the sake of visualization, we also use the symbolic notation $\{f,\ g,\ h\}$ to denote the equation

$$y''_{xx} = f_1(x)g_1(y)h_1(y'_x). \tag{1}$$

1. Taking y as the independent variable and x as the dependent one, we obtain an equation of the similar form for $x = x(y)$:

$$x''_{yy} = g_1(y)f_1(x)h_1^*(x'_y), \qquad \text{where} \quad h_1^*(w) = -w^3 h_1(1/w).$$

Denote this transformation by $\mathcal{F}$.

2. The Bäcklund transformation

$$\bar{x} = \int \frac{dw}{h_1(w)}, \qquad \bar{y} = \int f_1(x)\,dx, \qquad \text{where} \quad w = y'_x \tag{2}$$

leads to an equation of the similar form for function $\bar{y} = \bar{y}(\bar{x})$:

$$\bar{y}''_{\bar{x}\bar{x}} = f_2(\bar{x})g_2(\bar{y})h_2(\bar{y}'_{\bar{x}}),$$

where the functions f_2, g_2, and h_2 are defined in terms of the original functions f_1, g_1, and h_1 parametrically by the formulae

$$\begin{aligned} f_2(\bar{x}) &= w, & \bar{x} &= \int \frac{dw}{h_1(w)}, \\ g_2(\bar{y}) &= \frac{1}{f_1(x)}, & \bar{y} &= \int f_1(x)\,dx, \\ h_2(\bar{w}) &= -\frac{1}{[g_1(y)]^3}\frac{dg_1}{dy}, & \bar{w} &= \frac{1}{g_1(y)}. \end{aligned}$$

Denote transformation (2) by $\mathcal{G}$. For equations of the form (1) wherein f_1, g_1, and h_1 are power functions of their arguments, transformation $\mathcal{G}$ (to a precision of constant factors) is considered in Subsection 2.5.3. For equations (1) with exponential functions f_1 and g_1, transformation $\mathcal{G}$ is discussed in Subsection 2.7.3.

When found the solution $\bar{y} = \bar{y}(\bar{x})$ of the transformed equation, the formulae

$$\bar{y} = \int f_1(x)\,dx, \qquad \bar{y}'_{\bar{x}} = \frac{1}{g_1(y)},$$

make it possible to obtain the solution of the original equation (1) in the parametric form $x = x(\bar{x})$, $y = y(\bar{x})$.

3. The twofold application of transformation $\mathcal{G}$ to the original equation yields an equation of the similar form:

$$\bar{\bar{y}}''_{\bar{\bar{x}}\bar{\bar{x}}} = f_3(\bar{\bar{x}})g_3(\bar{\bar{y}})h_3(\bar{\bar{y}}'_{\bar{\bar{x}}}),$$

where the functions f_3, g_3, and h_3 are defined in terms of the original functions f_1, g_1, and h_1 parametrically by the formulae

$$\begin{aligned} f_3(\bar{\bar{x}}) &= \frac{1}{g_1(y)}, & \bar{\bar{x}} &= \int g_1(y)\,dy, \\ g_3(\bar{\bar{y}}) &= \frac{1}{w}, & \bar{\bar{y}} &= \int \frac{w\,dw}{h_1(w)}, \\ h_3(\bar{\bar{w}}) &= \frac{df_1}{dx}, & \bar{\bar{w}} &= f_1(x). \end{aligned}$$

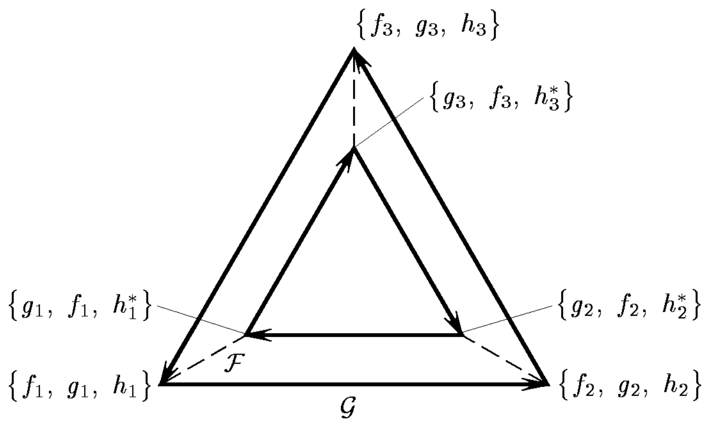

FIGURE 4

The treefold transformation $\mathcal{G}$ yields the original equation.

Different compositions of transformations $\mathcal{F}$ and $\mathcal{G}$ generate six different equations of the analogous form which are shown in Figure 4.

4. In the special case $g(y) = y^m$, $h = 1$, the transformation $x = \dfrac{1}{\tau}$, $y = \dfrac{u}{\tau}$ leads to an equation of the similar form:

$$u''_{\tau\tau} = \tau^{-m-3} f\Big(\frac{1}{\tau}\Big) u^m.$$

Denote this transformation by $\mathcal{H}$.

For $g(y) = y^m$ and $h = 1$, different compositions of transformations $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ generate twelve different equations of the form (1).

2.8. Some Nonlinear Equations with Arbitrary Parameters

2.8.1. Equations Containing Power Functions

1. $\boldsymbol{y''_{xx} = (ay + bx + c)^n}$.

This is a special case of equation 2.9.1.2 with $f(\xi) = \xi^n$.

2. $\boldsymbol{y''_{xx} = (ay + bx^2)^n + c}$.

The substitution $aw = ay + bx^2$ leads to an equation of the form 2.9.1.1:

$$w''_{xx} = a^n w^n + c + \frac{2b}{a}.$$

3. $\boldsymbol{y''_{xx} = \lambda x^{-2n-3}(xy + a)^n}$.

This is a special case of equation 2.9.1.7 with $f(\xi) = \lambda\xi^n$, $b = c = 0$.

4. $\boldsymbol{y''_{xx} = (ax^2 + bx + c)y^{-5}}$.

This is a special case of equation 2.9.1.9 with $f(\xi) = \xi^{-2}$.

5. $\boldsymbol{y''_{xx} = A(ax+b)^n(cx+d)^{-n-m-3}y^m}$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{cx+d}$ leads to the Emden—Fowler equation (see Section 2.3): $w''_{\xi\xi} = A\Delta^{-2}\xi^n w^m$, where $\Delta = ad - bc$.

6. $\boldsymbol{y''_{xx} = cx^m y^{-nk-m-3}(ay^n + bx^n)^k}$.

This is a special case of equation 2.9.1.5 with $f(\xi) = c\xi^{-nk-m-3}(a\xi^n + b)^k$.

7. $\boldsymbol{y''_{xx} = cx^m y^{-2nk-2m-3}(ay^{2n} + bx^n)^k}$.

This is a special case of equation 2.9.1.6 with $f(\xi) = c\xi^{-2nk-2m-3}(a\xi^{2n} + b)^k$.

8. $\boldsymbol{y''_{xx} = (ay^2 + bxy + cx^2 + \alpha y + \beta x + \gamma)^{-3/2}}, \qquad \boldsymbol{a \neq 0}$.

The substitution $2au = 2ay + bx + \alpha$ leads to an equation of the form 2.9.1.9:

$$u''_{xx} = u^{-3} f\left(\frac{u}{\sqrt{Ax^2 + Bx + C}}\right),$$

where $f(\xi) = \xi^3(a\xi^2 + 1)^{-3/2}$, $A = \dfrac{4ac - b^2}{4a}$, $B = \dfrac{2a\beta - b\alpha}{2a}$, $C = \dfrac{4a\gamma - \alpha^2}{4a}$.

9. $\boldsymbol{y''_{xx} = \lambda y^{-1/3} + (ax^2 + bx + c)y^{-5/3}}$.

The transformation $x = x(t)$, $y = (x'_t)^{3/2}$ leads to a third order equation:

$$2x'_t x'''_{ttt} - (x''_{tt})^2 = \tfrac{4}{3}\lambda(x'_t)^2 + \tfrac{4}{3}(ax^2 + bx + c).$$

Differentiating the latter equation with respect to t and dividing it by x'_t, we arrive at a constant-coefficient fourth-order linear equation: $3x''''_{tttt} = 2\lambda x''_{tt} + 4ax + 2b$.

10. $\boldsymbol{y''_{xx} = \lambda x^{-8/3} y^{-1/3} + (ax^{-10/3} + bx^{-7/3} + cx^{-4/3})y^{-5/3}}$.

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 2.8.1.9:

$$w''_{tt} = \lambda w^{-1/3} + (at^2 + bt + c)w^{-5/3}.$$

11. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + ax^n y = 0}$.

This is a special case of equation 2.9.2.1 with $f(x) = ax^n$.

12. $\boldsymbol{y''_{xx} + (2ay + bx^n)y'_x + bnx^{n-1}y = 0}$.

This is a special case of equation 2.9.2.13 with $f(x) = bx^n$.

13. $\boldsymbol{y''_{xx} = ax^n(xy'_x - y)^2 + bx^m}$.

This is a special case of equation 2.9.3.4 with $f(x) = bx^m$, $g(x) = 0$, $h(x) = ax^n$.

14. $\boldsymbol{y''_{xx} = ax^n(xy'_x - y)^m}$.

This is a special case of equation 2.9.4.30 with $f(x) = ax^n$, $g(\xi) = \xi^m$.

15. $y''_{xx} = ax^{-n-3}y^n(xy'_x - y)^m.$

This is a special case of equation 2.9.4.31 with $f(\xi) = a\xi^m$.

16. $y''_{xx} = ax^{-1}y^n y'_x(xy'_x - y)^m.$

This is a special case of equation 2.9.4.33 with $f(y) = ay^n$, $g(\xi) = \xi^m$.

17. $y''_{xx} = ax^{nk-1}y^{mk-1}y'_x(xy'_x - y)^{\frac{2n+m}{n}}.$

This is a special case of equation 2.9.4.32 with $f(\xi) = a\xi^k$.

18. $y''_{xx} = kx^\alpha(y'_x)^\beta(xy'_x - y)^\gamma.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$, where $w = w(t)$, leads to the generalized Emden—Fowler equation

$$w''_{tt} = \frac{1}{k}t^{-\beta}w^{-\gamma}(w'_t)^{-\alpha}.$$

A good deal of solvable equations of this type are outlined in Section 2.3 and Section 2.5.

19. $y''_{xx} = ax^{n-1}y^{m-1}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k.$

This is a special case of equation 2.9.4.36 with $f(\xi) = a\xi$.

20. $y''_{xx} = ax^n(xy'_x - y) + bx^m(xy'_x - y)^k.$

This is a special case of equation 2.9.4.39 with $f(x) = ax^n$, $g(x) = bx^m$.

21. $xy''_{xx} = ny'_x + ax^{2n+1} + bx^{2n+1}y^m.$

This is a special case of equation 2.9.2.4 with $f(y) = a + by^m$.

22. $xy''_{xx} = -(n+1)y'_x + ax^{n-1} + bx^{nm+n-1}y^m.$

This is a special case of equation 2.9.2.5 with $f(\xi) = a + b\xi^m$.

23. $yy''_{xx} - \frac{1}{4}(y'_x)^2 = ax^2 + bx + c.$

The substitution $y = w^{4/3}$ leads to a special case of the equation 2.8.1.9 with $\lambda = 0$:

$$4w''_{xx} = 3(ax^2 + bx + c)w^{-5/3}.$$

24. $3yy''_{xx} - 2(y'_x)^2 = ax^2 + bx + c.$

The substitution $y = w^3$ leads to the equation 2.8.1.4: $9w''_{xx} = (ax^2 + bx + c)w^{-5}$.

25. $2yy''_{xx} = (y'_x)^2 + bx^n y^2 - a.$

This is a special case of equation 2.9.3.7 with $f(x) = -bx^n$.

26. $\boldsymbol{yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + bx^m y^2.}$

This is a special case of equation 2.9.3.8 with $f(x) = -bx^m$.

27. $\boldsymbol{yy''_{xx} = n(y'_x)^2 + ax^k y^2 + bx^m y^{n+1}.}$

This is a special case of equation 2.9.3.9 with $f(x) = -ax^k$, $g(x) = -bx^m$.

28. $\boldsymbol{(n+2)yy''_{xx} - (n+1)(y'_x)^2 = (ax^2 + bx + c)^n.}$

The substitution $y = w^{n+2}$ leads to an equation of the form 2.7.1.36:

$$w''_{xx} = \frac{1}{(n+2)^2}(ax^2 + bx + c)^n w^{-2n-3}.$$

29. $\boldsymbol{yy''_{xx} = (y'_x)^2 + ax^n yy'_x + bx^m y^2.}$

This is a special case of equation 2.9.3.6 with $f(x) = -ax^n$, $g(x) = -bx^m$.

30. $\boldsymbol{ayy''_{xx} + b(y'_x)^2 + (x^n + \lambda)^m yy'_x = 0.}$

Solution: $\quad y^{\frac{a+b}{a}} = C_1 \displaystyle\int \exp\left[-\frac{1}{a}\int (x^n + \lambda)^m\,dx\right] dx + C_2.$

31. $\boldsymbol{(y + ax)y''_{xx} = bx^n(xy'_x - y)^2.}$

The substitution $y = -ax + xz$ leads to the equation

$$xzz''_{xx} + 2zz'_x - bx^{n+3}(z'_x)^2 = 0.$$

Having set $w = z'_x/z$, we obtain the Beroulli equation

$$xw'_x + 2w + x(1 - bx^{n+2})w^2 = 0.$$

32. $\boldsymbol{x^2 y''_{xx} = n(n+1)y + ax^{3n+2} + bx^{nm+3n+2}y^m.}$

This is a special case of equation 2.9.1.11 with $f(\xi) = a + b\xi^m$.

33. $\boldsymbol{x^2 y''_{xx} = k(k+1)y + ax^{km+3k+2}(bx^{2k+1} + c)^n y^m.}$

The transformation $\xi = bx^{2k+1} + c$, $\; w = yx^k$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = ab^{-2}(2k+1)^{-2}\xi^n w^m.$$

34. $\boldsymbol{x^2 y''_{xx} + xy'_x = ay^n + b.}$

This is a special case of equation 2.9.2.8 with $f(y) = ay^n + b$.

35. $\boldsymbol{x^2 y''_{xx} = -(n+m+1)xy'_x - nmy + ax^{nk+n-2m}y^k.}$

This is a special case of equation 2.9.2.9 with $f(\xi) = a\xi^k$.

36. $x^2 y''_{xx} + axy'_x + by = cx^n y^m$.

The transformation $x = \xi^\alpha$, $y = \xi^\beta w$, where $\alpha = \pm\dfrac{1}{\sqrt{D}}$, $\beta = \pm\dfrac{1-a}{2\sqrt{D}} - \dfrac{1}{2}$, $D = (1-a)^2 - 4b$, leads to the Emden—Fowler equation

$$w''_{\xi\xi} = c\alpha^2 \xi^{n\alpha+m\beta-\beta-2} w^m$$

whose solvable cases are outlined in Section 2.3.

37. $x^2 y''_{xx} = n(n-1)y + ax^n (xy'_x - ny)^m$.

This is a special case of equation 2.9.4.37 with $f(x) = ax^{n-2}$.

38. $(ax^2 + b)y''_{xx} + axy'_x + cy^n = 0$.

This is a special case of equation 2.9.2.11 with $f(y) = cy^n$.

39. $(ay + bx^2)y''_{xx} = 1$.

This is a special case of equation 2.8.1.2 with $n = -1$, $c = 0$.

40. $xyy''_{xx} = x(y'_x)^2 - yy'_x + ax^k y^s$.

This is a special case of equation 2.9.4.14 with $f(\xi) = a\xi$, $g(\xi) = 1$, $k = n-1$, $s = m+2$.

41. $y^2 y''_{xx} + y(y'_x)^2 = ax + b$.

Having set $\dfrac{1}{y} = u'_x(x)$, we obtain

$$-u'_x u'''_{xxx} + 3(u''_{xx})^2 = (ax+b)(u'_x)^5.$$

Taking u as the independent variable, we obtain a constant coefficient linear equation for $x = x(u)$: $x'''_{uuu} = ax + b$.

42. $(x+a)^2 y^2 y''_{xx} = bx$.

The transformation $\xi = \ln\left|\dfrac{x+a}{x}\right|$, $w = \dfrac{y}{x}$ leads to an equation of the form 2.2.1.7: $w''_{\xi\xi} - w'_\xi = a^{-2} w^{-2}$.

43. $(y^2 + ax^2 + 2bx + c)^2 y''_{xx} + sy = 0$.

Dividing by the coefficient of y''_{xx} and multiplying by $ax(xy'_x - y) + b(2xy'_x - y) + cy'_x$, we arrive at a total differential equation. Integrating the latter, we obtain

$$(ax^2 + 2bx + c)(y'_x)^2 - 2(ax+b)yy'_x + ay^2 + \frac{sy^2}{y^2 + ax^2 + 2bx + c} = C.$$

44. $\boldsymbol{(ax+b)^2(cx+d)^2y''_{xx} = sy + A(ax+b)^k(cx+d)^{1-m-k}y^m.}$

The transformation

$$\xi = \ln\Bigl(\frac{ax+b}{cx+d}\Bigr), \qquad w = \Bigl(\frac{ax+b}{cx+d}\Bigr)^{\frac{k}{m-1}} \frac{y}{cx+d}$$

leads to an autonomous equation:

$$w''_{\xi\xi} - (2n+1)w'_\xi + (n^2+n-s\Delta^{-2})w = A\Delta^{-2}w^m,$$

where $n = \dfrac{k}{m-1}$, $\Delta = ad - bc$.

45. $\boldsymbol{(a^2-x^2)(b^2-y^2)y''_{xx} + (a^2-x^2)y(y'_x)^2 = x(b^2-y^2)y'_x.}$

Solution: $\arcsin\dfrac{y}{b} = C_1 + C_2 \arcsin\dfrac{x}{b}$.

46. $\boldsymbol{(ay^n + bx^n)y''_{xx} + cx^{n-3} = 0.}$

This is a special case of equation 2.9.1.5 with $f(\xi) = -c(a\xi^n + b)^{-1}$.

47. $\boldsymbol{(ay^n + bx^n)y''_{xx} + cy^{n-3} = 0.}$

This is a special case of equation 2.9.1.5 with $f(\xi) = -c\xi^{n-3}(a\xi^n + b)^{-1}$.

48. $\boldsymbol{(ay^{2n} + bx^n)y''_{xx} + cy^{2n-3} = 0.}$

This is a special case of equation 2.9.1.6 with $f(\xi) = -c\xi^{2n-3}(a\xi^{2n} + b)^{-1}$.

49. $\boldsymbol{(ay^n + bx^n)y''_{xx} + cx^m y^{n-m-3} = 0.}$

This is a special case of equation 2.9.1.5 with $f(\xi) = -c\xi^{n-m-3}(a\xi^n + b)^{-1}$.

50. $\boldsymbol{(ay^{2n} + bx^n)y''_{xx} + cx^m y^{2n-2m-3} = 0.}$

This is a special case of equation 2.9.1.6 with $f(\xi) = -c\xi^{2n-2m-3}(a\xi^{2n} + b)^{-1}$.

51. $\boldsymbol{(y''_{xx})^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma.}$

Differentiating the equation with respect to x, we obtain

$$y''_{xx}(2y'''_{xxx} - \alpha x - \beta) = 0. \tag{1}$$

Equating the second factor to zero and integrating, we find

$$y = \frac{\alpha x^4}{48} + \frac{\beta x^3}{12} + C_2x^2 + C_1x + C_0. \tag{2}$$

The integration constants C_i and the parameters α, β, and γ are related by the constraint $4C_2^2 = \beta C_1 - \alpha C_0 + \gamma$ which is obtained by substituting the above solution (2) into the original equation.

In addition, there is the solution that corresponds to the first factor in (1):

$$y = \widetilde{C}_1 x + \widetilde{C}_0, \qquad \text{where} \quad \beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0.$$

2.8.2. Painlevé Equations

Preliminary comments.

Painlevé equations are met with in different fields of contemporary physics. They resulted from solving the problem of extracting equation classes of the form

$$\frac{d^2y}{dx^2} = R\Big(x,\ y,\ \frac{dy}{dx}\Big),$$

where R is a rational function of y and dy/dx with analytic coefficients whose integrals have no movable critical points. This problem was solved in works by P. Painlevé and B. Gambier. A total of 50 equations were extracted, of which 44 equations have the general solutions that are expressed in terms of elementary functions, solutions of some linear equations, or solutions of the other six equations.

