Supplement 1
Some Elementary Functions and Their Properties

In Supplement 1, n is a positive integer, unless otherwise specified.

1.1. Trigonometric Functions

1.1.1. Simplest Relations

$$\sin^2 x + \cos^2 x = 1, \qquad \tan x \cot x = 1,$$
$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x,$$
$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x},$$
$$\tan(-x) = -\tan x, \qquad \cot(-x) = -\cot x.$$

1.1.2. Relations Between Trigonometric Functions of Identical Argument

$$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\cot^2 x}},$$
$$\cos x = \pm\sqrt{1-\sin^2 x} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{\cot x}{\sqrt{1+\cot^2 x}},$$
$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x},$$
$$\cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x} = \pm\frac{\cos x}{\sqrt{1-\cos^x}} = \frac{1}{\tan x}.$$

1.1.3. Reduction Formulae

$$\sin(x \pm n\pi) = (-1)^n \sin x, \qquad \cos(x \pm n\pi) = (-1)^n \cos x,$$
$$\sin\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = \pm(-1)^n \cos x, \qquad \cos\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = \mp(-1)^n \sin x,$$
$$\tan(x \pm n\pi) = \tan x, \qquad \cot(x \pm n\pi) = \cot x,$$
$$\tan\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = -\cot x, \qquad \cot\Bigl(x \pm \frac{2n+1}{2}\pi\Bigr) = -\tan x,$$

$$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x \pm \cos x), \qquad \cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x \mp \sin x),$$
$$\tan\left(x \pm \frac{\pi}{4}\right) = \frac{\tan x \pm 1}{1 \mp \tan x}, \qquad \cot\left(x \pm \frac{\pi}{4}\right) = \frac{\cot x \mp 1}{1 \pm \cot x}.$$

1.1.4. Addition Formulae

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x, \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \qquad \cot(x \pm y) = \frac{1 \mp \tan x \tan y}{\tan x \pm \tan y}.$$

1.1.5. Addition and Subtraction of Trigonometric Functions

$$\begin{aligned}
\sin x \pm \sin y &= 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right),\\
\cos x + \cos y &= 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right),\\
\cos x - \cos y &= -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right),
\end{aligned}$$

$$a\cos x + b\sin x = r\sin(x + \varphi) = r\cos(x - \psi),$$

where $r = \sqrt{a^2 + b^2}$, $\sin\varphi = a/r$, $\cos\varphi = b/r$, $\sin\psi = b/r$, $\cos\psi = a/r$,

$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x + y)\sin(x - y),$$

$$\sin^2 x - \cos^2 y = -\cos(x + y)\cos(x - y),$$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \cot x \pm \cot y = -\frac{\sin(x \pm y)}{\sin x \sin y},$$

1.1.6. Product of Trigonometric Functions

$$\begin{aligned}
\sin x \sin y &= \tfrac{1}{2}[\cos(x - y) - \cos(x + y)],\\
\cos x \cos y &= \tfrac{1}{2}[\cos(x - y) + \cos(x + y)],\\
\sin x \cos y &= \tfrac{1}{2}[\sin(x - y) + \sin(x + y)].
\end{aligned}$$

1.1.7. Powers of Trigonometric Functions

$$\begin{aligned}
\cos^2 x &= \tfrac{1}{2}\cos 2x + \tfrac{1}{2}, & \sin^2 x &= -\tfrac{1}{2}\cos 2x + \tfrac{1}{2},\\
\cos^3 x &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x, & \sin^3 x &= -\tfrac{1}{4}\sin 3x + \tfrac{3}{4}\sin x,\\
\cos^4 x &= \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}, & \sin^4 x &= \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8},\\
\cos^5 x &= \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x, & \sin^5 x &= \tfrac{1}{16}\sin 5x - \tfrac{5}{16}\sin 3x + \tfrac{5}{8}\sin x,
\end{aligned}$$

$$\cos^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C_{2n}^{k} \cos[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^{n},$$

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} C_{2n+1}^{k} \cos[(2n-2k+1)x],$$

$$\sin^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n-k} C_{2n}^{k} \cos[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^{n},$$

$$\sin^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n-k} C_{2n+1}^{k} \sin[(2n-2k+1)x],$$

where $C_m^k = \frac{m!}{k!(m-k)!}$ are binomial coefficients.

