Chapter 1
First Order Differential Equations

1.1. Simplest Equations with Arbitrary Functions Integrable in a Closed Form*

1.1.1. Equations of the Form $y'_x = f(x)$

Solution:** $\quad y = \int f(x)\,dx + C.$

1.1.2. Equations of the Form $y'_x = f(y)$

Solution: $\quad x = \int \frac{dy}{f(y)} + C.$

Particular solutions: $y = A_k$, where A_k are roots of the algebraic equation $f(A_k) = 0$.

1.1.3. Separable Equations $y'_x = f(x)g(y)$

Solution: $\quad \int \frac{dy}{g(y)} = \int f(x)\,dx + C.$

Particular solutions: $y = A_k$, where A_k are roots of the algebraic equation $g(A_k) = 0$.

Remark. The equation of the form $f_1(x)g_1(y)y'_x = f_2(x)g_2(y)$ is reduced to the form 1.1.3 by dividing both sides by $f_1 g_1$.

1.1.4. Linear Equation $g(x)y'_x = f_1(x)y + f_0(x)$

Solution:

$$y = Ce^F + e^F \int e^{-F} \frac{f_0(x)}{g(x)}\,dx, \qquad \text{where} \quad F(x) = \int \frac{f_1(x)}{g(x)}\,dx.$$

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* Special cases of equations 1.1.1–1.1.5 for concrete functions f, f_0, f_1, f_α, and g are not discussed in this book; such cases can be readily recognized by the appearance of equations investigated, and the solution can be obtained using the general formulae given in Section 1.1.

** Hereinafter we shall often use the term “solution” to mean “general solution.”

1.1.5. Bernoulli Equation $g(x)y'_x = f_1(x)y + f_n(x)y^n$

Here, n is an arbitrary number.

The substitution $w(x) = y^{1-n}$ leads to a linear equation:

$$w'_x = (1-n)f_1(x)w + (1-n)f_n(x).$$

Solution:

$$y^{1-n} = Ce^F + (1-n)e^F \int e^{-F} \frac{f_n(x)}{g(x)}\,dx, \qquad \text{where} \quad F(x) = (1-n)\int \frac{f_1(x)}{g(x)}\,dx.$$

1.1.6. Homogeneous Equation $y'_x = f(y/x)$

The substitution $u(x) = y/x$ leads to an equation with separation of variables: $xu'_x = f(u) - u$.

Solution: $\displaystyle\int \frac{du}{f(u)-u} = \ln|x| + C.$

Particular solutions: $y = A_k x$, where A_k are roots of the algebraic equation $A_k - f(A_k) = 0$.

1.2. Riccati Equations: $g(y)y'_x = f_2(x)y^2 + f_1(x)y + f_0(x)$

1.2.1. Preliminary Comments

For $f_2 \equiv 0$, we obtain a linear equation (see 1.1.3), and for $f_0 \equiv 0$, we have the Bernoulli equation (see 1.1.4 with $n = 2$), whose solutions were given previously. Below we discuss equations with $f_0 f_2 \not\equiv 0$.

1. Given a particular solution $y_0 = y_0(x)$ of the Riccati equation, the general solution can be written as

$$y = y_0(x) + \Phi(x)\left[C - \int \Phi(x)\frac{f_2(x)}{g(x)}\,dx\right]^{-1},$$

where

$$\Phi(x) = \exp\left\{\int \bigl[2f_2(x)y_0(x) + f_1(x)\bigr]\frac{dx}{g(x)}\right\}.$$

To the particular solution $y_0(x)$ corresponds $C = \infty$.

2. The substitution

$$u(x) = \exp\left(-\int \frac{f_2}{g} y\,dx\right)$$

reduces the general Riccati equation to a second order linear equation:

$$g^2 f_2 u''_{xx} + g\bigl[f_2 g'_x - g(f_2)'_x - f_1 f_2\bigr]u'_x + f_0 f_2^2 u = 0,$$

which often may be easier to solve than the original Riccati equation.

3. Let $g = f_2 = 1$, $f_1(x)$ and $f_0(x)$ be polynomials. If the degree of the polynomial $\Delta = f_1^2 - 2(f_1)'_x - 4f_0$ is odd, the Riccati equation can not possess a polynomial solution.

If the degree of Δ is even, the equation involved may possess only the following polynomial solutions:

$$y = -\tfrac{1}{2}\left(f_1 \pm \left[\sqrt{\Delta}\,\right]\right),$$

where $\left[\sqrt{\Delta}\,\right]$ denotes an integer rational part of the expansion of $\sqrt{\Delta}$ in decreasing powers of x (for example, $\left[\sqrt{x^2 - 2x + 3}\,\right] = x - 1$).