The six equations not integrated by Painlevé and Gambier are referred to as Painlevé transcendents (or irreducible Painlevé equations), and their solutions are called transcendental Painlevé functions. The absence of movable critical points makes it possible to use solutions of Painlevé transcendents as basic functions for representing solutions of other nonlinear differential equations, along with quadratures and solutions of variable-coefficient linear differential equations. The canonical forms of Painlevé transcendents are specified below.

1. First Painlevé transcendent:

$$\boldsymbol{y''_{xx} = 6y^2 + x.} \tag{1}$$

The solution of the first Painlevé equation is a unique function of x. It can be presented, in the vicinity of movable pole x_0, in terms of the series

$$y = (x-x_0)^{-2} + \frac{x_0}{10}(x-x_0)^2 - \frac{1}{6}(x-x_0)^3 + C(x-x_0)^4 + \frac{x_0^2}{300}(x-x_0)^6 + \sum_{j=7}^{\infty} a_j(x-x_0)^j,$$

where x_0 and C are arbitrary constants; coefficients a_j $(j \geq 7)$ are uniquely defined in terms of x_0 and C.

For large values of $|z|$, the following asymptotic formulae holds:

$$y(x) \sim x^{1/2}\wp\big(\tfrac{4}{5}x^{5/4} - a,\ 12,\ b\big),$$

where the elliptic Weierstrass function $\wp(\zeta; 12, b)$ is defined implicitly by the integral

$$\zeta = \int \frac{d\wp}{\sqrt{4\wp^3 - 12\wp - b}};$$

a and b are some constants.

The first Painlevé transcendent (1) is invariant with respect to stretching variables $x = \lambda\bar{x}$, $y = \lambda^3\bar{y}$, where $\lambda^5 = 1$, i.e., it admits discrete symmetry of the fifth order.

2. Second Painlevé transcendent:

$$\boldsymbol{y''_{xx} = 2y^3 + xy + \alpha.} \tag{2}$$

The solution of the second Painlevé equation is a unique function of x. Denote the solution by $y(x,\alpha)$ with fixed parameter α. Then, the following relation holds:

$$y(x,-\alpha) = -y(x,\alpha),$$

while solutions $y(x,\alpha)$ and $y(x,\alpha-1)$ are related by the Bäcklund transformations

$$y(x,\alpha-1) = -y(x,\alpha) + \frac{2\alpha-1}{2y'_x(x,\alpha) - 2y^2(x,\alpha) - x},$$
$$y(x,\alpha) = -y(x,\alpha-1) - \frac{2\alpha-1}{2y'_x(x,\alpha-1) + 2y^2(x,\alpha-1) + x}.$$

Therefore, in order to study the general solution of equation (2) with arbitrary α it is sufficient to construct the solution for all α out of the band $0 \le \operatorname{Re}\alpha < \frac{1}{2}$.

Three solution corresponding to α and $\alpha \pm 1$ are related by the rational formulae

$$y_{\alpha+1} = -\frac{(y_{\alpha-1}+y_\alpha)(4y_\alpha^3 + 2xy_\alpha + 2\alpha + 1) + (2\alpha-1)y_\alpha}{2(y_{\alpha-1}+y_\alpha)(2y_\alpha^2 + x) + 2\alpha - 1},$$

where y_α stands for $y(x,\alpha)$.

Solutions $y(x,\alpha)$ and $y(x,-\alpha-1)$ are related by the Bäcklunds transformations

$$y(x,-\alpha-1) = y(x,\alpha) + \frac{2\alpha+1}{2y'_x(x,\alpha) + 2y^2(x,\alpha) + x},$$
$$y(x,\alpha) = y(x,-\alpha-1) - \frac{2\alpha+1}{2y'_x(x,-\alpha-1) + 2y^2(x,-\alpha-1) + x}.$$

The second Painlevé transcendent (2) for all $\alpha = n + \frac{1}{2}$ (n is an integer) possesses the one-parameter family of solutions which are generated by the general solution of the Fuchsian equation

$$(y'_x)^n + \sum_{j=1}^{n} P_j(x,y)(y'_x)^{n-j} = 0$$

and are expressed in terms of Airy functions and their derivatives.

There exist rational solutions of the form $y = R_1/R_2$, where R_1 and R_2 are polynomials in x, only when $a = 0,\ \pm1,\ \pm2,\ \pm3,\ \dots$ A rational solution can be written as

$$y_a = \frac{1}{P_a}\frac{dP_a}{dx} - \frac{1}{Q_a}\frac{dQ_a}{dx},$$

where P_a and Q_a are polynomials, a is an integer.

In the special case $a = 1$, we have

$$P_1 = 1, \quad Q_1 = x \qquad \text{hence} \qquad y_1 = -\frac{1}{x}.$$

In the special case $a = 2$, we have

$$P_2 = x, \quad Q_2 = x^3 + 4 \qquad \text{hence} \qquad y_2 = \frac{1}{x} - \frac{3x^2}{x^3+4}.$$

The following recurrence formulae take place:

$$P_{a+1} = Q_a, \quad Q_{a+1} = [xQ_a^2 + (Q'_a)^2 - 4Q_aQ''_a]/P_a,$$

where prime denotes differentiation with respect to x.

If we stretch and shift the variables in accordance with the rule $x = \varepsilon^2 \bar{x} - 6\varepsilon^{-10}$, $y = \varepsilon \bar{y} + \varepsilon^{-5}$ and let $\alpha = 4\varepsilon^{-15}$, equation (2), in the limit $\varepsilon \to 0$, transforms to equation (1).

3. Third Painlevé transcendent:

$$\boldsymbol{y''_{xx} = \frac{(y'_x)^2}{y} - \frac{y'_x}{y} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.} \tag{3}$$

The solution of the third Painlevé equation is a unique function of x.

Any solution of the Riccati equation

$$y'_x = ky^2 + \frac{\alpha - k}{kx} y + c, \tag{4}$$

where $k^2 = \gamma$, $c^2 = -\delta$, $k\beta + c(\alpha - 2k) = 0$, is a solution of equation (3). Substituting $x = \lambda\tau$, $y = -\dfrac{u'_x}{ku}$, where $\lambda^2 = \dfrac{1}{kc}$, into (4), we obtain a linear equation:

$$u''_{\tau\tau} + \frac{k - \alpha}{k\tau} u'_\tau + u = 0,$$

whose general solution is expressed in terms of Bessel functions:

$$u = \tau^{\frac{\alpha}{2k}} \left[C_1 J_{\frac{\alpha}{2k}}(\tau) + C_2 Y_{\frac{\alpha}{2k}}(\tau) \right].$$

In some special cases, equation (3) can be integrated by quadrature. Rewrite equation (3) in the form of integro-differential relations by two ways

$$\left(\frac{y'_z}{y}\right)^2 + \left(\frac{\delta}{y^2} - \gamma y^2\right) e^{2z} + 2\left(\frac{\beta}{y} - \alpha y\right) e^z = 2 \int \left[\left(\frac{\delta}{y^2} - \gamma y^2\right) e^{2z} + \left(\frac{\beta}{y} - \alpha y\right) e^z \right] dz \tag{5}$$

and

$$\frac{y'_z}{y} = \int \left[\left(\frac{\delta}{y^2} + \gamma y^2\right) e^{2z} + \left(\frac{\beta}{y} + \alpha y\right) e^z \right] dz, \qquad x = e^z. \tag{6}$$

It is obvious from (5) that for $\alpha = \beta = \gamma = \delta = 0$, the general solution has the form $y = C_1 x^{C_2}$. Adding (6) multiplied by 2 to (5), we obtain

$$\left(\frac{y'_z}{y}\right)^2 + 2\frac{y'_z}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right) e^{2z} + 2\left(\frac{\beta}{y} - \alpha y\right) e^z = 4 \int \frac{\delta e^{2z} + \beta e^z y}{y^2}\, dz, \tag{7}$$

Subtracting (6) times 2 from (5), we find

$$\left(\frac{y'_z}{y}\right)^2 - 2\frac{y'_z}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right) e^{2z} + 2\left(\frac{\beta}{y} - \alpha y\right) e^z = -4 \int (\gamma e^{2z} y^2 + \alpha e^z y)\, dz. \tag{8}$$

Substituting $\delta = \beta = 0$ into equation (7) and $\gamma = \alpha = 0$ into equation (8), we arrive at

$$\left(\frac{y'_z}{y}\right)^2 + 2\frac{y'_z}{y} - 2\alpha y e^z - \gamma y^2 e^{2z} = C_1, \tag{9}$$

$$\left(\frac{y'_z}{y}\right)^2 - 2\frac{y'_z}{y} + \frac{\delta}{y^2} e^{2z} + \frac{2\beta}{y} e^z = C_2. \tag{10}$$

Equations (9) and (10) are integrable by elementary functions. Substituting $y = e^{-z}/v$ into (9), we obtain

$$(v'_z)^2 = 2\alpha v + \gamma + (1 + C_1)v^2. \tag{11}$$

As a result find

$$y = \begin{cases} \dfrac{2\alpha}{z(\alpha^2 \ln^2 x + 2\alpha C \ln x + C^2 - \gamma)} & \text{if } C_1 + 1 = 0, \ \beta = \delta = 0; \\ \dfrac{1}{x\bigl(\sqrt{\gamma} \ln x + C\bigr)} & \text{if } C_1 + 1 = 0, \ \alpha = \beta = \delta = 0; \\ \dfrac{x^{m-1}}{Cx^{2m} + K_1 x^m + K_2} & \text{if } C_1 + 1 \neq 0, \ \beta = \delta = 0, \end{cases}$$

where $C \neq 0$, $K_1 = -\dfrac{\alpha}{C_1 + 1}$, $K_2 = \dfrac{\alpha^2 - \gamma(1 + C_1)}{4C(1 + C_1)^2}$, $m^2 = 1 + C_1$. Accordingly, equation (10) is reduced to equation (11) with the substitution $y = ve^z$.

If $\beta = -\alpha$ and $\delta = -\gamma$, the transformation $y = e^{-iw}$ brings equation (3) to the equivalent form

$$w''_{xx} + \frac{1}{x} w'_x = \frac{2\alpha}{x} \sin w + 2\gamma \sin 2w.$$

If we perform the transformation $x = 1 + \varepsilon^2 \bar{x}$, $y = 1 + 2\varepsilon \bar{y}$, $\alpha = -\frac{1}{2}\varepsilon^{-6}$, $\beta = \frac{1}{2}\varepsilon^{-6} + 2\bar{\beta}\varepsilon^{-3}$, $\gamma = \frac{1}{4}\varepsilon^{-6}$, $\delta = -\frac{1}{4}\varepsilon^{-6}$, then equation (3), in the limit $\varepsilon \to 0$, transforms to equation (2).

4. Fourth Painlevé transcendent:

$$\boldsymbol{y''_{xx} = \frac{(y'_x)^2}{2y} + \frac{3}{2} y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}.} \tag{12}$$

If we pass on to the new independent variable $x = e^z$, the solutions are unique functions of z.

The Laurent-series expansion of the solution of equation (12) in the vicinity of any pole $x = x_0$ has the form

$$y(x) = \frac{m}{x - x_0} - x_0 - \frac{m}{3}(x_0^2 + 2\alpha - 4m)(x - x_0) + C(x - x_0)^2 + \sum_{j=3}^{\infty} a_j (x - x_0)^j,$$

where $m = \pm 1$, C is an arbitrary number, and a_j ($j \geq 3$) are uniquely defined in terms of α, β, x_0, and C.

If the condition $\beta + 2(1 + \alpha m)^2 = 0$, where $m = \pm 1$, is satisfied, then every solution of the Riccati equation

$$y'_x = my^2 + 2mxy - 2(1 + \alpha m)$$

is simultaneously a solution of the fourth Painlevé transcendent (12).

Equation (12) is invariant with respect to the transformation $y = \lambda \bar{y}$, $x = \lambda \bar{x}$, $\alpha = \bar{\alpha}\lambda^2$, $\beta = \bar{\beta}$, where $\lambda^4 = 1$. Two solutions of equation (12) corresponding to different values of parameters α and β are related to each other by the Bäcklund transformations

$$\begin{aligned} \tilde{y} &= \frac{1}{2sy}(y'_x - q - 2sxy - sy^2), \qquad & q^2 &= -2\beta, \\ y &= -\frac{1}{2s\tilde{y}}(\tilde{y}'_x - p + 2sx\tilde{y} + s\tilde{y}^2), \qquad & p^2 &= -2\tilde{\beta}, \\ 2\beta &= -\bigl(\tilde{\alpha}s - 1 - \tfrac{1}{2}p\bigr)^2, \qquad & 4\alpha &= -2s - 2\tilde{\alpha} - 3sp, \end{aligned}$$

where $y=y(x,\alpha,\beta)$, $\tilde{y}=\tilde{y}(x,\tilde{\alpha},\tilde{\beta})$, s is an arbitrary parameter.

If we perform the transformation $x=2^{-2/3}\varepsilon\bar{x}-\varepsilon^{-3}$, $y=2^{2/3}\varepsilon\bar{y}+\varepsilon^{-3}$, $\alpha=-\bar{\alpha}-\frac{1}{2}\varepsilon^{-6}$, $\beta=-\frac{1}{2}\varepsilon^{-12}$, equation (12), in the limit $\varepsilon\to 0$, transforms to equation (3).

5. Fifth Painlevé transcendent:

$$\boldsymbol{y''_{xx}=\frac{3y-1}{2y(y-1)}(y'_x)^2-\frac{y'_x}{x}+\frac{(y-1)^2}{x^2}\Big(\alpha y+\frac{\beta}{y}\Big)+\gamma\frac{y}{x}+\frac{\delta y(y+1)}{y-1}.} \tag{13}$$

If we pass on to the new independent variable $x=e^z$, the solutions are unique functions of z.

Solutions of the fifth Painlevé transcendent (13) corresponding to different values of the parameters are related by two equalities

$$y(x,\alpha,\beta,\gamma,\delta)=y(-x,\alpha,\beta,-\gamma,\delta),$$
$$y(x,\alpha,\beta,\gamma,\delta)=\frac{1}{y(x,-\beta,-\alpha,-\gamma,\delta)}.$$

Having set $x=e^t$ in (13), we obtain

$$y''_{tt}=\frac{3y-1}{2y(y-1)}(y'_t)^2+(y-1)^2\Big(\alpha y+\frac{\beta}{y}\Big)+\gamma y e^t+\frac{\delta y(y+1)}{y-1}e^{2t}. \tag{14}$$

If $\gamma=\delta=0$, equation (14) is reduced, by means of integration, to a first order autonomous equation:

$$y'_t=(y-1)\sqrt{2\alpha y^2+Cy-2\beta}$$

which is readily integrable by quadrature.

If the condition

$$\gamma=\sqrt{-2\delta}\,\big(1+\sqrt{-2\beta}-\sqrt{2\alpha}\,\big)$$

is satisfied, any solution of the Riccati equation

$$xy'_x=\sqrt{2\alpha}\,y^2+\big(\sqrt{-2\delta}\,x-\sqrt{2\alpha}-\sqrt{-2\beta}\,\big)y+\sqrt{-2\beta} \tag{15}$$

is simultaneously a solution of the fifth Painlevé transcendent (13). equation (15) can be reduced to the degenerate hypergeometric equation 2.1.2.65.

If we perform the transformation $y=1+\varepsilon\bar{y}$, $\beta=-\varepsilon^{-2}\bar{\beta}$, $\alpha=\varepsilon^{-2}\bar{\beta}+\varepsilon^{-1}\bar{\alpha}$, $\gamma=\varepsilon\bar{\gamma}$, $\delta=\varepsilon\bar{\delta}$, equation (13), in the limit $\varepsilon\to 0$, transforms to equation (3). In a similar manner, as a result of the transformation $y=\sqrt{2}\,\varepsilon\bar{y}$, $x=1+\sqrt{2}\,\varepsilon\bar{x}$, $\alpha=\frac{1}{2}\varepsilon^{-4}$, $\gamma=-\varepsilon^{-4}$, $\delta=-\frac{1}{2}\varepsilon^{-4}-\varepsilon^{-2}\bar{\delta}$, equation (13), in the limit $\varepsilon\to 0$, transforms to equation (12).

6. Sixth Painlevé transcendent:

$$\boldsymbol{\begin{aligned} y''_{xx}=&\frac{1}{2}\Big(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\Big)(y'_x)^2-\Big(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\Big)y'_x\\ &+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\Big[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\Big].\end{aligned}} \tag{16}$$

In equation (16), the points $x=0$, $x=1$, and $x=\infty$ are critical. Painlevé found two integrable case of the equation. First, if $\alpha=\beta=\gamma=\delta=0$, the general solution of equation (16) has the form

$$y=E(C_1\omega_1+C_2\omega_2,\ x),$$

where $E(u,x)$ is the elliptic function, defined by the integral

$$u=\int_0^E \frac{dy}{\sqrt{y(y-1)(y-x)}}, \tag{17}$$

with periods $2\omega_1$ and $2\omega_1$ which are functions of x. Second, if $\alpha=\beta=\gamma=0$, $\delta=\frac{1}{2}$, the general solution of equation (16) has the form

$$y=E(w+C_1\omega_1+C_2\omega_2,\ x),$$

where $w\not\equiv 0$ is any particular solution of the linear equation

$$w''_{xx}-\frac{2x-1}{x(x-1)}w'_x+\frac{1}{4x(x-1)}w=0,$$

E is the elliptic function defined by formula (17).

Solutions of the sixth Painlevé transcendent (16) corresponding to different values of the parameters are related by three equalities

$$y(x,\ -\beta,\ -\alpha,\ \gamma,\ \delta)=\frac{1}{y\Big(\dfrac{1}{x},\ \alpha,\ \beta,\ \gamma,\ \delta\Big)},$$

$$y(x,\ -\beta,\ -\gamma,\ \alpha,\ \delta)=1-\frac{1}{y\Big(\dfrac{1}{1-x},\ \alpha,\ \beta,\ \gamma,\ \delta\Big)},$$

$$y\Big(x,\ -\beta,\ -\alpha,\ -\delta+\frac{1}{2},\ -\gamma+\frac{1}{2}\Big)=\frac{x}{y(x,\ \alpha,\ \beta,\ \gamma,\ \delta)}.$$

The consecutive use of these equalities yields 24 equations of the form (16) with different values of the parameters related by the known transformations.

All the solutions of the Riccati equation

$$y'_x=\frac{\sqrt{2\alpha}}{x(x-1)}y^2+\frac{\lambda x+\mu}{x(x-1)}y+\frac{\sqrt{-2\beta}}{x-1} \tag{18}$$

are simultaneously solutions of equation (16) if $\sqrt{2\alpha}-\sqrt{-2\beta}\neq 1$ and the condition

$$2\sqrt{2\alpha}\,(3\beta-\alpha+\gamma-\delta)+2\sqrt{-2\beta}\,(3\alpha-\beta-\gamma+\delta)+4\sqrt{-\alpha\beta}\,(\beta-\alpha+\gamma-\delta-1)$$
$$+(\alpha+\beta+\gamma+\delta)^2+2(\alpha-\beta-\gamma-4\alpha\beta-2\alpha\gamma-2\beta\delta)=0$$

is satisfied (one should take such a value of $\sqrt{-\alpha\beta}$ which coincides with $\sqrt{\alpha}\sqrt{-\beta}$). In equation (18),

$$\lambda=\frac{\sqrt{2\alpha}-(\alpha+\beta+\gamma+\delta)}{\sqrt{2\alpha}-\sqrt{-2\beta}-1},\qquad \mu=\frac{\sqrt{-2\beta}-(\alpha+\beta-\gamma-\delta)}{\sqrt{2\alpha}-\sqrt{-2\beta}-1}.$$

If we perform the transformation $z=1+\varepsilon\bar{x}$, $\delta=\varepsilon^{-2}\bar{\delta}$, $\gamma=\varepsilon^{-1}\bar{\gamma}-\varepsilon^{-2}\bar{\delta}$, equation (16), in the limit $\varepsilon\to 0$, transforms to equation (13).

2.8.3. Equation Containing Exponential Functions

1. $y''_{xx} = \lambda^2 y + a\exp[\lambda(n+3)x]y^n.$

This is a special case of equation 2.9.1.15 with $f(\xi) = a\xi^n$.