1.1.8. Trigonometric Functions of Multiple Arguments

$$\cos 2x = 2\cos^2 x - 1, \qquad \sin 2x = 2\sin x \cos x,$$
$$\cos 3x = -3\cos x + 4\cos^3 x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$
$$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x, \qquad \sin 4x = 4\cos x(\sin x - 2\sin^3 x),$$
$$\cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x, \qquad \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$$

$$\cos(2nx) = 1 + \sum_{k=1}^{n} (-1)^k \frac{n^2(n^2-1)\dots[n^2-(k-1)^2]}{(2k)!} 4^k \sin^{2k} x,$$

$$\cos[2(n+1)x] = \cos x \Biggl\{ 1 + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k)!} \sin^{2k} x \Biggr\},$$

$$\sin(2nx) = 2n\cos x \Biggl[\sin x + \sum_{k=1}^{n} (-1)^k \frac{(n^2-1)(n^2-2^2)\dots(n^2-k^2)}{(2k-1)!} 4^k \sin^{2k-1} x \Biggr],$$

$$\sin[2(n+1)x] = (2n+1) \Biggl\{ \sin x + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\dots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \sin^{2k+1} x \Biggr\},$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}, \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}, \quad \tan 4x = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}.$$

1.1.9. Euler and de Moivre Formulae, Relation to Hyperbolic Functions

$$e^{z+ix} = e^z(\cos x + i\sin x), \quad (\cos x + i\sin x)^n = \cos(nx) + i\sin(nx), \quad i^2 = -1,$$

$$\sin(ix) = i\sinh x, \quad \cos(ix) = \cosh x, \quad \tan(ix) = i\tanh x, \quad \cot(ix) = -i\coth x.$$

1.1.10. Differentiation and Integration Formulae

$$\frac{d\sin x}{dx} = \cos x, \qquad \frac{d\cos x}{dx} = -\sin x, \qquad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}, \qquad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x},$$

$$\int \sin x\, dx = -\cos x + C,$$

$$\int \cos x\, dx = \sin x + C,$$

$$\int \tan x\, dx = -\ln|\cos x| + C,$$

$$\int \cot x\, dx = \ln|\sin x| + C,$$

$$\int \sin^{2n} x\, dx = \frac{1}{2^{2n}} C_{2n}^n x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=1}^{n-1} (-1)^k C_{2n}^k \frac{\sin[(2n-2k)x]}{2n-2k} + C,$$

$$\int \sin^{2n+1} x\, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} C_n^k \cos^{2k+1} x + C,$$

$$\int \cos^{2n} x\, dx = \frac{1}{2^{2n}} C_{2n}^n x + \frac{1}{2^{2n-1}} \sum_{k=1}^{n-1} C_{2n}^k \frac{\sin[(2n-2k)x]}{2n-2k} + C,$$

$$\int \cos^{2n+1} x\, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} C_n^k \sin^{2k+1} x + C,$$

$$\int \tan^{2n} x\, dx = (-1)^n x - \sum_{k=1}^{n} \frac{(-1)^k (\tan x)^{2n-2k+1}}{2n-2k+1} + C,$$

$$\int \tan^{2n+1} x\, dx = (-1)^{n+1} \ln|\cos x| - \sum_{k=1}^{n} \frac{(-1)^k (\tan x)^{2n-2k+2}}{2n-2k+2} + C,$$

$$\int \cot^{2n} x\, dx = (-1)^n x + \sum_{k=1}^{n} \frac{(-1)^k (\cot x)^{2n-2k+1}}{2n-2k+1} + C,$$

$$\int \cot^{2n+1} x\, dx = (-1)^n \ln|\sin x| + \sum_{k=1}^{n} \frac{(-1)^k (\coth x)^{2n-2k+2}}{2n-2k+2} + C,$$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