4. The general Riccati equation, with the aid of the substitution

$$x = \varphi(\xi), \quad y = \frac{1}{F_2}w - \frac{1}{2}\frac{F_1}{F_2} + \frac{1}{2}\left(\frac{1}{F_2}\right)'_\xi, \qquad \text{where} \quad F_i(\xi) = \frac{f_i(\varphi)}{g(\varphi)}\varphi'_\xi, \tag{1}$$

is reduced to the canonical form

$$w'_\xi = w^2 + \Psi(\xi), \tag{2}$$

where function Ψ is defined by the formula

$$\Psi(\xi) = F_0F_2 - \frac{1}{4}F_1^2 + \frac{1}{2}F_1' - \frac{1}{2}F_1\frac{F_2'}{F_2} - \frac{3}{4}\left(\frac{F_2'}{F_2}\right)^2 + \frac{1}{2}\frac{F_2''}{F_2}.$$

(prime denotes differentiation with respect to ξ).

Substitution (1) is determined by function $\varphi = \varphi(\xi)$ which may be arbitray. For a specific original Riccati equation, different functions φ in (1) will generate different functions Ψ in equation (2).

In the special case where the original equation has the canonical form

$$y'_x = y^2 + f(x),$$

transformation (1) is written as

$$x = \varphi(\xi), \quad y = \frac{1}{\varphi'_\xi}w - \frac{1}{2}\frac{\varphi''_{\xi\xi}}{(\varphi'_\xi)^2},$$

and the transformed equation (2) is determined by function Ψ:

$$\Psi(\xi) = (\varphi'_\xi)^2 f(\varphi) - \frac{3}{4}\left(\frac{\varphi''_{\xi\xi}}{\varphi'_\xi}\right)^2 + \frac{1}{2}\frac{\varphi'''_{\xi\xi\xi}}{\varphi'_\xi}.$$

If the original Riccati equation is integrable by quadrature, we may obtain, specifying different functions φ, a variety of different integrable equations of the form (2). In Subsection 1.2.8, some useful transformations are given for specific functions φ.

5. The transformation (φ, ψ_1, ψ_2, ψ_3, and ψ_4 are arbitrary functions)

$$x = \varphi(\xi), \quad y = \frac{\psi_4(\xi)u + \psi_3(\xi)}{\psi_2(\xi)u + \psi_1(\xi)}$$

reduces the general Riccati equation to the Riccati equation.

1.2.2. Equations Containing Power Functions

1. $y'_x = ay^2 + bx + c.$

For $b = 0$, we have an equation of the form 1.1.2. For $b \neq 0$, the substitution $bt = bx + c$ leads to an equation of the form 1.2.2.4: $y'_t = ay^2 + bt$.

2. $y'_x = y^2 - a^2x^2 + 3a.$

Particular solution: $y_0 = ax - x^{-1}$.

3. $y'_x = y^2 + ax^2 + bx + c.$

This is a special case of equation 1.2.2.9 with $\alpha = 0$, $\beta = 0$.

4. $y'_x = ay^2 + bx^n.$

Special Riccati equation, n is an arbitrary number.

Solution: $y = -\dfrac{1}{a}\dfrac{u'_x}{u}$, where

$$u(x) = \sqrt{x}\left[C_1 J_{\frac{1}{2q}}\left(\frac{1}{q}\sqrt{ab}\,x^q\right) + C_2 Y_{\frac{1}{2q}}\left(\frac{1}{q}\sqrt{ab}\,x^q\right)\right], \quad q = \frac{n+2}{2};$$

J_m and Y_m are Bessel functions, $n \neq -2$. With $n = -2$, see equation 1.2.2.36.

5. $y'_x = y^2 + anx^{n-1} - a^2x^{2n}.$

Particular solution: $y_0 = ax^n$.

6. $y'_x = ay^2 + bx^{2n} + cx^{n-1}.$

For $n = -1$, we have 1.2.2.36. For $n \neq -1$, the substitution

$$\xi = \frac{1}{n+1}x^{n+1}, \quad \eta = yx^{-n}$$

leads to an equation of the form 1.2.2.25:

$$\xi\eta'_\xi + a\xi\eta^2 + \frac{n}{n+1}\eta = b\xi + \frac{c}{n+1}.$$

7. $y'_x = y^2 + ax^ny + ax^{n-1}.$

Particular solution: $y_0 = -1/x$.