2. $y''_{xx} = \lambda^2 y + ae^{\mu x}y^m, \qquad \lambda \neq 0.$

The transformation $\xi = e^{2\lambda x}$, $w = ye^{\lambda x}$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = \frac{a}{4\lambda^2}\xi^n w^m, \qquad \text{where} \quad n = \frac{\mu - 3\lambda - m\lambda}{2\lambda}.$$

3. $y''_{xx} = \lambda^2 y + ae^{\lambda(m+3)x}(be^{2\lambda x} + c)^n y^m, \qquad \lambda \neq 0.$

The transformation $\xi = be^{2\lambda x} + c$, $w = ye^{\lambda x}$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = \frac{a}{4b^2\lambda^2}\xi^n w^m.$$

4. $y''_{xx} = \lambda^2 y + Ae^{\lambda(m+3)x}(ae^{2\lambda x} + b)^n (ce^{2\lambda x} + d)^{-n-m-3} y^n.$

The transformation

$$\xi = \frac{ae^{2\lambda x} + b}{ce^{2\lambda x} + d}, \quad w = \frac{ye^{\lambda x}}{ce^{2\lambda x} + d}$$

leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = A(2\Delta\lambda)^{-2}\xi^n w^m, \qquad \text{where} \quad \Delta = ad - bc.$$

5. $y''_{xx} = ay'_x + be^{2ax}y^n.$

This is a special case of equation 2.9.2.17 with $f(y) = by^n$.

6. $y''_{xx} = -ay'_x + be^{anx}y^{n-1}.$

This is a special case of equation 2.9.2.18 with $f(\xi) = b\xi^{n-1}$.

7. $y''_{xx} + ay'_x + by = ce^{\lambda x}y^m.$

The substitution $\xi = e^x$ leads to an equation of the form 2.8.1.36:

$$\xi^2 y''_{\xi\xi} + (a+1)\xi y'_\xi + by = c\xi^\lambda y^m.$$

8. $y''_{xx} = -(\mu + \nu)y'_x - \nu\mu y + ae^{(n\mu - 2\nu)x}y^{n-1}.$

This is a special case of equation 2.9.2.19 with $f(\xi) = a\xi^{n-1}$.

9. $y''_{xx} = \lambda y'_x + bxy + ae^{2\lambda x}y^{-3}.$

This is a special case of equation 2.9.2.20 with $f(x) = -bx$.

10. $y''_{xx} = \lambda y'_x + be^{\mu x}y + ae^{2\lambda x}y^{-3}.$

This is a special case of equation 2.9.2.20 with $f(x) = -be^{\mu x}$.

11. $\boldsymbol{y''_{xx} = ay'_x + b\exp(2ax + cy^n).}$

This is a special case of equation 2.9.2.17 with $f(y) = b\exp(cy^n)$.

12. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + ae^{\lambda x}y = 0.}$

This is a special case of equation 2.9.2.1 with $f(x) = ae^{\lambda x}$.

13. $\boldsymbol{y''_{xx} = axe^y y'_x + ae^y.}$

Solution: $y = C_1 x - \ln(-a\int xe^{C_1 x}dx + C_2)$.

14. $\boldsymbol{y''_{xx} = 2ae^x yy'_x + ae^x y^2.}$

Solution in the parametric form:

$$x = \ln\left(\frac{\tau^2}{2C_1}\right), \qquad y = -a^{-1}C_1\tau^{-2}Z^{-1}(\tau Z'_\tau + Z),$$

where $Z = C_1 J_1(\tau) + C_2 Y_1(\tau)$ or $Z = C_1 I_1(\tau) + C_2 K_1(\tau)$, J_1 and Y_1 are Bessel functions, I_1 and K_1 are modified Bessel functions.

15. $\boldsymbol{y''_{xx} = ax^n e^y y'_x + anx^{n-1}e^y.}$

Solution: $\quad y = C_1 x - \ln\left[C_2 - a\displaystyle\int x^n \exp(C_1 x)\,dx\right].$

16. $\boldsymbol{y''_{xx} = ae^x y^{-1/2} y'_x + 2ae^x y^{1/2}.}$

Solution in the parametric form:

$$x = \ln\left(\pm\frac{C_1}{a}f\right) \mp \tau^2, \qquad y = C_1^2\bigl[2\tau \pm \exp(\mp\tau^2)f\bigr]^2,$$

where $f = \left[\displaystyle\int \exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}$.

17. $\boldsymbol{y''_{xx} + (2ay + be^{\lambda x})y'_x + \lambda be^{\lambda x}y = 0.}$

Integrating the equation, we obtain the Riccati equation:

$$y'_x + ay^2 + be^{\lambda x}y = C.$$

18. $\boldsymbol{y''_{xx} = ae^{x+y}(y'_x + 1).}$

Solution:

$$y = -\ln\left(C_1 e^{-C_2 x} - \frac{a}{1 + C_2}e^x\right).$$

To the limiting case $C_2 \to -1$ corresponds $y = -x - \ln(C_1 - ax)$.

19. $\boldsymbol{y''_{xx} = a(y'_x)^2 - be^{4ay} + cx^n.}$

This is a special case of equation 2.9.3.16 with $f(x) = -cx^n$.

20. $y''_{xx} = a(y'_x)^2 - be^{4ay} + ce^{\lambda x}$.

This is a special case of equation 2.9.3.16 with $f(x) = -ce^{\lambda x}$.

21. $y''_{xx} = a(y'_x)^2 + bx^n e^{ay} + cx^m$.

This is a special case of equation 2.9.3.15 with $f(x) = -bx^n$, $g(x) = -cx^m$.

22. $y''_{xx} = a(y'_x)^2 + be^{ay+cx} + kx^m$.

This is a special case of equation 2.9.3.15 with $f(x) = -be^{cx}$, $g(x) = -kx^m$.

23. $y''_{xx} = a(y'_x)^2 + be^{ay+\lambda x} + ce^{\mu x}$.

This is a special case of equation 2.9.3.15 with $f(x) = -be^{\lambda x}$, $g(x) = -ce^{\mu x}$.

24. $y''_{xx} + ay^n(y'_x)^2 + be^{\lambda y} + c = 0$.

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = be^{\lambda y} + c$.

25. $y''_{xx} = ay^n(y'_x)^2 + be^{\lambda x}y'_x$.

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = be^{\lambda x}$.

26. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + by^n + c = 0$.

This is a special case of equation 2.9.3.1 with $f(y) = ae^{\lambda y}$, $g(y) = by^n + c$.

27. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + be^{\mu y} + c = 0$.

This is a special case of equation 2.9.3.1 with $f(y) = ae^{\lambda y}$, $g(y) = be^{\mu y} + c$.

28. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + bx^n y'_x$.

This is a special case of equation 2.9.3.2 with $f(y) = ae^{\lambda y}$, $g(x) = bx^n$.

29. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + be^{\mu x}y'_x$.

This is a special case of equation 2.9.3.2 with $f(y) = ae^{\lambda y}$, $g(x) = be^{\mu x}$.

30. $y''_{xx} = ae^{\lambda x}(xy'_x - y)^2 + be^{\mu x}$.

This is a special case of equation 2.9.3.4 with $f(x) = be^{\mu x}$, $g(x) = 0$, $h(x) = ae^{\lambda x}$.

31. $y''_{xx} + be^{ax}y^m(y'_x)^3 + ay'_x = 0$.

This is a special case of equation 2.9.3.24 with $f(y) = by^m$.

32. $y''_{xx} + be^{ax+\lambda y}(y'_x)^3 + ay'_x = 0$.

This is a special case of equation 2.9.3.24 with $f(y) = be^{\lambda y}$.

33. $y''_{xx} = ae^x(y'_x)^3 + ae^x y(y'_x)^2$.

Solution: $x = C_1 y - \ln\left(a \int ye^{C_1 y}\,dy + C_2\right)$.

34. $y''_{xx} = axe^y (y'_x)^3 + ae^y (y'_x)^2.$

Solution in the parametric form:

$$x = C_1 e^{-a\tau} \biggl(\int \tau^{-1} e^{a\tau}\, d\tau + C_2 \biggr), \qquad y = \ln \tau.$$

35. $y''_{xx} = ax^2 e^y (y'_x)^3 + 2axe^y (y'_x)^2.$

Solution in the parametric form:

$$x = a^{-1} C_1 \tau^{-2} Z^{-1} (\tau Z'_\tau + Z), \qquad y = \ln\Bigl(\frac{\tau^2}{2C_1}\Bigr),$$

where $Z = C_1 J_1(\tau) + C_2 Y_1(\tau)$ or $Z = C_1 I_1(\tau) + C_2 K_1(\tau)$, J_1 and Y_1 are Bessel functions, I_1 and K_1 are modified Bessel functions.

36. $y''_{xx} = 2ax^{1/2} e^y (y'_x)^3 + ax^{-1/2} e^y (y'_x)^2.$

Solution in the parametric form:

$$x = C_1^2 \bigl[2\tau \pm \exp(\mp\tau^2) f\bigr]^2, \qquad y = \ln\Bigl(\mp \frac{C_1}{a} f\Bigr) \mp \tau^2,$$

where $f = \biggl[\int \exp(\mp\tau^2)\, d\tau + C_2 \biggr]^{-1}$.

37. $y''_{xx} = ane^x y^{n-1} (y'_x)^3 + ae^x y^n (y'_x)^2.$

Solution: $x = C_1 y - \ln\Bigl(a \int y^n e^{C_1 y}\, dy + C_2 \Bigr)$.

38. $y''_{xx} = ae^{x+y} [(y'_x)^3 + (y'_x)^2].$

Solution:

$$x = -\ln\Bigl(C_1 e^{C_2 y} + \frac{a}{1 - C_2} e^y \Bigr).$$

To the limiting case $C_2 \to 1$ corresponds $x = -y - \ln(C_1 + ay)$.

39. $y''_{xx} = ae^x (y'_x)^{3/2} + ae^x y (y'_x)^{1/2}.$

Solution in the parametric form:

$$x = \ln \tau^2, \qquad y = -2a^{-2} \tau^{-4} \bigl[Z^{-1} (\tau Z'_\tau + 2Z) \mp \tfrac{1}{2} \tau^2 \bigr],$$

where

$$Z = \begin{cases} C_1 J_2(\tau) + C_2 Y_2(\tau) & \text{for the upper sign,} \\ C_1 I_2(\tau) + C_2 K_2(\tau) & \text{for the lower sign,} \end{cases}$$

J_2 and Y_2 are Bessel functions, I_2 and K_2 are modified Bessel functions.

40. $y''_{xx} = axe^y (y'_x)^{5/2} + ae^y (y'_x)^{3/2}.$

Solution in the parametric form:

$$x = -2a^{-2}\tau^{-4}\bigl[Z^{-1}(\tau Z'_\tau + 2Z) \mp \tfrac{1}{2}\tau^2\bigr], \qquad y = \ln \tau^2,$$

where

$$Z = \begin{cases} C_1 J_2(\tau) + C_2 Y_2(\tau) & \text{for the upper sign,} \\ C_1 I_2(\tau) + C_2 K_2(\tau) & \text{for the lower sign,} \end{cases}$$

J_2 and Y_2 are Bessel functions, I_2 and K_2 are modified Bessel functions.

41. $y''_{xx} = -ay'_x + be^{amx} y^k (y'_x)^{m+2}.$

This is a special case of equation 2.9.4.49 with $f(y) = -by^k$, $n = m + 2$.

42. $y''_{xx} = -\dfrac{a}{m}\dfrac{2-k}{1-k} y'_x + be^{ax} y^{m-k+1} (y'_x)^k.$

This is a special case of equation 2.9.4.52 with $f(\xi) = b\xi$.

43. $y''_{xx} = y'_x + A \exp[(n + 2 - l)x] y^m (y'_x)^l.$

The substitution $\xi = e^x$ leads to the generalized Emden—Fowler equation

$$y''_{\xi\xi} = A\xi^n y^m (y'_\xi)^l$$

which is outlined in Section 2.5.

44. $y''_{xx} = -(y'_x)^2 + Ax^n \exp[(m + l - 1)y]\,(y'_x)^l.$

The substitution $w = e^y$ leads to the generalized Emden—Fowler equation

$$w''_{xx} = Ax^n w^m (w'_x)^l$$

which is outlined in Section 2.5.

45. $y''_{xx} = \dfrac{a}{n}\dfrac{1-k}{2-k} (y'_x)^2 + bx^{n+k-2} e^{ay} (y'_x)^k.$

This is a special case of equation 2.9.4.51 with $f(\xi) = b\xi$.

46. $y''_{xx} = ay^n (y'_x)^2 + be^{\lambda y} (y'_x)^k.$

This is a special case of equation 2.9.4.10 with $f(y) = ay^n$, $g(y) = be^{\lambda y}$.

47. $y''_{xx} = ae^{\lambda y} (y'_x)^2 + by^n (y'_x)^k.$

This is a special case of equation 2.9.4.10 with $f(y) = ae^{\lambda y}$, $g(y) = by^n$.

48. $y''_{xx} = ae^{\lambda y} (y'_x)^2 + be^{\mu y} (y'_x)^k.$

This is a special case of equation 2.9.4.10 with $f(y) = ae^{\lambda y}$, $g(y) = be^{\mu y}$.

49. $y''_{xx} = ax^n(xy'_x - y) + be^{\lambda x}(xy'_x - y)^k$.

This is a special case of equation 2.9.4.39 with $f(x) = ax^n$, $g(x) = be^{\lambda x}$.

50. $y''_{xx} = ae^{\lambda x}(xy'_x - y) + bx^n(xy'_x - y)^k$.

This is a special case of equation 2.9.4.39 with $f(x) = ae^{\lambda x}$, $g(x) = bx^n$.

51. $y''_{xx} = ae^{x+y}\left[(y'_x)^l + (y'_x)^{l-1}\right], \quad l \neq 2$.

Solution in the parametric form:

$$x = \tau - \int \frac{d\tau}{F} - C_2, \qquad y = \int \frac{d\tau}{F} + C_2,$$

where $F = \left[a(2-l)e^{\tau} + C_1\right]^{\frac{1}{l-2}} + 1$.

52. $y''_{xx} = ae^{\lambda x}(xy'_x - y) + be^{\mu x}(xy'_x - y)^k$.

This is a special case of equation 2.9.4.39 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu y}$.

53. $xy''_{xx} + y'_x = ax^n e^{\lambda y}$.

This equation is encountered in the combustion theory and hydrodynamics.

The transformation $\xi = \ln x$, $w = \lambda y + (n+1)\ln x$ leads to an autonomous equation of the form 2.9.1.1: $w''_{\xi\xi} = a\lambda e^w$. Having integrated the latter equation, we obtain the solution of the original equation in the parametric form:

$$x = \exp[C_1 \pm f(t)], \qquad y = \frac{t}{\lambda} - \frac{n+1}{\lambda}[C_1 \pm f(t)],$$

where

$$f(t) = \begin{cases} \dfrac{1}{\sqrt{C_2}} \ln \dfrac{\sqrt{C_2 + 2a\lambda e^t} - \sqrt{C_2}}{\sqrt{C_2 + 2a\lambda e^t} + \sqrt{C_2}} & \text{if } C_2 > 0, \\ -\dfrac{2}{\sqrt{2a\lambda e^t}} & \text{if } C_2 = 0, \\ \dfrac{2}{\sqrt{-C_2}} \arctan \dfrac{\sqrt{C_2 + 2a\lambda e^t}}{\sqrt{-C_2}} & \text{if } C_2 < 0. \end{cases}$$

54. $xy''_{xx} = ny'_x + ax^{2n+1}e^{\lambda y}$.

This is a special case of equation 2.9.2.4 with $f(y) = ae^{\lambda y}$.

55. $xy''_{xx} = ny'_x + ax^{2n+1}\exp(\lambda y^m)$.

This is a special case of equation 2.9.2.4 with $f(y) = a\exp(\lambda y^m)$.

56. $xy''_{xx} + y'_x = ax^n e^{\lambda y}(y'_x)^m$.

The transformation $\zeta = xy'_x$, $w = x^{n-m+1}e^{\lambda y}$ after dividing by w leads to a first order linear equation:

$$a\zeta^m w'_\zeta = \lambda\zeta + n - m + 1.$$

57. $xy''_{xx} + my'_x + ax^{nm-2m+1}e^{\lambda y}(y'_x)^n = 0.$

This is a special case of equation 2.9.4.11 with $f(y) = ae^{\lambda y}$.

58. $xy''_{xx} + y'_x = (ax^n e^{\lambda y} + bx^{m-1})(y'_x)^m.$

The transformation $\zeta = xy'_x$, $w = x^{n-m+1}e^{\lambda y}$ leads to a first order equation with separation of variables:

$$\zeta^m (aw + b) w'_\zeta = (\lambda\zeta + n - m + 1)w.$$

59. $2yy''_{xx} = (y'_x)^2 + be^{\lambda x}y^2 - a.$

This is a special case of equation 2.9.3.7 with $f(x) = -be^{\lambda x}$.

60. $yy''_{xx} = n(y'_x)^2 - ay^{4n-2} + be^{\lambda x}y^2.$

This is a special case of equation 2.9.3.8 with $f(x) = -be^{\lambda x}$.

61. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + be^{\lambda x}y^{n+1}.$

This is a special case of equation 2.9.3.9 with $f(x) = -ax^m$, $g(x) = -be^{\lambda x}$.

62. $yy''_{xx} = n(y'_x)^2 + ae^{\lambda x}y^2 + bx^m y^{n+1}.$

This is a special case of equation 2.9.3.9 with $f(x) = -ae^{\lambda x}$, $g(x) = -bx^m$.

63. $yy''_{xx} = n(y'_x)^2 + ae^{\lambda x}y^2 + be^{\mu x}y^{n+1}.$

This is a special case of equation 2.9.3.9 with $f(x) = -ae^{\lambda x}$, $g(x) = -be^{\mu x}$.

64. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + be^{\lambda x}y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -ax^n$, $g(x) = -be^{\lambda x}$.

65. $yy''_{xx} = (y'_x)^2 + ae^{\lambda x}yy'_x + bx^n y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -ae^{\lambda x}$, $g(x) = -bx^n$.

66. $yy''_{xx} = (y'_x)^2 + ae^{\lambda x}yy'_x + be^{\mu x}y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -ae^{\lambda x}$, $g(x) = -be^{\mu x}$.

67. $yy''_{xx} = (y'_x)^2 + be^{ax}y^n(y'_x)^k.$

This is a special case of equation 2.9.4.53 with $f(\xi) = b\xi$, $g(\zeta) = \zeta^k$, $n = m - k + 2$.

68. $yy''_{xx} = (y'_x)^2 + (ae^{\lambda x}y^n + by^{2-m})(y'_x)^m.$

The transformation $\xi = y'_x/y$, $w = e^{\lambda x}y^{n+m-2}$ leads to a first order equation with separation of variables:

$$\xi^m (aw + b) w'_\xi = [(n + m - 2)\xi + \lambda]w.$$

69. $x^2 y''_{xx} = ax^{n+2} e^y + n.$

This is a special case of equation 2.9.1.17 with $f(\xi) = a\xi$.

70. $x^2 y''_{xx} + xy'_x = ae^{\lambda y} + b.$

This is a special case of equation 2.9.2.8 with $f(y) = ae^{\lambda y} + b$.

71. $x^2 y''_{xx} + xy'_x = kx^n e^{ay} + b.$

This is a special case of equation 2.9.2.23 with $f(\xi) = k\xi + b$.

72. $(ax^2 + b)y''_{xx} + axy'_x + ce^{\lambda y} = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = ce^{\lambda y}$.

73. $(ae^{2x} + b)y''_{xx} + ae^{2x} y'_x + cy^n = 0.$

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x} + b$, $f(y) = -cy^n$.

74. $(ae^{2x} + b)y''_{xx} + ae^{2x} y'_x + ce^{\lambda y} = 0.$

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x} + b$, $f(y) = -ce^{\lambda y}$.

2.8.4. Equations Containing Hyperbolic Functions

1. $y''_{xx} = \lambda^2 y + a(\cosh \lambda x)^{-n-3} y^n.$

This is a special case of equation 2.9.1.21 with $f(\xi) = a\xi^n$.

2. $y''_{xx} = \lambda^2 y + a(\sinh \lambda x)^{-n-3} y^n.$

This is a special case of equation 2.9.1.20 with $f(\xi) = a\xi^n$.

3. $y''_{xx} = \lambda^2 y + a \sinh^n(\lambda x) \cosh^{-n-m-3}(\lambda x) y^m.$

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to the Emden—Fowler equation

$$w''_{\xi\xi} = a\lambda^{-2} \xi^n w^m$$

which is outlined in Section 2.3.

4. $y''_{xx} = \lambda^2 y + a \cosh^n(\lambda x) \sinh^{-n-m-3}(\lambda x) y^m.$

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to the Emden—Fowler equation

$$w''_{\xi\xi} = a\lambda^{-2} \xi^n w^m$$

which is outlined in Section 2.3.