1.2. Hyperbolic Functions

1.2.1. Simplest Relations

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh x \cdot \coth x = 1,$$
$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x,$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$
$$\tanh(-x) = -\tanh x, \qquad \coth(-x) = -\coth x.$$

1.1.2. Relations Between Hyperbolic Functions of Identical Argument

$$\sinh x = \pm\sqrt{\cosh^2 x - 1} = \pm\frac{\tanh x}{\sqrt{1-\tanh^2 x}} = \pm\frac{1}{\sqrt{\coth^2 x - 1}},$$
$$\cosh x = \sqrt{\sinh^2 x + 1} = \pm\frac{1}{\sqrt{1-\tanh^2 x}} = \pm\frac{\coth x}{\sqrt{\coth^2 x - 1}},$$
$$\tanh x = \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} = \pm\frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x},$$
$$\coth x = \frac{\sqrt{\sin^2 x + 1}}{\sinh x} = \pm\frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}.$$

1.2.3. Addition Formulae

$$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \qquad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

1.2.4. Addition and Subtraction of Hyperbolic Functions

$$\sinh x \pm \sinh y = 2\sinh\left(\frac{x \pm y}{2}\right)\cosh\left(\frac{x \mp y}{2}\right),$$
$$\cosh x + \cosh y = 2\cosh\left(\frac{x + y}{2}\right)\cosh\left(\frac{x - y}{2}\right),$$
$$\cosh x - \cosh y = 2\sinh\left(\frac{x + y}{2}\right)\sinh\left(\frac{x - y}{2}\right),$$
$$\sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y = \sinh(x+y)\sinh(x-y),$$
$$\sinh^2 x + \cosh^2 y = \cosh(x+y)\cosh(x-y),$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}, \quad \coth x \pm \coth y = \pm\frac{\sinh(x \pm y)}{\sinh x \sinh y},$$

1.2.5. Product of Hyperbolic Functions

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)],$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)],$$
$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)].$$

1.2.6. Powers of Hyperbolic Functions

$$\cosh^2 x = \tfrac{1}{2}\cosh 2x + \tfrac{1}{2}, \qquad \sinh^2 x = \tfrac{1}{2}\cosh 2x - \tfrac{1}{2},$$
$$\cosh^3 x = \tfrac{1}{4}\sinh 3x + \tfrac{3}{4}\sinh x, \qquad \sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x,$$
$$\cosh^4 x = \tfrac{1}{8}\cosh 4x + \tfrac{1}{2}\cosh 2x + \tfrac{3}{8}, \qquad \sinh^4 x = \tfrac{1}{8}\cosh 4x - \tfrac{1}{2}\cosh 2x + \tfrac{3}{8},$$

$$\cosh^5 x = \tfrac{1}{16}\sinh 5x + \tfrac{5}{16}\sinh 3x + \tfrac{5}{8}\sinh x,$$
$$\sinh^5 x = \tfrac{1}{16}\sinh 5x - \tfrac{5}{16}\sinh 3x + \tfrac{5}{8}\sinh x,$$

$$\cosh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cosh[2(n-k)x] + \frac{1}{2^{2n}} C_{2n}^n,$$
$$\cosh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \cosh[(2n-2k+1)x],$$
$$\sinh^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} (-1)^k C_{2n}^k \cosh[2(n-k)x] + \frac{(-1)^n}{2^{2n}} C_{2n}^n,$$
$$\sinh^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} (-1)^k C_{2n+1}^k \sinh[(2n-2k+1)x],$$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