8. $y'_x = y^2 + ax^ny + bx^{n-1}.$

The substitution $y = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.42:

$$u''_{xx} - ax^nu'_x + bx^{n-1}u = 0.$$

9. $y'_x = y^2 + (\alpha x + \beta)y + ax^2 + bx + c.$

The substitution $y = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.28:

$$u''_{xx} - (\alpha x + \beta)u'_x + (ax^2 + bx + c)u = 0.$$

10. $\boldsymbol{y'_x = y^2 + ax^n y - abx^n - b^2}$.

Particular solution: $y_0 = b$.

11. $\boldsymbol{y'_x = ax^n y^2 + bx^{-n-2}}$.

Solution:

$$\sqrt{ab}\ln x = \int \frac{du}{u^2 + \beta u + 1} + C, \qquad \text{where} \quad u = \frac{\sqrt{a}}{\sqrt{b}} x^{n+1} y, \quad \beta = \frac{n+1}{\sqrt{ab}}.$$

12. $\boldsymbol{y'_x = ax^n y^2 + bx^m}$.

1°. For $n \neq -1$, the substitution $\xi = x^{n+1}$ leads to an equation of the form 1.2.2.4:

$$y'_\xi = \frac{a}{n+1} y^2 + \frac{b}{n+1} \xi^{\frac{m-n}{n+1}}.$$

2°. For $n = -1$ and $m \neq -1$, the transformation $\zeta = x^{m+1}$, $w = -1/y$ yields an equation of the form 1.2.2.4:

$$w'_\zeta = \frac{b}{m+1} w^2 + \frac{a}{m+1} \zeta^{-1}.$$

3°. For $n = m = -1$, the original equation is an equation with separation of variables. In this case we have

$$\ln|x| = \int \frac{dy}{ay^2 + b} + C.$$

13. $\boldsymbol{y'_x = y^2 + k(ax + b)^n (cx + d)^{-n-4}}$.

The transformation

$$\xi = \frac{ax+b}{cx+d}, \qquad u = \frac{1}{\Delta}\left[(cx+d)^2 y + c(cx+d)\right], \qquad \text{where} \ \ \Delta = ad - bc,$$

leads to an equation of the form 1.2.2.4: $u'_\xi = u^2 + k\Delta^{-2}\xi^n$.

14. $\boldsymbol{y'_x = ax^n y^2 + bmx^{m-1} - ab^2 x^{n+2m}}$.

Particular solution: $y_0 = bx^m$.

15. $\boldsymbol{y'_x = -(n+1)x^n y^2 + ax^{n+m+1} y - ax^m}$.

Particular solution: $y_0 = x^{-n-1}$.

16. $\boldsymbol{y'_x = ax^n y^2 + bx^m y + bcx^m - ac^2 x^n}$.

Particular solution: $y_0 = -c$.

17. $\boldsymbol{y'_x = ax^n y^2 - ax^n (bx^m + c) y + bmx^{m-1}}$.

Particular solution: $y_0 = bx^m + c$.

18. $\boldsymbol{y'_x = -anx^{n-1} y^2 + cx^m (ax^n + b) y - cx^m}$.

Particular solution: $y_0 = (ax^n + b)^{-1}$.

19. $y'_x = ax^n y^2 + bx^m y + ckx^{k-1} - bcx^{m+k} - ac^2 x^{n+2k}$.

Particular solution: $y_0 = cx^k$.

20. $y'_x = (ax^{2n} + bx^{n-1})y^2 + c$.

The substitution $y = -1/w$ leads to an equation of the form 1.2.2.8:

$$w'_x = cw^2 + ax^{2n} + bx^{n-1}.$$

21. $xy'_x = ay^2 + by + cx^{2b}$.

The transformation $t = x^b$, $w = x^{-b}y$ leads to an equation with separation of variables: $bw'_t = aw^2 + c$.

22. $xy'_x = ay^2 + by + cx^n$.

The transformation $\xi = x^b$, $\eta = yx^{-b}$ reduces this equation to the special Riccati equation 1.2.2.4:

$$\eta'_\xi = \frac{a}{b}\eta^2 + \frac{c}{b}\xi^m, \qquad \text{where} \quad m = \frac{n}{b} - 2.$$

23. $xy'_x = ay^2 + (n + bx^n)y + cx^{2n}$.

The substitution $y = wx^n$ leads to an equation with separation of variables:

$$w'_x = x^{n-1}(aw^2 + bw + c).$$

24. $xy'_x = xy^2 + ay + bx^n$.

The substitution $y = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.62: $xu''_{xx} - au'_x + bx^n u = 0$.

25. $xy'_x + a_3 xy^2 + a_2 y + a_1 x + a_0 = 0$.