5. $y''_{xx} = b \cosh(\lambda x) y + ay^{-3}.$

This is a special case of equation 2.9.1.12 with $f(x) = -b\cosh(\lambda x)$.

6. $\boldsymbol{y''_{xx} = b\sinh(\lambda x)y + ay^{-3}}$.

This is a special case of equation 2.9.1.12 with $f(x) = -b\sinh(\lambda x)$.

7. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + a\cosh(\lambda x)y = 0}$.

This is a special case of equation 2.9.2.1 with $f(x) = a\cosh(\lambda x)$.

8. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + (a\sinh x + b)y = 0}$.

This is a special case of equation 2.9.2.1 with $f(x) = a\sinh x + b$.

9. $\boldsymbol{y''_{xx} + ay^n(y'_x)^2 + b\cosh^m y + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = b\cosh^m y + c$.

10. $\boldsymbol{y''_{xx} + ay^n(y'_x)^2 + b\tanh^m y + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = b\tanh^m y + c$.

11. $\boldsymbol{y''_{xx} = ay^n(y'_x)^2 + b\sinh^m(\lambda x)y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = b\sinh^m(\lambda x)$.

12. $\boldsymbol{y''_{xx} = ay^n(y'_x)^2 + b\tanh^m(\lambda x)y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = b\tanh^m(\lambda x)$.

13. $\boldsymbol{y''_{xx} + a\cosh^n y\,(y'_x)^2 + by^m + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = a\cosh^n y$, $g(y) = by^m + c$.

14. $\boldsymbol{y''_{xx} + a\cosh^n y\,(y'_x)^2 + b\cosh^m(\lambda y) + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = a\cosh^n y$, $g(y) = b\cosh^m(\lambda y) + c$.

15. $\boldsymbol{y''_{xx} = a\sinh^n y\,(y'_x)^2 + bx^m y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = a\sinh^n y$, $g(x) = bx^m$.

16. $\boldsymbol{y''_{xx} = a\sinh^n y\,(y'_x)^2 + b\sinh^m(\lambda x)y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = a\sinh^n y$, $g(x) = b\sinh^m(\lambda x)$.

17. $\boldsymbol{y''_{xx} + a\tanh^n y\,(y'_x)^2 + by^m + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = a\tanh^n y$, $g(y) = by^m + c$.

18. $\boldsymbol{y''_{xx} + a\tanh^n y\,(y'_x)^2 + b\tanh^m(\lambda y) + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = a\tanh^n y$, $g(y) = b\tanh^m(\lambda y) + c$.

19. $\boldsymbol{y''_{xx} = a\tanh^n y\,(y'_x)^2 + bx^m y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = a\tanh^n y$, $g(x) = bx^m$.

20. $y''_{xx} = a \tanh^n y \, (y'_x)^2 + b \tanh^m(\lambda x) y'_x$.

This is a special case of equation 2.9.3.2 with $f(y) = a \tanh^n y$, $g(x) = b \tanh^m(\lambda x)$.

21. $y''_{xx} = a y^n (y'_x)^2 + b \cosh^m y \, (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a y^n$, $g(y) = b \cosh^m y$.

22. $y''_{xx} = a y^n (y'_x)^2 + b \tanh^m y \, (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a y^n$, $g(y) = b \tanh^m y$.

23. $y''_{xx} = a \cosh^n y \, (y'_x)^2 + b y^m (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a \cosh^n y$, $g(y) = b y^m$.

24. $y''_{xx} = a \tanh^n y \, (y'_x)^2 + b y^m (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a \tanh^n y$, $g(y) = b y^m$.

25. $x y''_{xx} = n y'_x + a x^{2n+1} \cosh^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \cosh^m(\lambda y)$.

26. $x y''_{xx} = n y'_x + a x^{2n+1} \sinh^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \sinh^m(\lambda y)$.

27. $x y''_{xx} = n y'_x + a x^{2n+1} \tanh^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \tanh^m(\lambda y)$.

28. $x y''_{xx} = n y'_x + a x^{2n+1} \coth^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \coth^m(\lambda y)$.

29. $2 y y''_{xx} = (y'_x)^2 + b \cosh^m(\lambda x) y^2 - a$.

This is a special case of equation 2.9.3.7 with $f(x) = -b \cosh^m(\lambda x)$.

30. $y y''_{xx} = n (y'_x)^2 - a y^{4n-2} + b \cosh^m(\lambda x) y^2$.

This is a special case of equation 2.9.3.8 with $f(x) = -b \cosh^m(\lambda x)$.

31. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \cosh^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \cosh^k(\lambda x)$.

32. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \sinh^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \sinh^k(\lambda x)$.

33. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \tanh^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \tanh^k(\lambda x)$.

34. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\cosh^m(\lambda x)y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -ax^n$, $g(x) = -b\cosh^m(\lambda x)$.

35. $yy''_{xx} = (y'_x)^2 + a\cosh^n(\lambda x)yy'_x + bx^m y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -a\cosh^n(\lambda x)$, $g(x) = -bx^m$.

36. $x^2 y''_{xx} + xy'_x = a\cosh^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\cosh^n(\lambda y) + b$.

37. $x^2 y''_{xx} + xy'_x = a\sinh^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\sinh^n(\lambda y) + b$.

38. $x^2 y''_{xx} + xy'_x = a\tanh^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\tanh^n(\lambda y) + b$.

39. $x^2 y''_{xx} + xy'_x = a\coth^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\coth^n(\lambda y) + b$.

40. $(ax^2 + b)y''_{xx} + axy'_x + \cosh^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \cosh^n(\lambda y) + c$.

41. $(ax^2 + b)y''_{xx} + axy'_x + \sinh^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \sinh^n(\lambda y) + c$.

42. $(ax^2 + b)y''_{xx} + axy'_x + \tanh^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \tanh^n(\lambda y) + c$.

43. $(ax^2 + b)y''_{xx} + axy'_x + \coth^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \coth^n(\lambda y) + c$.

2.8.5. Equations Containing Logarithmic Functions

1. $y''_{xx} = ax^{-3}(\ln y - \ln x).$

This is a special case of equation 2.9.1.5 with $f(\xi) = a\ln\xi$.

2. $y''_{xx} = ax^{-3/2}(2\ln y - \ln x).$

This is a special case of equation 2.9.1.6 with $f(\xi) = 2a\ln\xi$.

3. $y''_{xx} + ay^n(y'_x)^2 + b\ln^m y + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = b\ln^m y + c$.

4. $\boldsymbol{y''_{xx} = ay^n (y'_x)^2 + b\ln^m(\lambda x) y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = b\ln^m(\lambda x)$.

5. $\boldsymbol{y''_{xx} + a\ln^n y\,(y'_x)^2 + by^m + c = 0}$.

This is a special case of equation 2.9.3.1 with $f(y) = a\ln^n y$, $g(y) = by^m + c$.

6. $\boldsymbol{y''_{xx} = a\ln^n y\,(y'_x)^2 + bx^m y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = a\ln^n y$, $g(x) = bx^m$.

7. $\boldsymbol{y''_{xx} = a\ln^n y\,(y'_x)^2 + b\ln^m(\lambda x) y'_x}$.

This is a special case of equation 2.9.3.2 with $f(y) = a\ln^n y$, $g(x) = b\ln^m(\lambda x)$.

8. $\boldsymbol{y''_{xx} = ax^{-2} y^{-1} (y'_x)^4 - 2ax^{-3}\ln y\,(y'_x)^3}$.

Solution in the parametric form:

$$x = \lambda\bigl[(F \pm 2\tau)^2 \pm 4\ln(C_1 F)\bigr]^{1/2}, \qquad y = C_1 F,$$

where $F = \exp(\mp\tau^2)\left[\int \exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}$, $\lambda = \bigl(\pm\frac{1}{2}aC_1^2\bigr)^{1/4}$.

9. $\boldsymbol{y''_{xx} = 2a\ln x\,y^{-3} - ax^{-1} y^{-2} (y'_x)^{-1}}$.

Solution in the parametric form:

$$x = C_1 F, \qquad y = \lambda\bigl[(F \pm 2\tau)^2 \pm 4\ln(C_1 F)\bigr]^{1/2},$$

where $F = \exp(\mp\tau^2)\left[\int \exp(\mp\tau^2)\,d\tau + C_2\right]^{-1}$, $\lambda = \bigl(\pm\frac{1}{2}aC_1^2\bigr)^{1/4}$.

10. $\boldsymbol{y''_{xx} = ay^n (y'_x)^2 + b\ln^m y\,(y'_x)^k}$.

This is a special case of equation 2.9.4.10 with $f(y) = ay^n$, $g(y) = b\ln^m y$.

11. $\boldsymbol{y''_{xx} = a\ln^n y\,(y'_x)^2 + by^m (y'_x)^k}$.

This is a special case of equation 2.9.4.10 with $f(y) = a\ln^n y$, $g(y) = by^m$.

12. $\boldsymbol{xy''_{xx} = ny'_x + ax^{2n+1}\ln^m(\lambda y)}$.

This is a special case of equation 2.9.2.4 with $f(y) = a\ln^m(\lambda y)$.

13. $\boldsymbol{xy''_{xx} = -(n+1)y'_x + ax^{n-1}(\ln y + n\ln x)}$.

This is a special case of equation 2.9.2.5 with $f(\xi) = a\ln\xi$.

14. $\boldsymbol{xy''_{xx} = (ay + n\ln x)y'_x}$.

This is a special case of equation 2.9.2.22 with $f(\xi) = \ln\xi$.

15. $xy''_{xx} + x(2ay + \ln x + b)y'_x + y = 0.$

Integrating the equation, we obtain Riccati equation

$$y'_x + ay^2 + (\ln x + b)y = C.$$

16. $yy''_{xx} = n(y'_x)^2 + ax^m y^2 + b\ln^k(\lambda x)y^{n+1}.$

This is a special case of equation 2.9.3.9 with $f(x) = -ax^m$, $g(x) = -b\ln^k(\lambda x)$.

17. $yy''_{xx} = (y'_x)^2 + a\ln^n(\lambda x)yy'_x + bx^m y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -a\ln^n(\lambda x)$, $g(x) = -bx^m$.

18. $yy''_{xx} = (y'_x)^2 + ax^n yy'_x + b\ln^m(\lambda x)y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -ax^n$, $g(x) = -b\ln^m(\lambda x)$.

19. $yy''_{xx} = (ax + n\ln y)(y'_x)^2.$

This is a special case of equation 2.9.3.17 with $f(\xi) = \ln\xi$.

20. $x^2y''_{xx} = x^2(y + a\ln x + b)^n + a.$

This is a special case of equation 2.9.1.22 with $f(\xi) = \xi^n$.

21. $x^2y''_{xx} = n(n+1)y + ax^{3n+2}(\ln y + n\ln x).$

This is a special case of equation 2.9.1.11 with $f(\xi) = a\ln\xi$.

22. $x^2y''_{xx} + \frac{1}{4}y + Ax^{\frac{1-m}{2}}(a\ln x + b)^n y^m = 0.$

The transformation $\xi = a\ln x + b$, $w = \dfrac{y}{\sqrt{x}}$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} + Aa^{-2}\xi^n w^m = 0.$$

23. $x^2y''_{xx} + xy'_x = a\ln^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\ln^n(\lambda y) + b$.

24. $(ax^2 + b)y''_{xx} + axy'_x + c\ln^n(\lambda y) = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = c\ln^n(\lambda y)$.

2.8.6. Equations Containing Trigonometric Functions

1. $y''_{xx} = -\lambda^2 y + a(\cos\lambda x)^n y^{-n-3}.$

This is a special case of equation 2.9.1.27 with $f(\xi) = a\xi^{-n-3}$.

2. $y''_{xx} = -\lambda^2 y + a(\sin\lambda x)^n y^{-n-3}.$

This is a special case of equation 2.9.1.26 with $f(\xi) = a\xi^{-n-3}$.

3. $\boldsymbol{y''_{xx} = -\lambda^2 y + a\cos^n(\lambda x)\sin^m(\lambda x) y^{-n-m-3}}$.

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = a\lambda^{-2}\xi^n w^{-n-m-3}.$$

4. $\boldsymbol{y''_{xx} = -\lambda^2 y + a\sin^n(\lambda x)[\sin(\lambda x) + b\cos(\lambda x)]^m y^{-n-m-3}}$.

The transformation $\xi = 1 + b\cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = a(b\lambda)^{-2}\xi^m w^{-n-m-3}.$$

5. $\boldsymbol{y''_{xx} = -\lambda^2 y + A\sin^n(\lambda x + a)\sin^m(\lambda x + b) y^{-n-m-3}}$.

The transformation

$$\xi = \frac{\sin(\lambda x + a)}{\sin(\lambda x + b)}, \quad w = \frac{y}{\sin(\lambda x + b)}$$

leads to the Emden—Fowler equation (see Section 2.3)

$$w''_{\xi\xi} = A[\lambda\sin(b - a)]^{-2}\xi^n w^{-n-m-3}.$$

6. $\boldsymbol{y''_{xx} = b\cos(\lambda x) y + a y^{-3}}$.

This is a special case of equation 2.9.1.12 with $f(x) = -b\cos(\lambda x)$.

7. $\boldsymbol{y''_{xx} = b\sin(\lambda x) y + a y^{-3}}$.

This is a special case of equation 2.9.1.12 with $f(x) = -b\sin(\lambda x)$.

8. $\boldsymbol{y''_{xx} = 2(\cos x)^{-2} y + a(\cot x)^{n+3} y^n}$.

This is a special case of equation 2.9.1.29 with $f(\xi) = a\xi^n$.

9. $\boldsymbol{y''_{xx} = 2(\sin x)^{-2} y + a(\tan x)^{n+3} y^n}$.

This is a special case of equation 2.9.1.28 with $f(\xi) = a\xi^n$.

10. $\boldsymbol{y''_{xx} = (n+1)(\tan x) y'_x + ny + a(\cos x)^{nm-2} y^{m-1}}$.

This is a special case of equation 2.9.2.30 with $f(\xi) = a\xi^{m-1}$.

11. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + a\sin(\lambda x) y = 0}$.

This is a special case of equation 2.9.2.1 with $f(x) = a\sin(\lambda x)$.

12. $\boldsymbol{y''_{xx} + 3yy'_x + y^3 + (a\cos x + b) y = 0}$.

This is a special case of equation 2.9.2.1 with $f(x) = a\cos x + b$.

13. $\boldsymbol{y''_{xx} + (2ay + b\sin x) y'_x + b(\cos x) y = 0}$.

Integrating, we obtain the Riccati equation $y'_x + ay^2 + b(\sin x) y = C$.

14. $y''_{xx} + ay^n (y'_x)^2 + b\sin^m y + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = b\sin^m y + c$.

15. $y''_{xx} + ay^n (y'_x)^2 + b\tan^n y + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = ay^n$, $g(y) = b\tan^m y + c$.

16. $y''_{xx} = ay^n (y'_x)^2 + b\sin^m(\lambda x) y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = b\sin^m(\lambda x)$.

17. $y''_{xx} = ay^n (y'_x)^2 + b\tan^m(\lambda x) y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = ay^n$, $g(x) = b\tan^m(\lambda x)$.

18. $y''_{xx} + a\sin^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\sin^n y$, $g(y) = by^m + c$.

19. $y''_{xx} + a\sin^n y\,(y'_x)^2 + b\sin^m(\lambda y) + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\sin^n y$, $g(y) = b\sin^m(\lambda y) + c$.

20. $y''_{xx} = a\sin^n y\,(y'_x)^2 + bx^m y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\sin^n y$, $g(x) = bx^m$.

21. $y''_{xx} = a\sin^n y\,(y'_x)^2 + b\sin^m(\lambda x) y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\sin^n y$, $g(x) = b\sin^m(\lambda x)$.

22. $y''_{xx} + a\tan^n y\,(y'_x)^2 + by^m + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\tan^n y$, $g(y) = by^m + c$.

23. $y''_{xx} + a\tan^n y\,(y'_x)^2 + b\tan^m(\lambda y) + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\tan^n y$, $g(y) = b\tan^m(\lambda y) + c$.

24. $y''_{xx} = a\tan^n y\,(y'_x)^2 + bx^m y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\tan^n y$, $g(x) = bx^m$.

25. $y''_{xx} = a\tan^n y\,(y'_x)^2 + b\tan^m(\lambda x) y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\tan^n y$, $g(x) = b\tan^m(\lambda x)$.

26. $y''_{xx} = ay^n (y'_x)^2 + b\sin^m y\,(y'_x)^k.$

This is a special case of equation 2.9.4.10 with $f(y) = ay^n$, $g(y) = b\sin^m y$.

27. $y''_{xx} = ay^n (y'_x)^2 + b\tan^m y\,(y'_x)^k.$

This is a special case of equation 2.9.4.10 with $f(y) = ay^n$, $g(y) = b\tan^m y$.

28. $y''_{xx} = a \sin^n y \, (y'_x)^2 + b y^m (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a \sin^n y$, $g(y) = b y^m$.

29. $y''_{xx} = a \tan^n y \, (y'_x)^2 + b y^m (y'_x)^k$.

This is a special case of equation 2.9.4.10 with $f(y) = a \tan^n y$, $g(y) = b y^m$.

30. $x y''_{xx} = n y'_x + a x^{2n+1} \cos^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \cos^m(\lambda y)$.

31. $x y''_{xx} = n y'_x + a x^{2n+1} \sin^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \sin^m(\lambda y)$.

32. $x y''_{xx} = n y'_x + a x^{2n+1} \tan^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \tan^m(\lambda y)$.

33. $x y''_{xx} = n y'_x + a x^{2n+1} \cot^m(\lambda y)$.

This is a special case of equation 2.9.2.4 with $f(y) = a \cot^m(\lambda y)$.

34. $2 y y''_{xx} = (y'_x)^2 + b \sin(\lambda x) y^2 - a$.

This is a special case of equation 2.9.3.7 with $f(x) = -b \sin(\lambda x)$.

35. $y y''_{xx} = n (y'_x)^2 - a y^{4n-2} + b \sin(\lambda x) y^2$.

This is a special case of equation 2.9.3.8 with $f(x) = -b \sin(\lambda x)$.

36. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \cos^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \cos^k(\lambda x)$.

37. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \sin^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \sin^k(\lambda x)$.

38. $y y''_{xx} = n (y'_x)^2 + a x^m y^2 + b \tan^k(\lambda x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a x^m$, $g(x) = -b \tan^k(\lambda x)$.

39. $y y''_{xx} = n (y'_x)^2 + a \sin(\lambda x) y^2 + b \sin(\mu x) y^{n+1}$.

This is a special case of equation 2.9.3.9 with $f(x) = -a \sin(\lambda x)$, $g(x) = -b \sin(\mu x)$.

40. $y y''_{xx} = (y'_x)^2 + a x^n y y'_x + b \sin^m(\lambda x) y^2$.

This is a special case of equation 2.9.3.6 with $f(x) = -a x^n$, $g(x) = -b \sin^m(\lambda x)$.

41. $y y''_{xx} = (y'_x)^2 + a x^n y y'_x + b \tan^m(\lambda x) y^2$.

This is a special case of equation 2.9.3.6 with $f(x) = -a x^n$, $g(x) = -b \tan^m(\lambda x)$.

42. $yy''_{xx} = (y'_x)^2 + a\sin^n(\lambda x)yy'_x + bx^m y^2.$

This is a special case of equation 2.9.3.6 with $f(x) = -a\sin^n(\lambda x)$, $g(x) = -bx^m$.

43. $x^2 y''_{xx} + xy'_x = a\sin^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\sin^n(\lambda y) + b$.

44. $x^2 y''_{xx} + xy'_x = a\tan^n(\lambda y) + b.$

This is a special case of equation 2.9.2.8 with $f(y) = a\tan^n(\lambda y) + b$.

45. $x^2 y''_{xx} + ax^2 \tan x\, y'_x + n(ax\tan x - n - 1)y = bx^{nm+2}(\cos x)^{2a} y^{m-3}.$

This is a special case of equation 2.9.2.33 with $f(\xi) = b\xi^{m-3}$.

46. $(ax^2 + b)y''_{xx} + axy'_x + \cos^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \cos^n(\lambda y) + c$.

47. $(ax^2 + b)y''_{xx} + axy'_x + \sin^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \sin^n(\lambda y) + c$.

48. $(ax^2 + b)y''_{xx} + axy'_x + \tan^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \tan^n(\lambda y) + c$.

49. $(ax^2 + b)y''_{xx} + axy'_x + \cot^n(\lambda y) + c = 0.$

This is a special case of equation 2.9.2.11 with $f(y) = \cot^n(\lambda y) + c$.

50. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = ay^n + b.$

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = ay^n + b$.

51. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\sin^n(\lambda y) + b.$

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\sin^n(\lambda y) + b$.

52. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\cos^n(\lambda y) + b.$

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\cos^n(\lambda y) + b$.

53. $\sin x\, y''_{xx} + \frac{1}{2}\cos x\, y'_x = a\tan^n(\lambda y) + b.$

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\tan^n(\lambda y) + b$.

54. $\sin^2 x\, y''_{xx} = n(n + 1 - n\sin^2 x)y + a(\sin x)^{nm+3n+2} y^m.$

This is a special case of equation 2.9.1.30 with $f(\xi) = a\xi^m$.

55. $\cos^2 x\, y''_{xx} = n(n + 1 - n\cos^2 x)y + a(\cos x)^{nm+3n+2} y^m.$

This is a special case of equation 2.9.1.31 with $f(\xi) = a\xi^m$.

2.8.7. Equations Containing the Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $y''_{xx} = \lambda^2 y + ae^{3\lambda x}(\ln y + \lambda x)$.

This is a special case of equation 2.9.1.15 with $f(\xi) = a\ln\xi$.

2. $y''_{xx} = -ay'_x + be^{ax}(\ln y + ax)$.

This is a special case of equation 2.9.2.18 with $f(\xi) = b\ln\xi$.

3. $y''_{xx} = ay'_x + be^{2ax}\ln^n(\lambda y)$.

This is a special case of equation 2.9.2.17 with $f(y) = b\ln^n(\lambda y)$.

4. $y''_{xx} = ay'_x + be^{2ax}\sin^n(\lambda y)$.

This is a special case of equation 2.9.2.17 with $f(y) = b\sin^n(\lambda y)$.

5. $y''_{xx} = ay'_x + be^{2ax}\tan^n(\lambda y)$.

This is a special case of equation 2.9.2.17 with $f(y) = b\tan^n(\lambda y)$.

6. $y''_{xx} + a\tan x\, y'_x + b(a\tan x - b)y = ce^{bmx}(\cos x)^{2a}y^{m-3}$.

This is a special case of equation 2.9.2.32 with $f(\xi) = c\xi^{m-3}$.

7. $y''_{xx} = a(y'_x)^2 - be^{4ay} + c\sinh(\lambda x)$.

This is a special case of equation 2.9.3.16 with $f(x) = -c\sinh(\lambda x)$.

8. $y''_{xx} = a(y'_x)^2 + b\cosh^n(\lambda x)e^{ay} + cx^m$.

This is a special case of equation 2.9.3.15 with $f(x) = -b\cosh^n(\lambda x)$, $g(x) = -cx^m$.

9. $y''_{xx} = a(y'_x)^2 + b\ln^n(\lambda x)e^{ay} + cx^m$.

This is a special case of equation 2.9.3.15 with $f(x) = -b\ln^n(\lambda x)$, $g(x) = -cx^m$.

10. $y''_{xx} = a(y'_x)^2 + b\ln^n(\lambda x)e^{ay} + ce^{\nu x}$.

This is a special case of equation 2.9.3.15 with $f(x) = -b\ln^n(\lambda x)$, $g(x) = -ce^{\nu x}$.

11. $y''_{xx} = a(y'_x)^2 - be^{4ay} + c\sin(\lambda x)$.

This is a special case of equation 2.9.3.16 with $f(x) = -c\sin(\lambda x)$.

12. $y''_{xx} = a(y'_x)^2 + b\sin^n(\lambda x)e^{ay} + cx^m$.

This is a special case of equation 2.9.3.15 with $f(x) = -b\sin^n(\lambda x)$, $g(x) = -cx^m$.

13. $y''_{xx} = a(y'_x)^2 + b\sin^n(\lambda x)e^{ay} + ce^{\nu x}$.

This is a special case of equation 2.9.3.15 with $f(x) = -b\sin^n(\lambda x)$, $g(x) = -ce^{\nu x}$.

14. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + b\ln^n y + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = ae^{\lambda y}$, $g(y) = b\ln^n y + c$.

15. $y''_{xx} + ae^{\lambda y}(y'_x)^2 + b\sin^n y + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = ae^{\lambda y}$, $g(y) = b\sin^n y + c$.

16. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\ln^n(\mu x)y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = ae^{\lambda y}$, $g(x) = b\ln^n(\mu x)$.

17. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\sin^n(\mu x)y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = ae^{\lambda y}$, $g(x) = b\sin^n(\mu x)$.

18. $y''_{xx} = ae^{\lambda y}(y'_x)^2 + b\tan^n(\mu x)y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = ae^{\lambda y}$, $g(x) = b\tan^n(\mu x)$.

19. $y''_{xx} + a\ln^n y\,(y'_x)^2 + be^{\lambda y} + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\ln^n y$, $g(y) = be^{\lambda y} + c$.

20. $y''_{xx} = a\ln^n y\,(y'_x)^2 + be^{\lambda x}y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\ln^n y$, $g(x) = be^{\lambda x}$.

21. $y''_{xx} = a\ln^n y\,(y'_x)^2 + b\sin^m(\lambda x)y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\ln^n y$, $g(x) = b\sin^m(\lambda x)$.

22. $y''_{xx} + a\sin^n y\,(y'_x)^2 + be^{\lambda y} + c = 0.$

This is a special case of equation 2.9.3.1 with $f(y) = a\sin^n y$, $g(y) = be^{\lambda y} + c$.

23. $y''_{xx} = a\sin^n y\,(y'_x)^2 + be^{\lambda x}y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\sin^n y$, $g(x) = be^{\lambda x}$.

24. $y''_{xx} = a\sin^n y\,(y'_x)^2 + b\ln^m(\lambda x)y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\sin^n y$, $g(x) = b\ln^m(\lambda x)$.

25. $y''_{xx} = a\tan^n y\,(y'_x)^2 + be^{\lambda x}y'_x.$

This is a special case of equation 2.9.3.2 with $f(y) = a\tan^n y$, $g(x) = be^{\lambda x}$.

26. $y''_{xx} + be^{ax}\cosh(\lambda y)(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 2.9.3.24 with $f(y) = b\cosh(\lambda y)$.

27. $y''_{xx} + be^{ax}\sin(\lambda y)(y'_x)^3 + ay'_x = 0.$

This is a special case of equation 2.9.3.24 with $f(y) = b\sin(\lambda y)$.

28. $\boldsymbol{xy''_{xx} = ax\ln x\, e^y y'_x + ae^y}$.

Solution: $\quad y = -\ln\left[e^{C_1 x}\left(C_2 - \dfrac{a}{C_1}\displaystyle\int x^{-1}e^{-C_1 x}\,dx\right) + \dfrac{a}{C_1}\ln x\right].$

29. $\boldsymbol{yy''_{xx} = ae^x (y'_x)^3 + ae^x y\ln y\,(y'_x)^2}$.

Solution: $\quad x = -\ln\left[e^{C_1 y}\left(C_2 + \dfrac{a}{C_1}\displaystyle\int y^{-1}e^{-C_1 y}\,dy\right) - \dfrac{a}{C_1}\ln y\right].$

30. $\boldsymbol{yy''_{xx} = (y'_x)^2 + ae^{\lambda x}yy'_x + b\sin^n(\nu x)y^2}$.

This is a special case of equation 2.9.3.6 with $f(x) = -ae^{\lambda x}$, $g(x) = -b\sin^n(\nu x)$.

31. $\boldsymbol{(ae^{2x}+b)y''_{xx} + ae^{2x}y'_x = \cosh^n(\lambda y) + c}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x}+b$, $f(y) = \cosh^n(\lambda y) + c$.

32. $\boldsymbol{(ae^{2x}+b)y''_{xx} + ae^{2x}y'_x = \tanh^n(\lambda y) + c}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x}+b$, $f(y) = \tanh^n(\lambda y) + c$.

33. $\boldsymbol{(ae^{2x}+b)y''_{xx} + ae^{2x}y'_x = \ln^n(\lambda y) + c}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x}+b$, $f(y) = \ln^n(\lambda y) + c$.

34. $\boldsymbol{(ae^{2x}+b)y''_{xx} + ae^{2x}y'_x = \sin^n(\lambda y) + c}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x}+b$, $f(y) = \sin^n(\lambda y) + c$.

35. $\boldsymbol{(ae^{2x}+b)y''_{xx} + ae^{2x}y'_x = \tan^n(\lambda y) + c}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2x}+b$, $f(y) = \tan^n(\lambda y) + c$.

36. $\boldsymbol{\sin x\, y''_{xx} + \tfrac{1}{2}\cos x\, y'_x = ae^{\lambda y} + b}$.

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = ae^{\lambda y} + b$.

37. $\boldsymbol{\sin x\, y''_{xx} + \tfrac{1}{2}\cos x\, y'_x = a\cosh^n(\lambda y) + b}$.

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\cosh^n(\lambda y) + b$.

38. $\boldsymbol{\sin x\, y''_{xx} + \tfrac{1}{2}\cos x\, y'_x = a\sinh^n(\lambda y) + b}$.

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\sinh^n(\lambda y) + b$.

39. $\boldsymbol{\sin x\, y''_{xx} + \tfrac{1}{2}\cos x\, y'_x = a\tanh^n(\lambda y) + b}$.

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\tanh^n(\lambda y) + b$.

40. $\boldsymbol{\sin x\, y''_{xx} + \tfrac{1}{2}\cos x\, y'_x = a\ln^n y + b}$.

This is a special case of equation 2.9.2.14 with $g(x) = \sin x$, $f(y) = a\ln^n y + b$.

2.9. Equations Containing Arbitrary Functions

Notation: f, g, h, φ, and ψ are arbitrary composite functions of their arguments indicated in parentheses just after the function name (the arguments may depend on x, y, y'_x).

2.9.1. Equations of the Form $F(x,y)y''_{xx} + G(x,y) = 0$

1. $y''_{xx} = f(y)$.

This autonomous equation is met with in mechanics, the combustion theory, and the theory of mass transfer with chemical reactions.

The substitution $u(y) = y'_x$ leads to a first order equation with separated variables: $uu'_y = f(y)$.

Solution: $\displaystyle\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = C_2 \pm x.$

Particular solutions: $y = A_k$, where A_k are roots of the equation $f(A_k) = 0$.

2. $y''_{xx} = f(ay + bx + c)$.

The substitution $w = ay+bx+c$ leads to an equation of the form 2.9.1.1: $w''_{xx} = af(w)$.

3. $y''_{xx} = f(y + ax^2 + bx + c)$.

The substitution $w = y + ax^2 + bx + c$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w) + 2a$.

4. $y''_{xx} = f(y + ax^n + bx^2 + cx) - an(n-1)x^{n-2}$.

The substitution $w = y + ax^n + bx^2 + cx$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w) + 2b$.

5. $y''_{xx} = \dfrac{1}{x^3} f\left(\dfrac{y}{x}\right)$.

The transformation $\xi = 1/x$, $w = y/x$ leads to an equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

6. $y''_{xx} = x^{-3/2} f(yx^{-1/2})$.

Having set $w = yx^{-1/2}$, we obtain

$$\frac{d}{dx}(xw'_x)^2 = \tfrac{1}{2}ww'_x + 2f(w)w'_x.$$

Integrating the latter equation, we arrive at an equation with separation of variables.

Solution: $\displaystyle\int \left[C_1 + \tfrac{1}{4}w^2 + 2\int f(w)\,dw\right]^{-1/2} dw = C_2 \pm \ln|x|.$

7. $y''_{xx} = \dfrac{1}{x^3} f\left(\dfrac{y}{x} + \dfrac{a}{x^2} + \dfrac{b}{x} + c\right)$.

The transformation $\xi = 1/x$, $w = y/x$ leads to an equation of the form 2.9.1.3: $w''_{\xi\xi} = f(w + a\xi^2 + b\xi + c)$.

8. $y''_{xx} = x^{-3/2} f(ayx^{-1/2} + bx^{1/2})$.

The substitution $w = ay + bx$ leads to an equation of the form 2.9.1.6: $w''_{xx} = ax^{-3/2} f(wx^{-1/2})$.

9. $y''_{xx} = y^{-3} f\left(\dfrac{y}{\sqrt{ax^2 + bx + c}}\right)$.

Setting $u(x) = y(ax^2 + bx + c)^{-1/2}$ and integrating the equation, we obtain a first order equation with separation of variables:

$$(ax^2 + bx + c)^2 (u'_x)^2 = (\tfrac{1}{4}b^2 - ac)u^2 + 2\int u^{-3} f(u)\,du + C_1.$$

10. $(ax^2 + bx + c)^{3/2} y''_{xx} = f\left(\dfrac{\alpha y + \beta x + \gamma}{\sqrt{ax^2 + bx + c}}\right)$.

Setting $w = \alpha y + \beta x + \gamma$ and denoting $f(z) = \dfrac{1}{\alpha z^3}\varphi(z)$, we obtain an equation of the form 2.9.1.9:

$$w''_{xx} = w^{-3}\varphi\left(\frac{w}{\sqrt{ax^2 + bx + c}}\right).$$

11. $y''_{xx} = n(n+1)x^{-2}y + x^{3n} f(yx^n)$.

This is a special case of equation 2.9.1.14 with $\psi = x^{-n}$.

12. $y''_{xx} + f(x)y = ay^{-3}$.

Yermakov's equation.

Let $w = w(x)$ be a nontrivial solution of the linear equation $w''_{xx} + f(x)w = 0$. The transformation $\xi = \displaystyle\int \frac{dx}{w^2}$, $z = \dfrac{y}{w}$ leads to an autonomous equation of the form 2.9.1.1: $z''_{\xi\xi} = az^{-3}$.

Solution: $\quad C_1 y^2 = aw^2 + w^2\left(C_2 + C_1 \displaystyle\int \frac{dx}{w^2}\right)^2.$

13. $(ax^n + b)y''_{xx} = an(n-1)x^{n-2}y + y^{-2} f\left(\dfrac{y}{ax^n + b}\right)$.

The transformation $\xi = \displaystyle\int \frac{dx}{(ax^n + b)^2}$, $w = \dfrac{y}{ax^n + b}$ leads to an equation of the form 2.9.1.1: $w''_{\xi\xi} = w^{-2} f(w)$.

14. $y''_{xx} = \dfrac{\psi''_{xx}}{\psi} y + \psi^{-3} f\left(\dfrac{y}{\psi}\right), \qquad \psi = \psi(x)$.

The transformation $\xi = \displaystyle\int \frac{dx}{\psi^2}$, $w = \dfrac{y}{\psi}$ leads to an equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

Solution: $\quad \displaystyle\int \left[C_1 + 2\int f(w)\,dw\right]^{-1/2} dw = C_2 \pm \int \frac{dx}{\psi^2(x)}.$

15. $y''_{xx} = \lambda^2 y + e^{3\lambda x} f(ye^{\lambda x})$.

This is a special case of equation 2.9.1.14 with $\psi = e^{-\lambda x}$.

16. $y''_{xx} = f(y + ae^{\lambda x} + b) - a\lambda^2 e^{\lambda x}$.

The substitution $w = y + ae^{\lambda x} + b$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

17. $x^2 y''_{xx} = x^2 f(x^n e^y) + n$.

The substitution $y = w - n \ln x$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(e^w)$.

18. $y''_{xx} = f(y + a \sinh x + b) - a \sinh x$.

The substitution $w = y + a \sinh x + b$ yields an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

19. $y''_{xx} = f(y + a \cosh x + b) - a \cosh x$.

The substitution $w = y + a \cosh x + b$ yields an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

20. $y''_{xx} = \lambda^2 y + (\sinh \lambda x)^{-3} f\left(\dfrac{y}{\sinh \lambda x}\right)$.

This is a special case of equation 2.9.1.14 with $\psi = \sinh \lambda x$.

21. $y''_{xx} = \lambda^2 y + (\cosh \lambda x)^{-3} f\left(\dfrac{y}{\cosh \lambda x}\right)$.

This is a special case of equation 2.9.1.14 with $\psi = \cosh \lambda x$.

22. $x^2 y''_{xx} = x^2 f(y + a \ln x + b) + a$.

The substitution $w = y + a \ln x + b$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

23. $y''_{xx} = -\dfrac{y}{x^2 \ln x} + \dfrac{1}{(\ln x)^3} f\left(\dfrac{y}{\ln x}\right)$.

This is a special case of equation 2.9.1.14 with $\psi = \ln x$.

24. $y''_{xx} = f(y + a \sin x + b) + a \sin x$.

The substitution $w = y + a \sin x + b$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

25. $y''_{xx} = f(y + a \cos x + b) + a \cos x$.

The substitution $w = y + a \cos x + b$ leads to an equation of the form 2.9.1.1: $w''_{xx} = f(w)$.

26. $y''_{xx} = -\lambda^2 y + (\sin \lambda x)^{-3} f\left(\dfrac{y}{\sin \lambda x}\right)$.

This is a special case of equation 2.9.1.14 with $\psi = \sin \lambda x$.

27. $y''_{xx} = -\lambda^2 y + (\cos \lambda x)^{-3} f\left(\dfrac{y}{\cos \lambda x}\right)$.

This is a special case of equation 2.9.1.14 with $\psi = \cos \lambda x$.

28. $y''_{xx} = 2(\sin x)^{-2}y + (\tan x)^3 f(y\tan x)$.

This is a special case of equation 2.9.1.14 with $\psi = \cot x$.

29. $y''_{xx} = 2(\cos x)^{-2}y + (\cot x)^3 f(y\cot x)$.

This is a special case of equation 2.9.1.14 with $\psi = \tan x$.

30. $\sin^2 x\, y''_{xx} = n(n+1-n\sin^2 x)y + \sin^{3n+2} x\, f(y\sin^n x)$.

The substitution $x = \xi + \frac{\pi}{2}$ leads to an equation of the form 2.9.1.31:

$$\cos^2\xi\, y''_{\xi\xi} = n(n+1-n\cos^2\xi)y + \cos^{3n+2}\xi\, f(y\cos^n\xi).$$

31. $\cos^2 x\, y''_{xx} = n(n+1-n\cos^2 x)y + \cos^{3n+2} x\, f(y\cos^n x)$.

The transformation $\xi = \displaystyle\int \cos^{2n} x\,dx$, $w = y\cos^n x$ leads to an autonomous equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

32. $y''_{xx} = \varphi^{-3} f\Big(\dfrac{y}{\varphi} + \psi\Big) + \dfrac{\varphi''_{xx}}{\varphi} y - \varphi\psi''_{xx} - 2\varphi'_x\psi'_x, \qquad \varphi = \varphi(x),\ \psi = \psi(x)$.

The transformation

$$t = \int \frac{dx}{\varphi^2}, \qquad w = \frac{y}{\varphi} + \psi$$

leads to an autonomous equation of the form 2.9.1.1: $w''_{tt} = f(w)$.

Solution:

$$\int \frac{dw}{\sqrt{2F(w) + C_1}} = \pm\int \frac{dx}{\varphi^2} + C_2, \qquad \text{where} \quad F(w) = \int f(w)\,dw.$$

2.9.2. Equations of the Form $F(x,y)y''_{xx} + G(x,y)y'_x + H(x,y) = 0$

1. $y''_{xx} + 3yy'_x + y^3 + f(x)y = 0$.

The substitution $y = w(\int w\,dx)^{-1}$ leads to the linear equation: $w''_{xx} + f(x)w = 0$.

2. $y''_{xx} + [2ay + f(x)]y'_x + af(x)y^2 = g(x)$.

Having set $u = y'_x + ay^2$, we obtain $u'_x + f(x)u = g(x)$. Thus, the original equation is reduced to the first order linear equation and the Riccati equation.

3. $y''_{xx} + [3y + f(x)]y'_x + y^3 + f(x)y^2 + g(x)y + h(x) = 0$.

The substitution $y = u'_x/u$ leads to a third order linear equation:

$$u'''_{xxx} + f(x)u''_{xx} + g(x)u'_x + h(x)u = 0.$$

4. $\boldsymbol{xy''_{xx} = ny'_x + x^{2n+1}f(y)}$.

Multiplying both sides by x^{-2n-1}, we obtain an equation of the form 2.9.2.14.

1°. Solution with $n \neq -1$:

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = \pm\frac{x^{n+1}}{n+1} + C_2.$$

2°. Solution with $n = -1$:

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = \pm\ln|x| + C_2.$$

5. $\boldsymbol{xy''_{xx} + (n+1)y'_x = x^{n-1}f(yx^n)}$.