1.2.7. Hyperbolic Functions of Multiple Argument

$$\cosh 2x = 2\cosh^2 x - 1, \qquad \sinh 2x = 2\sinh x \cosh x,$$
$$\cosh 3x = -3\cosh x + 4\cosh^3 x, \qquad \sinh 3x = 3\sinh x + 4\sinh^3 x,$$
$$\cosh 4x = 1 - 8\cosh^2 x + 8\cosh^4 x, \qquad \sinh 4x = 4\cosh x(\sinh x + 2\sinh^3 x),$$

$$\cosh 5x = 5\cosh x - 20\cosh^3 x + 16\cosh^5 x,$$
$$\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x.$$

$$\cosh(nx) = 2^{n-1}\cosh^n x + \frac{n}{2}\sum_{k=0}^{[n/2]} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^{k-2} 2^{n-2k-2} (\cosh x)^{n-2k-2},$$

$$\sinh(nx) = \sum_{k=0}^{[(n-1)/2]} 2^{n-k-1} C_{n-k-1}^k (\cosh x)^{n-2k-1},$$

where $C_m^k = \dfrac{k!}{m!\,(k-m)!}$ are binomial coefficients, $[A]$ stands for the integer part of number A.

1.2.8. Relation to Trigonometric Functoins

$$\sinh(ix) = i\sin x, \quad \cosh(ix) = \cos x, \quad \tanh(ix) = i\tan x, \quad \coth(ix) = -i\cot x,$$

where $i^2 = -1$.

1.2.9. Differentiation and Integration Formulae

$$\frac{d\,\sinh x}{dx} = \cosh x, \quad \frac{d\,\cosh x}{dx} = \sinh x, \quad \frac{d\,\tanh x}{dx} = \frac{1}{\cosh^2 x}, \quad \frac{d\,\coth x}{dx} = -\frac{1}{\sinh^2 x},$$

$$\int \sinh x\,dx = \cosh x + C,$$

$$\int \cosh x\,dx = \sinh x + C,$$

$$\int \tanh x\,dx = \ln|\cosh x| + C,$$

$$\int \coth x\,dx = \ln|\sinh x| + C,$$

$$\int \sinh^{2n} x\,dx = \frac{(-1)^n}{2^{2n}} C_{2n}^n x + \frac{1}{2^{2n-1}} \sum_{k=1}^{n-1} (-1)^k C_{2n}^k \frac{\sinh[(2n-2k)x]}{2n-2k} + C,$$

$$\int \sinh^{2n+1} x\,dx = \sum_{k=0}^{n} \frac{(-1)^{n+k}}{2k+1} C_n^k \cosh^{2k+1} x + C,$$

$$\int \cosh^{2n} x\,dx = \frac{1}{2^{2n}} C_{2n}^n x + \frac{1}{2^{2n-1}} \sum_{k=1}^{n-1} C_{2n}^k \frac{\cosh[(2n-2k)x]}{2n-2k} + C,$$

$$\int \cosh^{2n+1} x\,dx = \sum_{k=0}^{n} \frac{1}{2k+1} C_n^k \sinh^{2k+1} x + C,$$

$$\int \tanh^{2n} x\,dx = x - \sum_{k=1}^{n} \frac{(\tanh x)^{2n-2k+1}}{2n-2k+1} + C,$$

$$\int \tanh^{2n+1} x\,dx = \ln\cosh x - \sum_{k=1}^{n} \frac{(\tanh x)^{2n-2k+2}}{2n-2k+2} + C,$$

$$\int \coth^{2n} x\,dx = x - \sum_{k=1}^{n} \frac{(\coth x)^{2n-2k+1}}{2n-2k+1} + C,$$

$$\int \coth^{2n+1} x\,dx = \ln|\sinh x| - \sum_{k=1}^{n} \frac{(\coth x)^{2n-2k+2}}{2n-2k+2} + C,$$

where $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

1.3. Inverse Trigonometric Functions

1.3.1. Simplest Relations

Principal values of inverse trigonometric functions are defined by the inequalities

$$-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}, \qquad 0 \le \arccos x \le \pi, \qquad \text{where} \quad -1 \le x \le 1,$$
$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}, \qquad 0 < \operatorname{arccot} x < \pi, \qquad \text{where} \quad -\infty < x < +\infty.$$