The substitution $a_3 y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.59: $xu''_{xx} + a_2 u'_x + a_3(a_1 x + a_0)u = 0$.

26. $xy'_x = ax^n y^2 + by + cx^{-n}$.

The substitution $w = yx^n$ leads to an equation with separation of variables: $xw'_x = aw^2 + (b+n)w + c$.

27. $xy'_x = ax^n y^2 + my - ab^2 x^{n+2m}$.

Particular solution: $y_0 = bx^m$.

28. $xy'_x = x^{2n} y^2 + (m - n)y + x^{2m}$.

Solution: $y = x^{m-n} \tan\left(\dfrac{x^{n+m}}{n+m} + C\right)$.

29. $xy'_x = ax^n y^2 + by + cx^m$.

The transformation $\xi = x^{n-b}$, $\eta = yx^b$ leads to the special Riccati equation1.2.2.4:

$$(n+b)\eta'_\xi = a\eta^2 + c\xi^k, \qquad \text{where} \quad k = \frac{m-n-2b}{n+b}.$$

30. $xy'_x = ax^{2n}y^2 + (bx^n - n)y + c$.

For $n = 0$, this is an equation with separation of variables. For $n \neq 0$, the solution is

$$n\int \frac{dw}{aw^2 + bw + c} = x^n + C, \qquad \text{where} \quad w = yx^n.$$

31. $xy'_x = ax^{2n+m}y^2 + (bx^{n+m} - n)y + cx^m$.

The substitution $w = yx^n$ leads to an equation with separation of variables:

$$w'_x = x^{n+m-1}(aw^2 + bw + c).$$

32. $(a_2x + b_2)(y'_x + \lambda y^2) + (a_1x + b_1)y + a_0x + b_0 = 0$.

The substitution $\lambda y = u'_x/u$ leads to a second order linear equationof the form 2.1.2.103:

$$(a_2x + b_2)u''_{xx} + (a_1x + b_1)u'_x + \lambda(a_0x + b_0)u = 0.$$

33. $(ax + c)y'_x = \alpha(ay + bx)^2 + \beta(ay + bx) - bx + \gamma$.

The substitution $t = ay + bx$ leads to a linear equation with respect to $x = x(t)$:

$$(\alpha at^2 + \beta at + \gamma a + bc)x'_t = ax + c.$$

34. $2x^2y'_x = 2y^2 + xy - 2a^2x$.

Particular solution: $y_0 = a\sqrt{x}$.

35. $2x^2y'_x = 2y^2 + 3xy - 2a^2x$.

Particular solution: $y_0 = a\sqrt{x} - \dfrac{x}{2}$.

36. $x^2y'_x = ax^2y^2 + b$.

Solution:

$$y = \frac{\lambda}{x} - x^{2a\lambda}\left(\frac{ax}{2a\lambda + 1}x^{2a\lambda} + C\right)^{-1},$$

where λ is a root of the quadratic equation $a\lambda^2 + \lambda + b = 0$.

37. $x^2y'_x = ax^2y^2 + bxy + c$.

The substitution $w = xy$ leads to an equation with separation of variables:

$$xw'_x = aw^2 + (b+1)w + c.$$

38. $x^2y'_x = x^2y^2 - a^2x^4 + a(1-2b)x^2 - b(b+1)$.

Particular solution: $y_0 = ax + bx^{-1}$.

39. $\boldsymbol{x^2y'_x = cx^2y^2 + (ax^2 + bx)y + \alpha x^2 + \beta x + \gamma.}$

The substitution $cy = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.134:

$$x^2u''_{xx} - x(ax + b)u'_x + c(\alpha x^2 + \beta x + \gamma)u = 0.$$

40. $\boldsymbol{x^2y'_x = ax^2y^2 + bx^n + c.}$

Having set $w = xy + A$, where A is a root of the quadratic equation $aA^2 - A + c = 0$, we arrive at an equation of the form 1.2.2.22: $xw'_x = aw^2 + (1 - 2aA)w + bx^n$.

41. $\boldsymbol{x^2y'_x = x^2y^2 + \dfrac{1 - n^2}{4} + ax^{2m}(bx^m + c)^n.}$

The transformation

$$\xi = bx^m + c, \quad w = \frac{1}{bm}x^{1-m}y + \frac{1 - m}{2bm}x^{-m}$$

leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + a(bm)^{-2}\xi^n$.

42. $\boldsymbol{x^2y'_x = ax^2y^2 + bxy + cx^m + s.}$

The substitution $ay = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.127: $x^2u''_{xx} - bxu'_x + a(cx^m + s)u = 0$.