The transformation $\xi = x^n$, $w = yx^n$ leads to an autonomous equation of the form 2.9.1.1: $n^2 w''_{\xi\xi} = f(w)$.

6. $\boldsymbol{xy''_{xx} - ny'_x + f(x)y = ax^{2n+1}y^{-3}}$.

The substitution $w = yx^{-n/2}$ leads to Yermakov's equation 2.9.1.12:

$$w''_{xx} + x^{-2}[xf(x) - \tfrac{1}{4}n(n+2)]w = aw^{-3}.$$

7. $\boldsymbol{xy''_{xx} = f(y)y'_x}$.

The substitution $w(y) = xy'_x/y$ leads to a first order linear equation: $yw'_y = -w + 1 + f(y)$.

8. $\boldsymbol{x^2y''_{xx} + xy'_x = f(y)}$.

Substitutions $x = \pm e^t$ leads to an equation of the form 2.9.1.1: $y''_{tt} = f(y)$.

9. $\boldsymbol{x^2y''_{xx} + (n+m+1)xy'_x + nmy = x^{n-2m}f(yx^n)}$.

1°. For $n \neq m$, the transformation $\xi = x^{n-m}$, $w = yx^n$ leads to an autonomous equation of the form 2.9.1.1: $(n-m)^2 w''_{\xi\xi} = f(w)$.

2°. For $n = m$, the transformation $\xi = \ln x$, $w = yx^n$ leads to an autonomous equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

10. $\boldsymbol{x^2y''_{xx} = f(y/x)(xy'_x - y)}$.

This is a special case of equation 2.9.3.23 with $g(x) = h(x) = 0$.

11. $\boldsymbol{(ax^2+b)y''_{xx} + axy'_x + f(y) = 0}$.

The substitution $\xi = \displaystyle\int \frac{dx}{\sqrt{ax^2+b}}$ leads to an autonomous equation of the form 2.9.1.1: $y''_{\xi\xi} + f(y) = 0$.

12. $\boldsymbol{x^3y''_{xx} = f(y/x)(xy'_x - y)}$.

This is a special case of equation 2.9.4.34 with $g(z) = z$.

13. $y''_{xx} + [2ay + f(x)]y'_x + f'_x(x)y = 0.$

Integrating, we obtain the Riccati equation $y'_x + f(x)y + ay^2 = C$.

14. $gy''_{xx} + \frac{1}{2}g'_x y'_x = f(y), \qquad g = g(x).$

Intergating, we obtain an equation with separation of variables:

$$g(x)(y'_x)^2 = 2\int f(y)\,dy + C_1.$$

With $g(x) \geq 0$, the solution is

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = C_2 \pm \int \frac{dx}{\sqrt{g(x)}}.$$

15. $y''_{xx} + (2fy + g)y'_x + f'_x y^2 + g'_x y = 0, \qquad f = f(x), \quad g = g(x).$

Integrating, we obtain the Riccati equation $y'_x + fy^2 + gy = C$.

16. $y''_{xx} - \dfrac{\varphi'_x}{\varphi}y'_x + \left(\dfrac{\varphi'_x}{\varphi}\dfrac{\psi'_x}{\psi} - \dfrac{\psi''_{xx}}{\psi}\right)y = \dfrac{\varphi^2}{\psi^3}f\left(\dfrac{y}{\psi}\right), \qquad \varphi = \varphi(x), \quad \psi = \psi(x).$

The transformation $\xi = \displaystyle\int \frac{\varphi}{\psi^2}\,dx$, $w = \dfrac{y}{\psi}$ leads to an autonomous equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

17. $y''_{xx} = ay'_x + e^{2ax}f(y).$

Multiplying both sides by e^{-2ax}, we obtain an equation of the form 2.9.2.14.

Solution:

$$\int \left[C_1 + 2\int f(y)\,dy\right]^{-1/2} dy = C_2 \pm \frac{1}{a}e^{ax}.$$

18. $y''_{xx} = -ay'_x + e^{ax}f(ye^{ax}).$

The transformation $\xi = e^{ax}$, $w = ye^{ax}$ leads to an equation of the form 2.9.1.1: $w''_{\xi\xi} = a^{-2}f(w)$.

19. $y''_{xx} + (\mu + \nu)y'_x + \nu\mu y = e^{(\mu - 2\nu)x}f(ye^{\mu x}).$

1°. For $\mu \neq \nu$, the transformation $\xi = e^{(\mu-\nu)x}$, $w = ye^{\mu x}$ leads to an autonomous equation of the form 2.9.1.1: $(\mu - \nu)^2 w''_{\xi\xi} = f(w)$.

2°. For $\mu = \nu$, the substitution $w = ye^{\mu x}$ leads to an autonomous equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

20. $y''_{xx} - \lambda y'_x + f(x)y = ae^{2\lambda x}y^{-3}.$

The substitution $w = ye^{-\lambda x/2}$ leads to Yermakov's equation 2.9.1.12:

$$w''_{xx} + \left[f(x) - \tfrac{1}{4}\lambda^2\right]w = aw^{-3}.$$

21. $\boldsymbol{xy''_{xx} - ny'_x - a(ax+n)y = x^{2n+1}e^{3ax}f(ye^{ax})}$.

This is a special case of equation 2.9.2.16 with $\varphi = x^n$, $\psi = e^{-ax}$.

22. $\boldsymbol{xy''_{xx} = f(x^n e^{ay})y'_x}$.

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to a first order equation with separation of variables: $z(aw+n)w'_z = [f(z)+1]w$.

23. $\boldsymbol{x^2y''_{xx} + xy'_x = f(x^n e^{ay})}$.

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to a first order equation with separation of variables: $z(aw+n)w'_z = f(z)$.

24. $\boldsymbol{x^2y''_{xx} - ax^2y'_x - n(ax+n+1)y = x^{3n+2}e^{2ax}f(yx^n)}$.

This is a special case of equation 2.9.2.16 with $\varphi = e^{ax}$, $\psi = x^{-n}$.

25. $\boldsymbol{(ae^{2\lambda x}+b)y''_{xx} + a\lambda e^{2\lambda x}y'_x + f(y) = 0}$.

This is a special case of equation 2.9.2.14 with $g(x) = ae^{2\lambda x} + b$.

26. $\boldsymbol{y''_{xx} + g'_x y'_x + fy = ae^{-2g}y^{-3}, \qquad f = f(x),\ \ g = g(x)}$.

The substitution $w = ye^{g/2}$ leads to Yermakov's equation 2.9.1.12:

$$w''_{xx} + [f - \tfrac{1}{4}(g'_x)^2 - \tfrac{1}{2}g''_{xx}]w = aw^{-3}.$$

27. $\boldsymbol{y''_{xx} - \dfrac{\varphi'_x}{\varphi}y'_x - a\left(\dfrac{\varphi'_x}{\varphi} + a\right)y = e^{3ax}\varphi^2 f(ye^{ax}), \qquad \varphi = \varphi(x)}$.

The transformation $\xi = \int \varphi e^{2ax}\,dx$, $w = ye^{ax}$ leads to an equation of the form 2.9.1.1: $w''_{\xi\xi} = f(w)$.

28. $\boldsymbol{x^2y''_{xx} + xy'_x = f(y + a\ln x + b\ln^2 x)}$.

The substitution $x = e^t$ leads to an equation of the form 2.9.1.3: $y''_{tt} = f(y + at + bt^2)$.

29. $\boldsymbol{y''_{xx} + \lambda m\tan(\lambda x)y'_x + f(x)y = a[\cos(\lambda x)]^{2m}y^{-3}}$.

This is a special case of equation 2.9.2.26 with $g = -m\ln\cos(\lambda x)$.

30. $\boldsymbol{y''_{xx} - (n+1)\tan x\, y'_x - ny = \cos^{n-2} x\, f(y\cos^n x)}$.

This is a special case of equation 2.9.2.16 with $\varphi = \cos^{-n-1} x$, $\psi = \cos^{-n} x$.

31. $\boldsymbol{y''_{xx} + (m-n)\tan x\, y'_x - n[(m+1)\tan^2 x + 1]y = \cos^{2m+n} x\, f(y\cos^n x)}$.

This is a special case of equation 2.9.2.16 with $\varphi = \cos^{m-n} x$, $\psi = \cos^{-n} x$.

32. $\boldsymbol{y''_{xx} + a\tan x\, y'_x + b(a\tan x - b)y = \cos^{2a} x\, e^{3bx}f(ye^{bx})}$.

This is a special case of equation 2.9.2.16 with $\varphi = \cos^a x$, $\psi = e^{-bx}$.

33. $x^2 y''_{xx} + a x^2 \tan x \, y'_x + n(a x \tan x - n - 1) y = x^{3n+2} \cos^{2a} x \, f(y x^n)$.

This is a special case of equation 2.9.2.16 with $\varphi = \cos^a x$, $\psi = x^{-n}$.

34. $x^2 y''_{xx} - a x^2 \cot x \, y'_x - n(a x \cot x + n + 1) y = x^{3n+2} \sin^{2a} x \, f(y x^n)$.

This is a special case of equation 2.9.2.16 with $\varphi = \sin^a x$, $\psi = x^{-n}$.

35. $\sin x \, y''_{xx} + \frac{1}{2} \cos x \, y'_x = f(y)$.

This is a special case of equation 2.9.2.14 with $g = \sin x$.

36. $\cos x \, y''_{xx} - \frac{1}{2} \sin x \, y'_x = f(y)$.

This is a special case of equation 2.9.2.14 with $g = \cos x$.

2.9.3. Equations of the Form $F(x,y) y''_{xx} + \sum\limits_{m=0}^{M} G_m(x,y) (y'_x)^m = 0 \quad (M = 2,\ 3,\ 4)$

1. $y''_{xx} + f(y)(y'_x)^2 + g(y) = 0$.

The substitution $w(y) = (y'_x)^2$ leads to a first order linear equation:

$$w'_y + 2f(y)w + 2g(y) = 0.$$

2. $y''_{xx} = f(y)(y'_x)^2 + g(x) y'_x$.

Dividing by y'_x, we obtain a total differential equation. Its solution is found from the equation

$$\ln |y'_x| = \int f(y)\, dy + \int g(x)\, dx + C.$$

Solving the latter for y'_x, we arrive at an equation with separation of variables. In addition, $y = C_1$ is the solution with arbitrary constant C_1.

3. $y''_{xx} = \dfrac{f(x^n y^m)}{xy} (x y'_x - y) y'_x$.

The transformation $z = x^n y^m$, $w = \dfrac{x y'_x}{y}$ leads to an equation with separation of variables: $z(mw + n) w'_z = [f(z) - 1](w^2 - w)$.

4. $y''_{xx} = f(x) + g(x)(x y'_x - y) + h(x)(x y'_x - y)^2$.

The substitution $w(x) = x y'_x - y$ leads to the Riccati equation:

$$w'_x = x f(x) + x g(x) w + x h(x) w^2.$$

5. $y y''_{xx} + (y'_x)^2 + f(x) y y'_x + g(x) = 0$.

The substitution $u = y^2$ leads to a linear equation: $u''_{xx} + f(x) u'_x + 2g(x) = 0$.

6. $\boldsymbol{yy''_{xx} - (y'_x)^2 + f(x)yy'_x + g(x)y^2 = 0.}$

The substitution $u = y'_x/y$ leads to a first order linear equation: $u'_x + f(x)u + g(x) = 0$.

7. $\boldsymbol{2yy''_{xx} - (y'_x)^2 + f(x)y^2 + a = 0, \qquad a > 0.}$

If u and v are two solutions of the linear equation $4y''_{xx} + f(x)y = 0$, which satisfy the condition $(uv'_x - u'_x v)^2 = a$, then $y = uv$ is a solution of the original equation.

8. $\boldsymbol{yy''_{xx} - n(y'_x)^2 + f(x)y^2 + ay^{4n-2} = 0.}$

1°. With $n = 1$, this is an equation of the form 2.9.3.6.

2°. With $n \neq 1$, the substitution $w = y^{1-n}$ leads to Yermakov's equation 2.9.1.12:

$$w''_{xx} + (1-n)f(x)w + a(1-n)w^{-3} = 0.$$

9. $\boldsymbol{yy''_{xx} - n(y'_x)^2 + f(x)y^2 + g(x)y^{n+1} = 0.}$

The substitution $w = y^{1-n}$ leads to a nonhomogeneous linear equation:

$$w''_{xx} + (1-n)f(x)w + (1-n)g(x) = 0.$$

10. $\boldsymbol{yy''_{xx} + a(y'_x)^2 + f(x)y'_x + g(x)y^2 = 0.}$

The substitution $w = y^{a+1}$ leads to a linear equation: $w''_{xx} + f(x)w'_x + (a+1)g(x)w = 0$.

11. $\boldsymbol{yy''_{xx} - 2(y'_x)^2 - (fy + 2g)y'_x + f'_x y^2 + g'_x y = 0, \qquad f = f(x), \quad g = g(x).}$

Integrating, we obtain the Riccati equation: $y'_x + Cy^2 + fy + g = 0$.

12. $\boldsymbol{yy''_{xx} - (y'_x)^2 + (fy^2 + g)y'_x + f'_x y^3 - g'_x y = 0, \qquad f = f(x), \quad g = g(x).}$

Integrating, we obtain the Riccati equation: $y'_x + fy^2 + Cy - g = 0$, where C is an arbitrary constant.

13. $\boldsymbol{yy''_{xx} = f(x)(y'_x)^2.}$

The substitution $w(x) = xy'_x/y$ leads to the Bernoulli equation 1.1.5:

$$xw'_x = w + [f(x) - 1]w^2.$$

14. $\boldsymbol{yy''_{xx} + f(x)(y'_x)^2 + g(x)yy'_x + h(x)y^2 = 0.}$

The substitution $u = y'_x/y$ leads to the Riccati equation:

$$u'_x + (1+f)u^2 + gu + h = 0.$$

15. $\boldsymbol{y''_{xx} - a(y'_x)^2 + f(x)e^{ay} + g(x) = 0.}$

The substitution $w = e^{-ay}$ leads to a nonhomogeneous linear equation:

$$w''_{xx} - ag(x)w = af(x).$$

16. $y''_{xx} - a(y'_x)^2 + be^{4ay} + f(x) = 0.$

The substitution $w = e^{-ay}$ leads to Yermakov's equation 2.9.1.12:

$$w''_{xx} - af(x)w = abw^{-3}.$$

17. $yy''_{xx} = f(e^{ax}y^n)(y'_x)^2.$

The transformation $z = e^{ax}y^n$, $w = y'_x/y$ leads to a first order separable equation: $z(nw+a)w'_z = [f(z)-1]w^2$.

18. $y''_{xx} = xf(y)(y'_x)^3.$

Taking y as the independent variable, we obtain a linear equation for $x = x(y)$: $x''_{yy} = -f(y)x$.

19. $y''_{xx} + [xf(y) + g(y)](y'_x)^3 + h(y)(y'_x)^2 = 0.$

Taking y as the independent variable, we obtain a linear equation for $x = x(y)$: $x''_{yy} - h(y)x'_y - f(y)x - g(y) = 0$.

20. $x^2y''_{xx} = f\left(\dfrac{y}{x}\right)(xy'_x - y)(y'_x)^2.$

This is a special case of equation 2.9.4.35 with $k = 2$.

21. $x^3y''_{xx} + [x^4f(y) + a](y'_x)^3 = 0.$

Taking y as the independent variable, we obtain an equation of the form 2.9.1.12 for $x = x(y)$: $x''_{yy} - f(y)x - ax^{-3} = 0$.

22. $y''_{xx} = x^{-1}[f(y) + g(y)(xy'_x - y) + h(y)(xy'_x - y)^2]y'_x.$

The substitution $w(y) = xy'_x - y$ leads to the Riccati equation:

$$w'_y = f(y) + g(y)w + h(y)w^2.$$

23. $y''_{xx} = x^{-2}(xy'_x - y)\left[f\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right)y'_x + h\left(\dfrac{y}{x}\right)(y'_x)^2\right].$

The transformation $z = y/x$, $w = xy'_x/y$ leads to the Riccati equation:

$$zw'_z = f(z) + [zg(z) - 1]w + z^2h(z)w^2.$$

24. $y''_{xx} + e^{ax}f(y)(y'_x)^3 + ay'_x = 0.$

The substitution $\xi = e^{-ax}$ leads to an equation of the form 2.9.4.2 with $g(z) = a^3z^3$: $y''_{\xi\xi} - af(y)(y'_\xi)^3 = 0$.

25. $y''_{xx} = x^{-5/2}\varphi(yx^{-3/2})(y'_x)^4.$

This is a special case of equation 2.9.4.8 with $n = -3/2$, $m = 1$, $f(z) = z\varphi(z)$.

26. $y''_{xx} + f(y)(y'_x)^4 + g(y)(y'_x)^2 + h(y) = 0.$

The substitution $w(y) = (y'_x)^2$ leads to the Riccati equation:

$$w'_y + 2f(y)w^2 + 2g(y)w + 2h(y) = 0.$$

27. $xy''_{xx} + x^{2m+1}f(y)(y'_x)^4 + my'_x = 0.$

This is a special case of equation 2.9.4.46 with $n = 4$, $\varphi = x^{-m}$.

28. $(y + ax)y''_{xx} = f(x)(xy'_x - y)^2.$

The substitution $y = -ax + xz$ leads to the equation

$$xzz''_{xx} + 2zz'_x - x^3 f(x)(z'_x)^3 = 0.$$

Setting $w = z'_x/z$, we obtain the Bernoulli equation:

$$xw'_x + 2w + [x - x^3 f(x)]w^2 = 0.$$

2.9.4. Equations of the Form $F(x, y, y'_x)y''_{xx} + G(x, y, y'_x) = 0$

1. $y''_{xx} = f(x)g(y'_x).$

The substitution $u(x) = y'_x$ leads to a first order equation with separation of variables: $u'_x = f(x)g(u)$.

2. $y''_{xx} = f(y)g(y'_x).$

The substitution $u(y) = y'_x$ leads to a first order equation with separation of variables: $uu'_y = f(y)g(u)$.

In addition, there may exist solutions of the form $y = Ax + C$, where A are roots of the equation $g(A) = 0$, C is an arbitrary number, or $y = B$, where B are roots of the equation $f(B) = 0$.

3. $y''_{xx} = f(ax + by + c)g(y'_x).$

With $b = 0$, we have an equation of the form 2.9.4.1. With $b \neq 0$, the substitution $u(x) = y + (ax + c)/b$ leads to an equation of the form 2.9.4.2: $u''_{xx} = f(bu)g\Big(u'_x - \dfrac{a}{b}\Big)$.

4. $y''_{xx} = x^{-n-1}f(x^n y'_x).$

The substitution $w(x) = x^n y'_x$ leads to a first order equation with separation of variables: $xw'_x = f(w) + nw$.

5. $y''_{xx} = y^{-2n-1}f(y^n y'_x).$

The substitution $w(y) = y^n y'_x$ leads to a first order equation with separation of variables: $yww'_y = f(w) + nw^2$.

6. $y''_{xx} = \dfrac{y}{x^2} f\left(\dfrac{xy'_x}{y}\right).$

The substitution $w(x) = xy'_x/y$ leads to a first order equation with separation of variables: $xw'_x = f(w) + w - w^2$.

7. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{xy}}}$.

Setting $u = y'_x$ and passing on to new variables

$$t = \int \frac{du}{f(u)}, \quad w = 2\sqrt{x},$$

we have $y = (w'_t)^2$. Differentiating the latter with respect to x, we obtain a second order linear equation $w''_{tt} = g(t)w$, where function $g(t)$ is defined parametrically:

$$g = \tfrac{1}{4}u, \quad t = \int \frac{du}{f(u)}.$$

8. $\boldsymbol{y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m}{n+m}}}$.

The transformation $z = x^n y^m$, $w = xy'_x/y$ yields

$$z(mw+n)w'_z = z^{-\frac{1}{m+1}} f(z) w^{\frac{2n+m}{n+m}} + w - w^2.$$

Divide both sides of this equation by $w^{\frac{2n+m}{n+m}}$ and introduce a new variable

$$\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}.$$

As a result we obtain a first order linear equation:

$$(n+m)z\zeta'_z = -\zeta + z^{-\frac{1}{n+m}} f(z).$$

9. $\boldsymbol{y''_{xx} + f(x)y'_x + g(x)(y'_x)^k = 0}$.

The substitution $u(x) = y'_x$ leads to the Bernoulli equation $u'_x + f(x)u + g(x)u^k = 0$.

10. $\boldsymbol{y''_{xx} = f(y)(y'_x)^2 + g(y)(y'_x)^k}$.