$$\arcsin(-x) = -\arcsin x, \qquad \arccos(-x) = \pi - \arccos x,$$
$$\arctan(-x) = -\arctan x, \qquad \operatorname{arccot}(-x) = \pi - \operatorname{arccot} x,$$

$$\sin(\arcsin x) = x, \qquad \cos(\arccos x) = x,$$
$$\tan(\arctan x) = x, \qquad \cot(\operatorname{arccot} x) = x,$$

$$\arcsin(\sin x) = \begin{cases} x - 2n\pi & \text{if} \quad 2n\pi - \frac{1}{2}\pi \le x \le 2n\pi + \frac{1}{2}\pi, \\ -x + 2(n+1)\pi & \text{if} \quad (2n+1)\pi - \frac{1}{2}\pi \le x \le 2(n+1)\pi + \frac{1}{2}\pi, \end{cases}$$

$$\arccos(\cos x) = \begin{cases} x - 2n\pi & \text{if} \quad 2n\pi \le x \le 2n\pi, \\ -x + 2(n+1)\pi & \text{if} \quad (2n+1)\pi \le x \le 2(n+1)\pi, \end{cases}$$

$$\arctan(\tan x) = x - n\pi \quad \text{if} \quad n\pi - \tfrac{1}{2}\pi < x < n\pi + \tfrac{1}{2}\pi,$$
$$\operatorname{arccot}(\cot x) = x - n\pi \quad \text{if} \quad n\pi < x < (n+1)\pi.$$

1.3.2. Relation Between Inverse Trigonometric Functions

$$\arcsin x + \arccos x = \frac{\pi}{2}, \qquad \arctan x + \operatorname{arccot} x = \frac{\pi}{2}.$$

$$\arcsin x = \begin{cases} \arcsin\sqrt{1-x^2} & \text{if} \quad 0 \le x \le 1, \\ -\arccos\sqrt{1-x^2} & \text{if} \quad -1 \le x \le 0, \\ \arctan\dfrac{x}{\sqrt{1-x^2}} & \text{if} \quad -1 < x < 1, \\ \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} - \pi & \text{if} \quad -1 \le x < 0, \end{cases}$$

$$\arccos x = \begin{cases} \arcsin\sqrt{1-x^2} & \text{if} \quad 0 \le x \le 1, \\ \pi - \arcsin\sqrt{1-x^2} & \text{if} \quad -1 \le x \le 0, \\ \arctan\dfrac{\sqrt{1-x^2}}{x} & \text{if} \quad 0 < x \le 1, \\ \operatorname{arccot}\dfrac{x}{\sqrt{1-x^2}} & \text{if} \quad -1 \le x \le 1. \end{cases}$$

$$\arctan x = \begin{cases} \arcsin\dfrac{x}{\sqrt{1+x^2}} & \text{if} \quad x - \text{any}, \\ \arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if} \quad x \ge 0, \\ -\arccos\dfrac{1}{\sqrt{1+x^2}} & \text{if} \quad x \le 0, \\ \operatorname{arccot}\dfrac{1}{x} & \text{if} \quad x > 0. \end{cases}$$

$$\operatorname{arccot} x = \begin{cases} \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if} \quad x > 0, \\ \pi - \arcsin\dfrac{1}{\sqrt{1+x^2}} & \text{if} \quad x < 0, \\ \arctan\dfrac{1}{x} & \text{if} \quad x > 0, \\ \pi + \operatorname{arccot}\dfrac{1}{x} & \text{if} \quad x < 0. \end{cases}$$

1.3.3. Addition and Subtraction of Inverse Trigonometric Functions

$$
\begin{aligned}
\arcsin x + \arcsin y &= \arcsin\bigl(x\sqrt{1-y^2} + y\sqrt{1-x^2}\,\bigr) && \text{if } x^2+y^2<1,\\
\arccos x + \arccos y &= \arccos\bigl[xy - \sqrt{(1-x^2)(1-y^2)}\,\bigr] && \text{if } x+y>0,\\
\arccos x - \arccos y &= -\arccos\bigl[xy + \sqrt{(1-x^2)(1-y^2)}\,\bigr] && \text{if } x-y\ge 0,\\
\arctan x + \arctan y &= \arctan\frac{x+y}{1-xy} && \text{if } xy<1,\\
\arctan x - \arctan y &= \arctan\frac{x-y}{1+xy} && \text{if } xy\ge -1.
\end{aligned}
$$