43. $\boldsymbol{x^2y'_x = ax^2y^2 + bxy + cx^{2m} + sx^m.}$

The substitution $ay = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.128:

$$x^2u''_{xx} - bxu'_x + ax^m(cx^m + s)u = 0.$$

44. $\boldsymbol{x^2y'_x = cx^2y^2 + (ax^n + b)xy + \alpha x^{2n} + \beta x^n + \gamma.}$

The substitution $cy = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.141:

$$x^2u''_{xx} - (ax^n + b)xu'_x + c(\alpha x^{2n} + \beta x^n + \gamma)u = 0.$$

45. $\boldsymbol{x^2y'_x = (\alpha x^{2n} + \beta x^n + \gamma)y^2 + (ax^n + b)xy + cx^2.}$

The substitution $y = -1/w$ leads to an equation of the form 1.2.2.44:

$$x^2w'_x = cx^2w^2 - (ax^n + b)xw + \alpha x^{2n} + \beta x^n + \gamma.$$

46. $\boldsymbol{(x^2 - 1)y'_x + \lambda(y^2 - 2xy + 1) = 0.}$

The substitution $y = \dfrac{2\lambda - 1}{\lambda}x + \dfrac{1 - \lambda}{\lambda}\dfrac{1}{u(x)}$ leads to an equation of the same form:

$$(x^2 - 1)u'_x + (\lambda - 1)(u^2 - 2xu + 1) = 0.$$

If $\lambda = n$ is a positive integer, then by using the above substitution, the original equation can be reduced to an equation of the same form, wherein $\lambda = 1$, i.e., to an equation of the form 1.2.2.49 with $a = 1$, $b = -1$.

47. $\boldsymbol{(ax^2+b)y'_x+\alpha y^2+\beta xy+\dfrac{b}{\alpha}(a+\beta)=0.}$

Particular solution: $y_0=-\dfrac{a+\beta}{\alpha}x.$

48. $\boldsymbol{(ax^2+b)y'_x+\alpha y^2+\beta xy+\gamma=0.}$

The substitution $y=-\dfrac{a+\beta}{\alpha}x-\dfrac{1}{u(x)}$ leads to an equation of the same form:

$$(ax^2+b)u'_x+\Bigl(\gamma-\frac{a+\beta}{\alpha}b\Bigr)u^2+(2a+\beta)xu+\alpha=0.$$

49. $\boldsymbol{(ax^2+b)y'_x+y^2-2xy+(1-a)x^2-b=0.}$

Solution: $y=x+\left(\displaystyle\int\frac{dx}{ax^2+b}+C\right)^{-1}.$

50. $\boldsymbol{(ax^2+bx+c)y'_x=y^2+(2\lambda x+b)y+\lambda(\lambda-a)x^2+\mu.}$

Particular solutions:

$$y_0=-\lambda x+A,\qquad\text{where}\quad A=\tfrac{1}{2}\bigl(-b\pm\sqrt{b^2-4\mu-4\lambda c}\,\bigr).$$

51. $\boldsymbol{(ax^2+bx+c)y'_x=y^2+(ax+\mu)y-\lambda^2x^2+\lambda(b-\mu)x+\lambda c.}$

Particular solution: $y_0=\lambda x.$

52. $\boldsymbol{(a_2x^2+b_2x+c_2)y'_x=y^2+(a_1x+b_1)y-\lambda(\lambda+a_1-a_2)x^2+\lambda(b_2-b_1)x+\lambda c_2.}$

Particular solution: $y_0=\lambda x.$

53. $\boldsymbol{(a_2x^2+b_2x+c_2)y'_x=y^2+(a_1x+b_1)y+a_0x^2+b_0x+c_0.}$

Let λ and β be roots of the system of the quadratic equations

$$\lambda^2+\lambda(a_1-a_2)+a_0=0,\quad \beta^2+\beta b_1+c_0-\lambda c_2=0,$$

that are solved consecutively (in the general case there are four roots). If some of roots satisfy the condition $2\lambda\beta+\lambda b_1+\beta a_1+b_0-\lambda b_2=0$, the original equation posesses a paticular solution: $y_0=\lambda x+\beta$.