The substitution $w(y) = y'_x$ leads to the Bernoulli equation $w'_y = f(y)w + g(y)w^{k-1}$.

11. $\boldsymbol{xy''_{xx} + x^{nm-2m+1} f(y)(y'_x)^n + my'_x = 0}$.

This is a special case of equation 2.9.4.46 with $\varphi = x^{-m}$.

12. $\boldsymbol{y''_{xx} = \dfrac{n-kn-km}{km} x^{-1} y'_x + x^{k-1} y^{-k} f(x^n y^m)(y'_x)^{k+1}}$.

Passing on to new variables $z = x^n y^m$, $w = xy'_x/y$, we arrive at

$$z(mw+n)w'_z = \frac{n(1-k)}{km} w - w^2 + f(z)w^{k+1}.$$

Multiplying the latter by w^{-k-1}, we obtain, with the aid of the substitution

$$\zeta = \frac{m}{1-k} w^{1-k} - \frac{n}{k} w^{-k},$$

a first order linear equation:

$$z\zeta'_z = \frac{k-1}{m}\zeta + f(z).$$

13. $\boldsymbol{y''_{xx} = \dfrac{m+km+kn}{kn} y^{-1}(y'_x)^2 + x^k y^{-k-1} f(x^n y^m)(y'_x)^{k+2}}.$

Passing on to new variables $z = x^n y^m$, $w = xy'_x/y$, we arrive at

$$z(mw+n)w'_z = w + \frac{m(1+k)}{kn} w^2 + f(z)w^{k+2}.$$

Multiplying the latter by w^{-k-2}, we obtain, with the aid of the substitution

$$\zeta = \frac{m}{k} w^{-k} + \frac{n}{k+1} w^{-k-1},$$

a first order linear equation:

$$z\zeta'_z = -\frac{k+1}{n}\zeta - f(z).$$

14. $\boldsymbol{y''_{xx} = y^{-1}(y'_x)^2 - x^{-1}y'_x + x^{-2} y f(x^n y^m) g\Big(\dfrac{xy'_x}{y}\Big)}.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a first order separable equation:

$$z(mw+n)w'_z = f(z)g(w).$$

15. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy+b}}}.$

Setting $u = y'_x$, we rewrite the equation as follows:

$$\big[\sqrt{x}\, u'_x/f(u)\big]^{-2} = ay + bx^{-1}.$$

Differentiating both sides with respect to x and passing on to new variables

$$t = \frac{1}{2}\int \frac{du}{f(u)}, \qquad z = \sqrt{x},$$

we obtain an equation of the form 2.9.1.12:

$$z''_{tt} = au(t)z - bz^{-3}.$$

16. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{ay+bx^2}}}.$

Setting $u = y'_x$, we rewrite the equation as follows:

$$\big[u'_x/f(u)\big]^{-2} = ay + bx^2.$$

Differentiating both sides with respect to x and introducing a new independent variable $t = \displaystyle\int \frac{du}{f(u)}$, we obtain a second order linear equation for $x = x(t)$, integrable by quadrature:

$$2x''_{tt} = 2bx + au(t),$$

where function $u = u(t)$ is defined imlicitly: $t = \displaystyle\int \frac{du}{f(u)}$.

17. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{ax+by^2}}}.$

Taking y as the independent variable, we obtain an equation of the form 2.9.4.16 for $x = x(y)$:

$$x''_{yy} = -(ax+by^2)^{-1/2} f(1/x'_y)(x'_y)^3.$$

18. $\boldsymbol{y''_{xx} = (ax^2+bxy+cy^2+\alpha x+\beta y+\gamma)^{-1/2} f(y'_x)}.$

The transformation $x = At + Bu + C$, $y = Dt + Pu + Q$, where $u = u(t)$, reduces this equation, by selecting appropriate constants A, B, C, D, P, and Q, to an equation of the form 2.9.4.15, 2.9.4.16, or 2.9.4.17.

19. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy+bx^{3/2}+cx}}}.$

Setting $u = y'_x$, we rewrite the equation as follows:

$$\left[\sqrt{x}\, u'_x/f(u)\right]^{-2} = ay + b\sqrt{x} + c.$$

Differentiating both sides with respect to x and changing to new variables

$$t = \frac{1}{2}\int \frac{du}{f(u)}, \qquad z = \sqrt{x},$$

we obtain a second order linear equation:

$$2z''_{tt} = 2au(t)z + b.$$

where function $u = u(t)$ is defined in the inplicit form: $t = \dfrac{1}{2}\displaystyle\int \frac{du}{f(u)}$.

20. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy+by^{3/2}+cy}}}.$

Taking y as the independent variable, we obtain an equation of the form 2.9.4.19 for $x = x(y)$:

$$x''_{yy} = -(axy+by^{3/2}+cy)^{-1/2} f(1/x'_y)(x'_y)^3.$$

21. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy+bx^2+cx^{3/2}+dx}}}.$

The substitution $aw = ay + bx$ leads to an equation of the form 2.9.4.19:

$$w''_{xx} = \frac{f(w'_x - b/a)}{\sqrt{axw+cx^{3/2}+dx}}.$$

22. $\boldsymbol{y''_{xx} = \dfrac{f(y'_x)}{\sqrt{axy+by^2+cy^{3/2}+dy}}}.$

Taking y as the independent variable, we obtain an equation of the form 2.9.4.21 for $x = x(y)$:

$$x''_{yy} = -(axy+by^2+cy^{3/2}+dy)^{-1/2} f(1/x'_y)(x'_y)^3.$$

23. $\boldsymbol{y''_{xx} = \left[xf(y'_x) + yg(y'_x) + h(y'_x)\right]^{-1}}$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to a second order linear equation:

$$w''_{tt} = [f(t) + tg(t)]w'_t - g(t)w + h(t).$$

24. $\boldsymbol{y''_{xx} = a + f(x)\sqrt{(y'_x)^2 - 2ay}}$.

Setting $u = y'_x$, we rewrite the equation as follows:

$$(u'_x - a)^2[f(x)]^{-2} = u^2 - 2ay.$$

Differentiating both sides with respect to x and dividing by $(u'_x - a)$, we obtain a linear equation:

$$fu''_{xx} = f'_x u'_x + f^3 u - af'_x.$$

25. $\boldsymbol{y''_{xx} = f(x)\sqrt{y'_x(xy'_x - y)}}$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 2.9.4.7: $w''_{tt} = \dfrac{1}{f(w'_t)\sqrt{tw}}$.

26. $\boldsymbol{y''_{xx} = f(y)(xy'_x - y)^{1/2}(y'_x)^2}$.

The transformation $x = tw'_t - w$, $y = -w'_t$, where $w = w(t)$, leads to an equation of the form 2.9.4.7: $w''_{tt} = \dfrac{1}{f(-w'_t)\sqrt{tw}}$.

27. $\boldsymbol{y''_{xx} = x^{-2}(xy'_x - y)f(y'_x)}$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to an equation of the form 2.9.3.13: $w''_{tt} = \dfrac{(w'_t)^2}{f(t)w}$.

28. $\boldsymbol{y''_{xx} = x^{-3}f(xy'_x - y)(y'_x)^{-1}}$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 2.9.3.18: $w''_{tt} = t[f(w)]^{-1}(w'_t)^3$.

29. $\boldsymbol{y''_{xx} = x^{-1}f\left(y'_x - \dfrac{y}{x}\right)}$.

The substitution $w = y'_x - \dfrac{y}{x}$ leads to an equation with separation of variables: $xw'_x = -w + f(w)$.

30. $\boldsymbol{y''_{xx} = f(x)g(xy'_x - y)}$.

The substitution $w = xy'_x - y$ leads to an equation with separation of variables: $w'_x = xf(x)g(w)$.

31. $\boldsymbol{y''_{xx} = x^{-n-3}y^n f(xy'_x - y)}$.

The transformation $x = 1/t$, $y = w/t$ leads to an autonomous equation of the form 2.9.4.2: $w''_{tt} = w^n f(-w'_t)$.

32. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(xy'_x - y)^{\frac{2n+m}{n}}.$

This is a special case of equation 2.9.4.36 with $k = \dfrac{2n+m}{n}$.

33. $y''_{xx} = x^{-1} f(y) g(xy'_x - y) y'_x.$

The substitution $w(y) = xy'_x - y$ leads to an equation with separation of variables: $w'_y = f(y)g(w)$.

34. $y''_{xx} = x^{-3} f(y/x) g(xy'_x - y).$

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 2.9.4.2: $w''_{tt} = f(w)g(-w'_t)$.

35. $y''_{xx} = x^{-2} f(y/x)(xy'_x - y)(y'_x)^k.$

The transformation $z = y/x$, $w = xy'_x/y$ leads to the Bernoulli equation

$$zw'_z = -w + z^k f(z) w^k.$$

There are particular solutions $y = x$ and $y = Ax$, where A are roots of the equation $f(A) = 0$; with $k > 0$ we also have $y = C$, where C is an arbitrary number.

36. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to the equation

$$z(mw+n)w'_z = z^{\frac{k-1}{n+m}} f(z) w^{\frac{2n+m-nk}{n+m}}(w-1)^k + w - w^2.$$

Multiplying both sides by $w^{-\frac{2n+m}{n+m}}$ and passing on to a new variable

$$\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}},$$

we arrive at the Bernoulli equation

$$(n+m)z\zeta'_z = -\zeta + z^{\frac{k-1}{n+m}} f(z)\zeta^k.$$

37. $y''_{xx} = n(n-1)x^{-2}y + f(x)(xy'_x - ny)^m.$

The substitution $w = x^n y'_x - nx^{n-1}y$ leads to a first order equation with separation of variables: $w'_x = x^{n+m-nm} f(x) w^m$.

38. $y''_{xx} = n(n-1)x^{-2}y + f(x)g(x^n y'_x - nx^{n-1}y).$

The substitution $w = x^n y'_x - nx^{n-1}y$ leads to a first order equation with separation of variables: $w'_x = x^n f(x) g(w)$.

39. $y''_{xx} = f(x)(xy'_x - y) + g(x)(xy'_x - y)^k.$

The substitution $w(x) = xy'_x - y$ leads to the Bernoulli equation

$$w'_x = xf(x)w + xg(x)w^k.$$

40. $y''_{xx} = x^{-1}\left[f(y)(xy'_x - y) + g(y)(xy'_x - y)^k\right]y'_x.$

The substitution $w(y) = xy'_x - y$ leads to the Bernoulli equation $w'_y = f(y)w + g(y)w^k$.

41. $y''_{xx} = x^{-2}(xy'_x - y)\left[f\left(\frac{y}{x}\right)y'_x + g\left(\frac{y}{x}\right)(y'_x)^k\right].$

The transformation $z = y/x$, $w = xy'_x/y$ leads to the Bernoulli equation: $zw'_z = [zf(z) - 1]w + z^k g(z)w^k$.

42. $y''_{xx} = x^{-3}(xy'_x - y)^2 f\left(\frac{y}{x}\right) + x^{-3}(xy'_x - y)^k g\left(\frac{y}{x}\right).$

The transformation $x = -1/t$, $y = -w/t$ leads to an equation of the form 2.9.4.10:

$$w''_{tt} = f(w)(w'_t)^2 + g(w)(w'_t)^k.$$

43. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m-nk}{n+m}}(xy'_x - y)^k + \dfrac{g(x^n y^m)}{xy}y'_x(xy'_x - y).$

The transformation $z = x^n y^m$, $w = xy'_x/y$ followed by the substitution

$$\zeta = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$$

leads to the Bernoulli equation

$$(n+m)z\zeta'_z = [g(z) - 1]\zeta + z^{\frac{k-1}{n+m}} f(z)\zeta^k.$$

44. $y''_{xx} = x^n y^m (y'_x)^{\frac{2n+m+3}{n+m+2}} F(\zeta), \qquad \zeta = (xy'_x - y)(y'_x)^{-\frac{n+1}{n+m+2}}.$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ leads to an equation of the form 2.9.4.36:

$$w''_{tt} = \frac{1}{tw}\left[\frac{t^a w}{F(t^a w)}\right](w'_t)^{\frac{2a+1-ab}{a+1}}(tw'_t - w)^b,$$

where $a = -\frac{n+1}{n+m+2}$, $b = -m$.

45. $y''_{xx} = \dfrac{f(x^n y^m)}{xy}(y'_x)^{\frac{2n+m}{n+m}} + \dfrac{g(x^n y^m)}{xy}y'_x(xy'_x - y) + \dfrac{h(x^n y^m)}{xy}(y'_x)^{\frac{m}{n+m}}(xy'_x - y)^2.$

The transformation $z = x^n y^m$, $w = xy'_x/y$ followed by the substitution

$$\zeta(z) = w^{\frac{m}{n+m}} - w^{-\frac{n}{n+m}}$$

leads to the Riccati equation

$$(n+m)z\zeta'_z = z^{\frac{1}{n+m}} h(z)\zeta^2 + [g(z) - 1]\zeta + z^{-\frac{1}{n+m}} f(z).$$

46. $y''_{xx} + \varphi^{2-n} f(y)(y'_x)^n - \dfrac{\varphi'_x}{\varphi} y'_x = 0, \qquad \varphi = \varphi(x).$

The substitution $\xi = \int \varphi(x)\,dx$ leads to an equation of the form 2.9.4.2:

$$y''_{\xi\xi} + f(y)(y'_\xi)^n = 0.$$

47. $g y''_{xx} + \frac{1}{2} g'_x y'_x = f(y) h\bigl(y'_x \sqrt{g}\,\bigr), \qquad g = g(x).$

The substitution $w(y) = y'_x \sqrt{g}$ leads to an equation with separation of variables: $w w'_y = f(y) h(w)$.

48. $f y''_{xx} + \frac{1}{2} f'_x y'_x = f g(y)(y'_x)^2 + f^n h(y)(y'_x)^{2n}, \qquad f = f(x).$

The substitution $\xi = \displaystyle\int \frac{dx}{\sqrt{f(x)}}$ leads to an autonomous equation of the form 2.9.4.10: $y''_{\xi\xi} = g(y)(y'_\xi)^2 + h(y)(y'_\xi)^{2n}$.

49. $y''_{xx} + e^{a(n-2)x} f(y)(y'_x)^n + a y'_x = 0.$

This is a special case of equation 2.9.4.46 with $\varphi = e^{-ax}$.

50. $y''_{xx} = -x^{-1} y'_x + x^{-2} f(x^n e^{ay}) g(x y'_x).$

The transformation $z = x^n e^{ay}$, $w = x y'_x$ leads to an equation with separation of variables: $z(aw + n) w'_z = f(z) g(w)$.

51. $y''_{xx} = \dfrac{a}{n} \dfrac{1-k}{2-k} (y'_x)^2 + x^{k-2} f(x^n e^{ay})(y'_x)^k.$

Passing on to new variables $z = x^n e^{ay}$, $w = x y'_x$, we have

$$z(aw + n) w'_z = \frac{a}{n} \frac{1-k}{2-k} w^2 + w + f(z) w^k.$$

Multiplying both sides by w^{-k} and introducing a new variable

$$v = \frac{a}{2-k} w^{2-k} + \frac{n}{1-k} w^{1-k},$$

we obtain a first order linear equation:

$$z v'_z = \frac{1-k}{n} v + f(z).$$

52. $y''_{xx} = -\dfrac{a}{m} \dfrac{2-k}{1-k} y'_x + y^{1-k} f(e^{ax} y^m)(y'_x)^k.$

Passing on to new variables $z = e^{ax} y^m$, $w = y'_x / y$, we have

$$z(mw + a) w'_z = -w^2 - \frac{a}{m} \frac{2-k}{1-k} w + f(z) w^k.$$

Multiplying both sides by w^{-k} and introducing a new variable

$$v = \frac{m}{2-k} w^{2-k} + \frac{a}{1-k} w^{1-k},$$

we obtain a first order linear equation:

$$m z v'_z = (k-2) v + m f(z).$$

53. $y''_{xx} = y^{-1}(y'_x)^2 + yf(e^{ax}y^m)g(y'_x/y)$.

The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to an equation with separation of variables: $z(mw + a)w'_z = f(z)g(w)$.

54. $y''_{xx} = x^{-2}f(x^n e^{ay}) \exp\Bigl(-\dfrac{a}{n}xy'_x\Bigr)$.

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to the equation

$$z(aw + n)w'_z = w + f(z)\exp\Bigl(-\frac{a}{n}w\Bigr)$$

which can be reduced, with the aid of the substitution $\zeta = w\exp\Bigl(\dfrac{a}{n}w\Bigr)$, to a first order linear equation: $nz\zeta'_z = \zeta + f(z)$.

55. $y''_{xx} = -\dfrac{a}{m}y'_x \ln\Bigl(\dfrac{y'_x}{y}\Bigr) + f(e^{ax}y^m)y'_x$.

The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to an equation

$$z(mw + a)w'_z = -\frac{a}{m}w\ln w - w^2 + f(z)w.$$

Dividing both sides by w and passing on to a new variable $v = mw + a\ln w$, we obtain a first order linear equation: $mzv'_z = -v + mf(z)$.

56. $y''_{xx} = -\dfrac{a}{n}(y'_x)^2 \ln(xy'_x) + f(x^n e^{ay})(y'_x)^2$.

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to the equation

$$z(aw + n)w'_z = w - \frac{a}{n}w^2\ln w + w^2 f(z).$$

Dividing both sides by w^2 and passing on to a new variable $v = a\ln w - nw^{-1}$, we obtain a first order linear equation: $nzv'_z = -v + nf(z)$.

2.9.5. Equations of the Form $F(x,\ y,\ y'_x,\ y''_{xx}) = 0$

1. $y = f(y''_{xx})$.

The substitution $w(y) = \frac{1}{2}(y'_x)^2$ leads to an equation of the form 1.9.1.2: $y = f(w'_y)$.

2. $y'_x = yf\Bigl(\dfrac{y''_{xx}}{y}\Bigr)$.

The transformation $t = y^2$, $w = (y'_x)^2$ leads to an equation of the form 1.9.1.7: $w = tf^2(w'_t)$.

3. $y = xf(x^3 y''_{xx})$.

The transformation $x = 1/t$, $y = w/t$ leads to an equation of the form 2.9.5.1: $w = f(w''_{tt})$.

4. $\boldsymbol{y = ax^2 + bx + c + f(y''_{xx}).}$

The substitution $w = y - ax^2 - bx - c$ leads to an equation of the form 2.9.5.1: $w = f(w''_{xx} + 2a)$.

5. $\boldsymbol{xy'_x - y = f(x^n y''_{xx}).}$

This is a special case of equation 2.9.5.10 with $\varphi = x^n$.

6. $\boldsymbol{xy'_x = y + a(y'_x)^2 + by'_x + c + f(y''_{xx}).}$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ ($y'_x = t$, $y''_{xx} = 1/w''_{tt}$) leads to an equation of the form 2.9.5.4: $w = at^2 + bt + c + f(1/w''_{tt})$.

7. $\boldsymbol{f(y''_{xx}) + xy''_{xx} = y'_x.}$

Solution: $2y = C_1 x^2 + 2xf(C_1) + C_2$.

8. $\boldsymbol{F(y''_{xx} + y) = (y''_{xx})^2 + (y'_x)^2.}$

Differentiating with respect to x, we obtain

$$[f'(y''_{xx} + y) - 2y''_{xx}](y'''_{xxx} + y'_x) = 0.$$

From the equation $y'''_{xxx} + y'_x = 0$, we find

$$y = A\sin(x + C_1) + C_2, \qquad \text{where} \quad A^2 = f(C_2).$$

Equating the expression in the square brackets to zero, we arrive at another solution in the parametric form:

$$x = \int \frac{[2 - f''_{uu}(u)]\,du}{\sqrt{4f(u) - \left[f'_u(u)\right]^2}}, \qquad y = u - \tfrac{1}{2} f'_u(u).$$

9. $\boldsymbol{f(x)(y''_{xx} - a)^2 = (y'_x)^2 - 2ay.}$

Differentiating with respect to x, we obtain

$$(y''_{xx} - a)(2fy'''_{xxx} + f'_x y''_{xx} - 2y'_x - af'_x) = 0. \tag{1}$$

Equating the second factor to zero and making the transformation $\xi = \int \frac{dx}{\sqrt{f}}$, $w = y'_x$, we arrive at a second order linear equation of the form 2.1.9.1:

$$w''_{\xi\xi} - w = \tfrac{1}{2} af'_x,$$

whose right-hand side is to be expressed in terms of ξ. Substituting the solution of the latter equation into the original one, we obtain a relation connecting integration constants.

Equating the first factor in (1) to zero, we find the second solution $y = \frac{1}{2}a(x + C)^2$.