1.3.4. Differentiation and Integration Formulae

$$
\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}},
$$

$$
\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2},
$$

$$
\begin{aligned}
\int \arcsin x\,dx &= x\arcsin x + \sqrt{1-x^2} + C,\\
\int \arccos x\,dx &= x\arccos x - \sqrt{1-x^2} + C,\\
\int \arctan x\,dx &= x\arctan x - \frac{1}{2}\ln(x^2+1) + C,\\
\int \operatorname{arccot} x\,dx &= x\operatorname{arccot} x + \frac{1}{2}\ln(x^2+1) + C.
\end{aligned}
$$

1.4. Inverse Hyperbolic Functions

1.4.1. Relation to Logarithmic Functions and Simplest Relations

$$
\operatorname{Arsinh} x = \ln\bigl(x + \sqrt{x^2+1}\,\bigr), \qquad \operatorname{Arcosh} x = \pm\ln\bigl(x + \sqrt{x^2-1}\,\bigr),
$$

$$
\operatorname{Artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad \operatorname{Arcoth} x = \frac{1}{2}\ln\frac{1+x}{x-1},
$$

$$
\operatorname{Arsinh}(-x) = -\operatorname{Arsinh} x, \qquad \operatorname{Arcosh}(-x) = \operatorname{Arcosh} x,
$$

$$
\operatorname{Artanh}(-x) = -\operatorname{Artanh} x, \qquad \operatorname{Arcoth}(-x) = \operatorname{Arcoth} x,
$$

1.4.2. Relations Between Inverse Hyperbolic Functions

$$
\operatorname{Arsinh} x = \operatorname{Arcosh}\sqrt{x^2+1} = \operatorname{Artanh}\frac{x}{\sqrt{x^2+1}},
$$

$$
\operatorname{Arcosh} x = \operatorname{Arsinh}\sqrt{x^2-1} = \operatorname{Artanh}\frac{\sqrt{x^2-1}}{x},
$$

$$
\operatorname{Artanh} x = \operatorname{Arsinh}\frac{x}{\sqrt{x^2-1}} = \operatorname{Arcosh}\frac{1}{\sqrt{1-x^2}}x = \operatorname{Arcoth}\frac{1}{x}.
$$

1.4.3. Addition and Subtraction of Inverse Hyperbolic Functions

$$\operatorname{Arsinh} x \pm \operatorname{Arsinh} y = \operatorname{Arsinh}\bigl(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\,\bigr),$$
$$\operatorname{Arcosh} x \pm \operatorname{Arcosh} y = \operatorname{Arcosh}\bigl[xy \pm \sqrt{(x^2-1)(y^2-1)}\,\bigr],$$
$$\operatorname{Arsinh} x \pm \operatorname{Arcosh} y = \operatorname{Arsinh}\bigl[xy \pm \sqrt{(x^2+1)(y^2-1)}\,\bigr],$$
$$\operatorname{Artanh} x \pm \operatorname{Artanh} y = \operatorname{Artanh}\frac{x \pm y}{1 \pm xy},$$
$$\operatorname{Artanh} x \pm \operatorname{Arcoth} y = \operatorname{Artanh}\frac{xy \pm 1}{y \pm x}.$$