54. $\boldsymbol{(x-a)(x-b)y'_x+y^2+k(y+x-a)(y+x-b)=0.}$

To the case of $k=0$ corresponds an equation with separation of variables. To the case of $k=-1$ corresponds a linear equation. For $k\neq-1$ and 0, with the aid of the substitution $ku(x)=y+k(y+x)$, we obtain the general solution:

$$\frac{y+k(y+x-a)}{y+k(y+x-b)}\Bigl(\frac{x-a}{x-b}\Bigr)^k=C\qquad\text{if}\ \ a\neq b,$$
$$\frac{1}{y+k(y+x-a)}+\frac{1}{x-a}=C\qquad\text{if}\ \ a=b.$$

55. $(c_2x^2 + b_2x + a_2)(y'_x + \lambda y^2) + (b_1x + a_1)y + a_0 = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.166:

$$(c_2x^2 + b_2x + a_2)u''_{xx} + (b_1x + a_1)u'_x + \lambda a_0 u = 0.$$

56. $x^3 y'_x = ax^3y^2 + (bx^2 + c)y + sx.$

The substitution $ay = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.170:

$$x^3 u''_{xx} - (bx^2 + c)u'_x + asxu = 0.$$

57. $x^3 y'_x = ax^3y^2 + x(bx + c)y + \alpha x + \beta.$

The substitution $ay = -u'_x/u$ leads to a second order linear equation of the form 2.1.2.173:

$$x^3 u''_{xx} - x(bx + c)u'_x + a(\alpha x + \beta)u = 0.$$

58. $x(x^2 + a)(y'_x + \lambda y^2) + (bx^2 + c)y + sx = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.177:

$$x(x^2 + a)u''_{xx} + (bx^2 + c)u'_x + \lambda sxu = 0.$$

59. $x^2(x + a)(y'_x + \lambda y^2) + x(bx + c)y + \alpha x + \beta = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.181:

$$x^2(x + a)u''_{xx} + x(bx + c)u'_x + \lambda(\alpha x + \beta)u = 0.$$

60. $(ax^2 + bx + c)(xy'_x - y) - y^2 + x^2 = 0.$

Solution: $\ln\left|\dfrac{y - x}{y + x}\right| = C + 2\displaystyle\int \frac{dx}{ax^2 + bx + c}.$

61. $x^4 y'_x = -x^4y^2 - a^2.$

Solution: $y = \dfrac{1}{x} + \dfrac{a}{x^2}\tan\left(\dfrac{a}{x} + C\right).$

62. $x^2(x^2 + a)(y'_x + \lambda y^2) + x(bx^2 + c)y + s = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.206:

$$x^2(x^2 + a)u''_{xx} + x(bx^2 + c)u'_x + \lambda su = 0.$$

63. $ax^2(x - 1)^2(y'_x + \lambda y^2) + bx^2 + cx + s = 0.$

The substitution $\lambda y = u'_x/u$ leads to a second order linear equation of the form 2.1.2.205:

$$ax^2(x - 1)^2 u''_{xx} + \lambda(bx^2 + cx + s)u = 0.$$

64. $a(x^2-1)^2(y'_x+\lambda y^2)+bx(x^2-1)y+cx^2+dx+s=0.$

The substitution $\lambda y=u'_x/u$ leads to a second order linear equation of the form 2.1.2.213:

$$a(x^2-1)^2u''_{xx}+bx(x^2-1)u'_x+\lambda(cx^2+dx+s)u=0.$$

65. $(ax^2+bx+c)^2(y'_x+y^2)+A=0.$

The substitution $y=u'_x/u$ leads to a second order linear equation of the form 2.1.2.220:

$$(ax^2+bx+c)^2u''_{xx}+Au=0.$$

66. $x^{n+1}y'_x=ax^{2n}y^2+bx^ny+cx^m+d.$

Having set $w=x^ny+A$, where A is a root of the quadratic equation $aA^2-(b+n)A+d=0$, we arrive at an equation of the form 1.2.2.22:

$$xw'_x=aw^2+(n+b-2aA)w+cx^m.$$

67. $x(ax^k+b)y'_x=\alpha x^ny^2+(\beta-anx^k)y+\gamma x^{-n}.$

The transformation $t=x^ny$, $z=x^{-k}$ leads to an equation with separation of variables:

$$[\alpha t^2+(\beta+bn)t+\gamma]z'_t=-k(bz+a).$$

68. $x^2(ax^b-1)(y'_x+\lambda y^2)+(px^b+q)xy+rx^b+s=0.$

The substitution $\lambda y=u'_x/u$ leads to a second order linear equation of the form 2.1.2.238:

$$x^2(ax^b-1)u''_{xx}+(px^b+q)xu'_x+\lambda(rx^b+s)u=0.$$

69. $(ax^n+bx^m+c)y'_x=cy^2-bx^{m-1}y+ax^{n-2}.$

Particular solution: $y_0=-1/x.$

70. $(ax^n+bx^m+c)y'_x=ax^{n-2}y^2+bx^{m-1}y+c.$

Particular solution: $y_0=x.$

71. $(ax^n+bx^m+c)y'_x=\alpha x^ky^2+\beta x^sy-\alpha\lambda^2x^k+\beta\lambda x^s.$