10. $\boldsymbol{xy'_x - y = f(\varphi y''_{xx}), \qquad \varphi = \varphi(x).}$

The transformation $\xi = \int \frac{x}{\varphi}\,dx$, $w = xy'_x - y$ leads to an equation of the form 1.9.1.2: $w = f(w'_\xi)$.

11. $\boldsymbol{F\left(y+\dfrac{(y'_x)^2}{y''_{xx}},\ x+\dfrac{y'_x-(y'_x)^3}{2y''_{xx}}\right)=0}$.

Solutions can be found from the equality

$$(y-C_1)^2=2C_2(x-A)+C_2^2, \qquad \text{where} \quad F(C_1,\,A)=0.$$

The question of whether there are other solutions calls for further investigation.

12. $\boldsymbol{F(y''_{xx},\ xy''_{xx}-y'_x,\ x^2y''_{xx}-2xy'_x+2y)=0}$.

It is known that all the functions of the form

$$y=\tfrac{1}{2}Ax^2-C_1x+\tfrac{1}{2}C_2,$$

where $A=A(C_1,C_2)$ is determined from the equation $F(A,C_1,C_2)=0$, are the solutions of the original equation.

13. $\boldsymbol{F\left(x-\dfrac{(y'_x)^2+1}{y''_{xx}}y'_x,\ y+\dfrac{(y'_x)^2+1}{y''_{xx}},\ \dfrac{\left((y'_x)^2+1\right)^{3/2}}{y''_{xx}}\right)=0}$.

It is known that all the functions of the form

$$(x-C_1)^2+(y-C_2)^2=A^2,$$

where $A=A(C_1,C_2)$ is determined from the equation $F(C_1,C_2,A)=0$, are the solutions of the original equation.

14. $\boldsymbol{xy'_x-y=f(e^{\lambda x}y''_{xx})}$.

This is a special case of equation 2.9.5.10 with $\varphi=e^{\lambda x}$.

2.9.6. General Equations Admitting the Order Reduction

1. $\boldsymbol{y''_{xx}=F(x,\ y'_x)}$.

The substitution $w(x)=y'_x$ leads to a first order equation: $w'_x=F(x,w)$.

2. $\boldsymbol{y''_{xx}=F(y,\ y'_x)}$.

The substitution $w(y)=y'_x$ leads to a first order equation: $ww'_y=F(y,w)$.

3. $\boldsymbol{y''_{xx}=F(ax+by,\ y'_x)}$.

The substitution $bw=ax+by$ leads to the equation of the form 2.9.6.2:

$$w''_{xx}=F\left(bw,\ w'_x-\frac{a}{b}\right).$$

4. $\boldsymbol{y''_{xx}=x^{k-2}F(x^{-k}y,\ x^{1-k}y'_x)}$.

Homogeneous equation in the extended sense.

The transformation $t=\ln x$, $w=x^{-k}y$ leads to an equation of the form 2.9.6.2:

$$w''_{tt}+(2k-1)w'_t+k(k-1)w=F(w,\ w'_t+kw).$$

5. $y''_{xx} = \dfrac{y}{x^2} F\Big(x^n y^m,\ \dfrac{x}{y} y'_x\Big).$

Homogeneous equation in the extended sense.

The transformation $z = x^n y^m$, $w = xy'_x/y$ leads to a first order equation:

$$z(mw+n)w'_z = F(z,w) + w - w^2.$$

6. $y''_{xx} = F(x,\ xy'_x - y).$

The substitution $w(x) = xy'_x - y$ leads to a first order equation: $w'_x = xF(x,w)$.

7. $y''_{xx} = F\Big(x,\ y'_x - \dfrac{y}{x}\Big).$

The substitution $w(x) = y'_x - \dfrac{y}{x}$ leads to a first order equation: $xw'_x = -w + xF(x,w)$.

8. $y''_{xx} = x^{-2} F(y,\ xy'_x - y).$

The substitution $w(y) = xy'_x - y$ leads to a first order equation: $(y+w)w'_y = F(y,w)$.

9. $y''_{xx} + [f(x) + g(y)]y'_x + f'_x(x)y = 0.$

Integrating yields a first order equation: $y'_x + f(x)y + \displaystyle\int g(y)\,dy = C$.

10. $f_1 y'_x y''_{xx} + f_2 y y''_{xx} + f_3 (y'_x)^2 + f_4 y y'_x + f_5 y^2 = 0, \qquad f_k = f_k(x).$

The substitution $w(x) = y'_x/y$ leads to the Abel equation

$$(f_1 w + f_2)w'_x + f_1 w^3 + (f_2 + f_3)w^2 + f_4 w + f_5 = 0.$$

11. $f(y'_x)y''_{xx} + g(y)y'_x + h(x) = 0.$

Integrating, we obtain

$$\int f(u)\,du + \int g(y)\,dy + \int h(x)\,dx = C, \qquad \text{where} \quad u = y'_x.$$

12. $f^2 y''_{xx} + f f'_x y'_x = \Phi(y,\ f y'_x), \qquad f = f(x).$

The substitution $w(y) = f y'_x$ leads to a first order equation: $ww'_y = \Phi(y,w)$.

13. $x(x-a)^2 y''_{xx} = f(y/x).$

The transformation $\xi = \ln\Big(\dfrac{x-a}{x}\Big)$, $w = \dfrac{y}{x}$ leads to an equation of the form 2.9.6.2: $w''_{\xi\xi} - w'_\xi = a^{-2} f(w)$. For some specific functions f, the solutions of this equation are given in Subsection 2.2.1.

14. $y''_{xx} = \dfrac{(cx+d)^{n-1}}{(ax+b)^{n+2}} f\Big(\dfrac{(ax+b)^n y}{(cx+d)^{n+1}}\Big).$

The transformation

$$\xi = \ln\Big(\frac{ax+b}{cx+d}\Big), \qquad w = \frac{(ax+b)^n y}{(cx+d)^{n+1}}$$

leads to an equation of the form 2.9.6.2:

$$w''_{\xi\xi} - (2n+1)w'_\xi + n(n+1)w = \Delta^{-2} f(w), \qquad \text{where} \quad \Delta = ad - bc.$$

15. $\boldsymbol{y''_{xx} = 2ayy'_x + F(x,\ y'_x - ay^2).}$

This equation can be derived by eliminating subsidiary function z from the simultaneous first-order equations

$$z'_x = F(x, z), \qquad (1)$$
$$y'_x - ay^2 = z. \qquad (2)$$

If one succeed in finding the general solution $z = z(x, C_1)$ of equation (1), the original equation is reducible to the Riccati equation (2) with a known right-hand side.

16. $\boldsymbol{y''_{xx} = e^{-ax}F(e^{ax}y,\ e^{ax}y'_x).}$

The substitution $w = e^{ax}y$ leads to an equation of the form 2.9.6.2:

$$w''_{xx} - 2aw'_x + a^2 w = F(w,\ w'_x - aw).$$

17. $\boldsymbol{y''_{xx} = yF(e^{ax}y^m,\ y'_x/y).}$

Exponential homogeneous equation.

The transformation $z = e^{ax}y^m$, $w = y'_x/y$ leads to a first order equation:

$$z(mw + a)w'_z = F(z, w) - w^2.$$

18. $\boldsymbol{y''_{xx} = x^{-2}F(x^n e^{ay},\ xy'_x).}$

Exponential homogeneous equation.

The transformation $z = x^n e^{ay}$, $w = xy'_x$ leads to a first order equation:

$$z(aw + n)w'_z = F(z, w) + w.$$

19. $\boldsymbol{y''_{xx} = e^{2ay}F(xe^{ay},\ e^{-ay}y'_x).}$

The transformation $z = xe^{ay}$, $w = e^{-ay}y'_x$ leads to a first order equation:

$$(azw + 1)w'_z = F(z, w) - aw^2.$$

20. $\boldsymbol{y''_{xx} = ae^y y'_x + F(x,\ y'_x - ae^y).}$

This equation can be derived by eliminating subsidiary function z from the simultaneous first-order equations

$$z'_x = F(x, z), \qquad (1)$$
$$y'_x - ae^y = z. \qquad (2)$$

If one succeed in finding the general solution $z = z(x, C_1)$ of equation (1), the original equation is equation (2) of the form 1.7.2.5 which is readily integrable.

21. $\boldsymbol{y''_{xx} = x^{-2}F(ay + b\ln x,\ xy'_x).}$

The transformation $z = ay + b\ln x$, $w = xy'_x$ leads to a first order equation:

$$(aw + b)w'_z = F(z, w) + w.$$

22. $y''_{xx} = yF(ax + b\ln y,\ y'_x/y)$.

The transformation $z = ax + b\ln y$, $w = y'_x/y$ leads to a first order equation:

$$(bw + a)w'_z = F(z, w) - w^2.$$

23. $y''_{xx} = F(x, y)$.

Let $F \neq \varphi(x)y + \psi(x)$, i.e., the equation is a nonlinear one. Then, its order can be lowered by one if the right-hand side of the equation has the following form:

$$F(x,y) = f^{-3/2}E\biggl\{\Phi(u) + \int\Bigl[\frac{1}{2}ff'''_{xxx}(u+V) + f^{1/2}g''_{xx}E^{-1}\Bigr]\,dx\biggr\}, \qquad (1)$$

where

$$E = \exp\Bigl(k\int f^{-1}\,dx\Bigr), \quad V = \int f^{-3/2}gE^{-1}\,dx, \quad u = f^{-1/2}E^{-1}y - V;$$

$\Phi = \Phi(u)$, $f = f(x)$, $g = g(x)$ are arbitrary functions, k is an arbitrary constant.

The integral in (1) may always be expressed in terms of E and V. The following cases are possible:

Case 1. For $f'''_{xxx} \neq 0$,

$$F(x,y) = f^{-3/2}E\Phi(u) + \frac{1}{4f^2}\bigl[2ff''_{xx} - (f'_x)^2\bigr]y + \frac{1}{2f^2}\bigl[2fg'_x - f'_xg + 2kg\bigr] + k^2f^{-3/2}EV.$$

Case 2. For $f = ax^2 + bx + c$, $f'_x \neq -2k$, $f'_x \neq \frac{2}{3}k$,

$$F(x,y) = f^{-3/2}E\Phi(u) + \frac{1}{2f^2}\bigl[2fg'_x - f'_xg + 2kg\bigr] + \Bigl(k^2 + \frac{1}{4}\Delta\Bigr)f^{-3/2}EV,$$

where $\Delta = 4ac - b^2$.

Case 3. For $f = \beta - 2kx$,

$$F(x,y) = f^{-2}[\Phi(y+W) + fg'_x + 2kg], \qquad W = \int f^{-1}g\,dx.$$

Case 4. For $f = \frac{2}{3}kx + \beta$,

$$F(x,y) = \Phi\bigl(f^{-2}y - U\bigr) + f^{-2}\Bigl(fg'_x + \frac{2}{3}k\Bigr) + \frac{8}{3}k^2U, \qquad U = \int f^{-2}g\,dx.$$

In all these cases, the transformation

$$t = \int f^{-1}dx, \qquad u = f^{-1/2}E^{-1}y - V$$

leads to an autonomous equation:

$$u''_{tt} + 2ku'_t + k^2u = \Phi(u),$$

which is reducible, with the aid of the substitution $z(u) = u'_t$, to the Abel equation:

$$zz'_u + 2kz + k^2 u = \Phi(u)$$

(see Subsection 1.3.1).

With $k = 0$, the solution of the original equation for the first and second cases is as follows:

for case 1,

$$\int \frac{du}{\sqrt{2\Psi(u) + C_1}} = \pm \int \frac{dx}{f} + C_2, \qquad \text{where} \quad \Psi(w) = \int \Phi(w)\, dw.$$

for case 2,

$$\int \frac{du}{\sqrt{2\Psi(u) - \frac{1}{4}\Delta u^2 + C_1}} = \pm \int \frac{dx}{ax^2 + bx + c} + C_2, \qquad \text{where} \quad \Psi(w) = \int \Phi(w)\, dw.$$

Remark. The original equation can be reduced, with the aid of point transformations, to an autonomous form *only* for function F of the form (1).

2.9.7. Some Transformations

1. $\boldsymbol{y''_{xx} + x^{-3} F\Big(\frac{1}{x},\ \frac{y}{x}\Big) = 0.}$

The transformation $\xi = 1/x$, $w = y/x$ leads to the equation $w''_{\xi\xi} + F(\xi, w) = 0$.

2. $\boldsymbol{y''_{xx} = n(n+1)x^{-2}y + x^{3n} F(x^{2n+1},\ x^n y) = 0.}$

The transformation $\xi = x^{2n+1}$, $w = x^n y$ leads to the equation $(2n+1)^2 w''_{\xi\xi} = F(\xi, w)$.

3. $\boldsymbol{y''_{xx} + (ax+b)^{-3} F\Big(\frac{cx+d}{ax+b},\ \frac{y}{ax+b}\Big) = 0.}$

The transformation $\xi = \dfrac{cx+d}{ax+b}$, $w = \dfrac{y}{ax+b}$ leads to the equation

$$w''_{\xi\xi} + \Delta^{-2} F(\xi, w) = 0, \qquad \text{where} \quad \Delta = ad - bc.$$

4. $\boldsymbol{x^2 y''_{xx} + axy'_x + by + F(x,\ y) = 0.}$

The transformation $x = \xi^\nu$, $y = \xi^\mu w$, where parameters ν and μ are found from the simultaneous algebraic equations

$$2\mu + 1 + (a-1)\nu = 0, \qquad \mu^2 + (a-1)\mu\nu + b\nu^2 = 0,$$

leads to an equation of the form

$$w''_{\xi\xi} + \nu^2 \xi^{-\mu-2} F(\xi^\nu,\ \xi^\mu w) = 0.$$

5. $\boldsymbol{y''_{xx} = n(n+1)x^{-2}y + x^{3n}F(ax^{2n+1} + b,\ x^n y)}$.

The transformation $\xi = ax^{2n+1} + b$, $w = x^n y$ leads to the equation

$$a^2(2n+1)^2 w''_{\xi\xi} = F(\xi, w).$$

6. $\boldsymbol{y''_{xx} = \lambda^2 y + e^{3\lambda x}F(ae^{2\lambda x} + b,\ e^{\lambda x}y)}$.

The transformation $\xi = ae^{2\lambda x} + b$, $w = e^{\lambda x}y$ leads to the equation

$$w''_{\xi\xi} = (2a\lambda)^{-2}F(\xi, w).$$

7. $\boldsymbol{y''_{xx} = \lambda^2 y + \dfrac{e^{3\lambda x}}{(ce^{2\lambda x} + d)^3}F\left(\dfrac{ae^{2\lambda x} + b}{ce^{2\lambda x} + d},\ \dfrac{e^{\lambda x}y}{ce^{2\lambda x} + d}\right)}$.

The transformation

$$\xi = \frac{ae^{2\lambda x} + b}{ce^{2\lambda x} + d}, \qquad w = \frac{e^{\lambda x}y}{ce^{2\lambda x} + d}$$

leads to the equation

$$w''_{\xi\xi} = (2\Delta\lambda)^{-2}F(\xi, w), \qquad \text{where} \quad \Delta = ad - bc.$$

8. $\boldsymbol{y''_{xx} = \lambda^2 y + \sinh^{-3}(\lambda x)F\left(\coth(\lambda x),\ \dfrac{y}{\sinh(\lambda x)}\right)}$.

The transformation $\xi = \coth(\lambda x)$, $w = \dfrac{y}{\sinh(\lambda x)}$ leads to the equation

$$w''_{\xi\xi} = \lambda^{-2}F(\xi, w).$$

9. $\boldsymbol{y''_{xx} = \lambda^2 y + \cosh^{-3}(\lambda x)F\left(\tanh(\lambda x),\ \dfrac{y}{\cosh(\lambda x)}\right)}$.

The transformation $\xi = \tanh(\lambda x)$, $w = \dfrac{y}{\cosh(\lambda x)}$ leads to the equation

$$w''_{\xi\xi} = \lambda^{-2}F(\xi, w).$$

10. $\boldsymbol{x^2 y''_{xx} + \tfrac{1}{4}y + \sqrt{x}\,F\left(a\ln x + b,\ \dfrac{y}{\sqrt{x}}\right) = 0}$.

The transformation $\xi = a\ln x + b$, $w = \dfrac{y}{\sqrt{x}}$ leads to the equation

$$w''_{\xi\xi} + a^{-2}F(\xi, w) = 0.$$

11. $\boldsymbol{|x^2 - 1|^{3/2}y''_{xx} = F\left(\ln\dfrac{ax - a}{x + 1},\ \dfrac{y}{\sqrt{|x^2 - 1|}}\right)}$.

The transformation $\xi = \ln\dfrac{ax - a}{x + 1}$, $w = \dfrac{y}{\sqrt{|x^2 - 1|}}$ leads to the equation

$$4w''_{\xi\xi} = F(\xi, w) + w.$$

12. $\boldsymbol{y''_{xx} + \lambda^2 y + \sin^{-3}(\lambda x) F\Big(\cot(\lambda x),\ \dfrac{y}{\sin(\lambda x)}\Big) = 0}$.

The transformation $\xi = \cot(\lambda x)$, $w = \dfrac{y}{\sin(\lambda x)}$ leads to the equation

$$w''_{\xi\xi} + \lambda^{-2} F(\xi, w) = 0.$$

13. $\boldsymbol{y''_{xx} + \lambda^2 y + \cos^{-3}(\lambda x) F\Big(\tan(\lambda x),\ \dfrac{y}{\cos(\lambda x)}\Big) = 0}$.

The transformation $\xi = \tan(\lambda x)$, $w = \dfrac{y}{\cos(\lambda x)}$ leads to the equation

$$w''_{\xi\xi} + \lambda^{-2} F(\xi, w) = 0.$$

14. $\boldsymbol{y''_{xx} + \lambda^2 y + \sin^{-3}(\lambda x + b) F\Big(\dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)},\ \dfrac{y}{\sin(\lambda x + b)}\Big) = 0}$.

The transformation $\xi = \dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}$, $w = \dfrac{y}{\sin(\lambda x + b)}$ leads to the equation

$$w''_{\xi\xi} + [\lambda \sin(b - a)]^{-2} F(\xi, w) = 0.$$

15. $\boldsymbol{(x^2 + 1)^{3/2} y''_{xx} + F\Big(\arctan x + b,\ \dfrac{y}{\sqrt{x^2 + 1}}\Big) = 0}$.

The transformation $\xi = \arctan x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to the equation

$$w''_{\xi\xi} + w + F(\xi, w) = 0.$$

16. $\boldsymbol{(x^2 + 1)^{3/2} y''_{xx} + F\Big(\operatorname{arccot} x + b,\ \dfrac{y}{\sqrt{x^2 + 1}}\Big) = 0}$.

The transformation $\xi = \operatorname{arccot} x + b$, $w = \dfrac{y}{\sqrt{x^2 + 1}}$ leads to the equation

$$w''_{\xi\xi} + w + F(\xi, w) = 0.$$

17. $\boldsymbol{y''_{xx} + F(x,\ y) = 0}$.

The transformation $x = \varphi(\xi)$, $y = w\sqrt{a\varphi'_\xi}$ leads to the equation

$$w''_{\xi\xi} + \left[\frac{1}{2}\frac{\varphi'''_{\xi\xi\xi}}{\varphi'_\xi} - \frac{3}{4}\left(\frac{\varphi''_{\xi\xi}}{\varphi'_\xi}\right)^2\right] w + a^{-2}(a\varphi'_\xi)^{3/2} F\Big(\varphi,\ w\sqrt{a\varphi'_\xi}\Big) = 0.$$

The sign of parameter a must coincide with that of derivative φ'_ξ.

18. $\boldsymbol{y''_{xx} + f(x, y)(y'_x)^3 + g(x, y)(y'_x)^2 = 0}$.

Taking y as the independent variable, we obtain the following equation for $x = x(y)$: $x''_{yy} - g(x, y)x'_y - f(x, y) = 0$.

19. $\boldsymbol{F(x,\ y,\ y'_x,\ y''_{xx}) = 0}$.

The Legendre transformation $x = w'_t$, $y = tw'_t - w$, where $w = w(t)$, in view of the relations $y'_x = t$ and $y''_{xx} = 1/w''_{tt}$, leads to the equation

$$F\left(w'_t,\ tw'_t - w,\ t,\ \frac{1}{w''_{tt}}\right) = 0.$$

Given the solution of the original equation, the solution of the transformed equation is written in the parametric form:

$$t = y'_x, \quad w = xy'_x - y, \qquad \text{where} \quad y = y(x).$$