1.4.4. Differentiation and Integration Formulae

$$\frac{d}{dx}\operatorname{Arsinh} x = \frac{1}{\sqrt{x^2+1}}, \qquad \frac{d}{dx}\operatorname{Arcosh} x = \frac{\pm 1}{\sqrt{x^2-1}},$$
$$\frac{d}{dx}\operatorname{Artanh} x = \frac{1}{1-x^2}, \qquad \frac{d}{dx}\operatorname{Arcoth} x = -\frac{1}{1-x^2},$$

$$\int \operatorname{Arsinh} x\,dx = x\operatorname{Arsinh} x - \sqrt{x^2+1} + C,$$
$$\int \operatorname{Arcosh} x\,dx = x\operatorname{Arcosh} x \mp \sqrt{x^2-1} + C,$$
$$\int \operatorname{Artanh} x\,dx = x\operatorname{Artanh} x + \frac{1}{2}\ln(1-x^2) + C,$$
$$\int \operatorname{Arcoth} x\,dx = x\operatorname{Arcoth} x + \frac{1}{2}\ln(x^2-1) + C.$$

1.5. Some Conventional Symbols

1.5.1. Factorial

$$0! = 1! = 1, \quad n! = 1\cdot 2\cdot 3\ldots(n-1)n, \qquad n = 2,\ 3,\ 4,\ \ldots,$$
$$(2n)!! = 2\cdot 4\cdot 6\ldots(2n-2)(2n) = 2^n n!,$$
$$(2n+1)!! = 1\cdot 3\cdot 5\ldots(2n-1)(2n+1) = \frac{2^{n+1}}{\sqrt{\pi}}\Gamma\Bigl(n+\frac{3}{2}\Bigr),$$
$$n!! = \begin{cases} (2k)!! & \text{if } n = 2k, \\ (2k+1)!! & \text{if } n = 2k+1, \end{cases} \qquad 0!! = 1.$$

1.5.2. Binomial Coefficients

$$C_a^b = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)},$$

where $\Gamma(x)$ is the gamma-function.

$$C_a^k = (-1)^k\frac{(-a)_k}{k!} = \frac{a(a-1)\ldots(a-k+1)}{k!} \qquad k = 1,\ 2,\ 3,\ldots$$

$$C_n^k = \frac{n!}{k!(n-k)!}, \qquad k = 1,\ 2,\ 3,\ \ldots,\ n,$$

$$C_a^0 = 1, \quad C_n^k = 0 \quad \text{for } k = -1,\ -2,\ -3,\ \ldots \quad \text{or} \quad k > n,$$

$$C_a^{b+1} = \frac{a}{b+1} C_{a-1}^b = \frac{a-b}{b+1} C_a^b, \quad C_a^b + C_a^{b+1} = C_{a+1}^{b+1},$$

$$C_{-1/2}^n = \frac{(-1)^n}{2^{2n}} C_{2n}^n = (-1)^n \frac{(2n-1)!!}{(2n)!!},$$

$$C_{1/2}^n = \frac{(-1)^{n-1}}{n 2^{2n-1}} C_{2n-2}^{n-1} = \frac{(-1)^{n-1}}{n} \frac{(2n-3)!!}{(2n-2)!!},$$

$$C_{n+1/2}^{2n+1} = (-1)^n 2^{-4n-1} C_{2n}^n, \quad C_{2n+1/2}^n = 2^{-2n} C_{4n+1}^{2n},$$

$$C_n^{1/2} = \frac{2^{2n+1}}{\pi C_{2n}^n}, \quad C_n^{n/2} = \frac{2^{2n}}{\pi} C_n^{(n-1)/2}.$$

1.5.3. Pochhammer Symbol

$$(a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)},$$

$$(a)_0 = 1,\ (a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, \quad (n)_k = \frac{(n+k-1)!}{(n-1)!}$$

$$(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}, \quad \text{where } a \neq 1,\ 2,\ \ldots,\ n;\ k = 1,\ 2,\ 3,\ \ldots,$$

$$(1)_n = n!, \quad (1/2)_n = 2^{-2n} \frac{(2n)!}{n!}, \quad (3/2)_n = 2^{-2n} \frac{(2n+1)!}{n!},$$

$$(a+mk)_{nk} = \frac{(a)_{mk+nk}}{(a)_{mk}}, \quad (a+n)_n = \frac{(a)_{2n}}{(a)_n}, \quad (a+n)_k = \frac{(a)_n (a+k)_n}{(a)_n},$$