Particular solution: $y_0=-\lambda.$

72. $(ax^n+bx^m+c)(xy'_x-y)+sx^k(y^2-\lambda x^2)=0.$

Particular solutions: $y_0=\pm x\sqrt{\lambda}.$

73. $(ax^n+bx^m+c)(y'_x-y^2)+an(n-1)x^{n-2}+bm(m-1)x^{m-2}=0.$

Particular solution: $y_0=-\dfrac{anx^{n-1}+bmx^{m-1}}{ax^n+bx^m+c}.$

1.2.3. Equations Containing Exponential Functions

1. $y'_x = ay^2 + be^{\lambda x}$.

The substitution $t = e^{\lambda x}$ leads to an equation of the form 1.2.2.22: $\lambda t y'_t = ay^2 + bt$.

2. $y'_x = y^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

3. $y'_x = \sigma y^2 + a + be^{\lambda x} + ce^{2\lambda x}$.

The substitution $\sigma y = -u'_x/u$ leads to a second order linear equation of the form 2.1.3.5: $u''_{xx} + \sigma(a + be^{\lambda x} + ce^{2\lambda x})u = 0$.

4. $y'_x = \sigma y^2 + ay + be^x + c$.

The substitution $\sigma y = -u'_x/u$ leads to a second order linear equation of the form 2.1.3.10: $u''_{xx} - au'_x + \sigma(be^x + c)u = 0$.

5. $y'_x = y^2 + by + a(\lambda - b)e^{\lambda x} - a^2 e^{2\lambda x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

6. $y'_x = y^2 + ae^{\lambda x} y - abe^{\lambda x} - b^2$.

Particular solution: $y_0 = b$.

7. $y'_x = y^2 + ae^{2\lambda x}(e^{\lambda x} + b)^n - \frac{1}{4}\lambda^2$.

The transformation

$$\xi = e^{\lambda x} + b, \quad w = \frac{1}{\lambda}\Big(e^{-\lambda x} y - \frac{\lambda}{2} e^{-\lambda x}\Big)$$

leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + a\lambda^{-2}\xi^n$.

8. $y'_x = y^2 + ae^{8\lambda x} + be^{6\lambda x} + ce^{4\lambda x} - \lambda^2$.

The transformation

$$\xi = e^{2\lambda x}, \quad w = e^{-2\lambda x}\Big(\frac{y}{2\lambda} - \frac{1}{2}\Big)$$

leads to an equation of the form 1.2.2.3: $w''_{\xi\xi} = w^2 + (2\lambda)^{-2}(a\xi^2 + b\xi + c)$.

9. $y'_x = y^2 + axe^{\lambda x} y + ae^{\lambda x}$.

Particular solution: $y_0 = -1/x$.

10. $y'_x = ae^{\lambda x} y^2 + be^{-\lambda x}$.

Solution: $\displaystyle\int \frac{dz}{az^2 + \lambda z + b} = x + C$, where $z = e^{\lambda x} y$.

11. $y'_x = ae^{kx} y^2 + be^{sx}$, $k \neq 0$.

The substitution $t = e^{kx}$ leads to an equation of the form 1.2.2.4: $ky'_t = ay^2 + bt^{s-k}$.

12. $y'_x = be^{\mu x}y^2 + a\lambda e^{\lambda x} - a^2 b e^{(\mu+2\lambda)x}$.

Particular solution: $y_0 = ae^{\lambda x}$.

13. $y'_x = ae^{\lambda x}y^2 + by + ce^{-\lambda x}$.

The substitution $z = e^{\lambda x}y$ leads to an equation with separation of variables: $z'_x = az^2 + (b+\lambda)z + c$.

14. $y'_x = ae^{\mu x}y^2 + \lambda y - ab^2 e^{(\mu+2\lambda)x}$.

Particular solution: $y_0 = be^{\lambda x}$.

15. $y'_x = e^{\lambda x}y^2 + ae^{\mu x}y + a\lambda e^{(\mu-\lambda)x}$.

Particular solution: $y_0 = -\lambda e^{-\lambda x}$.

16. $y'_x = -\lambda e^{\lambda x}y^2 + ae^{\mu x}y - ae^{(\mu-\lambda)x}$.

Particular solution: $y_0 = e^{-\lambda x}$.

17. $y'_x = ae^{\mu x}y^2 + abe^{(\lambda+\mu)x}y - b\lambda e^{\lambda x}$.

Particular solution: $y_0 = -be^{\lambda x}$.

18. $y'_x = ae^{kx}y^2 + by + ce^{sx} + de^{-kx}$.

The substitution $t = e^{kx}$ leads to an equation of the form 1.2.2.42:

$$kt^2 y'_t = at^2y^2 + bty + ct^{(k+s)/k} + d.$$

19. $y'_x = ae^{(2\lambda+\mu)x}y^2 + [be^{(\lambda+\mu)x} - \lambda]y + ce^{\mu x}$.

The substitution $w = e^{\lambda x}y$ leads to an equation with separation of variables:

$$w'_x = e^{(\lambda+\mu)x}(aw^2 + bw + c).$$

20. $y'_x = ae^{kx}y^2 + by + ce^{knx} + de^{k(2n+1)x}$.

The substitution $t = e^{kx}$ leads to an equation of the form 1.2.2.43:

$$kt^2 y'_t = at^2y^2 + bty + ct^{n+1} + dt^{2(n+1)}.$$

21. $y'_x = e^{\mu x}(y - be^{\lambda x})^2 + b\lambda e^{\lambda x}$.

Particular solution: $y_0 = be^{\lambda x}$.

22. $y'_x = ae^{\lambda x}y^2 + bnx^{n-1} - ab^2 e^{\lambda x}x^{2n}$.

Particular solution: $y_0 = bx^n$.

23. $y'_x = e^{\lambda x}y^2 + ax^n y + a\lambda x^n e^{-\lambda x}$.

Particular solution: $y_0 = -\lambda e^{-\lambda x}$.

24. $y'_x = -\lambda e^{\lambda x} y^2 + a x^n e^{\lambda x} - a x^n.$

Particular solution: $y_0 = e^{-\lambda x}$.

25. $y'_x = a e^{\lambda x} y^2 - a b x^n e^{\lambda x} y + b n x^{n-1}.$

Particular solution: $y_0 = b x^n$.

26. $y'_x = a x^n y^2 + b\lambda e^{\lambda x} - a b^2 x^n e^{2\lambda x}.$

Particular solution: $y_0 = b e^{\lambda x}$.

27. $y'_x = a x^n y^2 + \lambda y - a b^2 x^n e^{2\lambda x}.$

Particular solution: $y_0 = b e^{\lambda x}$.

28. $y'_x = a x^n y^2 - a b x^n e^{\lambda x} y + b\lambda e^{\lambda x}.$

Particular solution: $y_0 = b e^{\lambda x}$.

29. $y'_x = -(k+1) x^k y^2 + a x^{k+1} e^{\lambda x} y - a e^{\lambda x}.$

Particular solution: $y_0 = x^{-k-1}$.

30. $y'_x = a x^n y^2 - a x^n (b e^{\lambda x} + c) y + b\lambda e^{\lambda x}.$

Particular solution: $y_0 = b e^{\lambda x} + c$.

31. $y'_x = a x^n e^{2\lambda x} y^2 + (b x^n e^{\lambda x} - \lambda) y + c x^n.$

The substitution $w = e^{\lambda x} y$ leads to an equation with separation of variables:

$$w'_x = x^n e^{\lambda x} (a w^2 + b w + c).$$

32. $y'_x = a e^{\lambda x} (y - b x^n - c)^2 + b n x^{n-1}.$

Particular solution: $y_0 = b x^n + c$.

33. $x y'_x = a e^{\lambda x} y^2 + k y + a b^2 x^{2k} e^{\lambda x}.$

Solution: $y = b x^k \tan\Bigl(ab \displaystyle\int x^{k-1} e^{\lambda x}\, dx + C\Bigr)$.

34. $x y'_x = a x^{2n} e^{\lambda x} y^2 + (b x^n e^{\lambda x} - n) y + c e^{\lambda x}.$

Solution: $\displaystyle\int \frac{dw}{a w^2 + b w + c} = \int x^{n-1} e^{\lambda x}\, dx + C, \qquad \text{where} \quad w = x^n y.$

35. $(a e^{\lambda x} + b e^{\mu x} + c) y'_x = y^2 + k e^{\nu x} y - m^2 + k m e^{\nu x}.$

Particular solution: $y_0 = -m$.

36. $(a e^{\lambda x} + b e^{\mu x} + c)(y'_x - y^2) + a\lambda^2 e^{\lambda x} + b\mu^2 e^{\mu x} = 0.$

Particular solution: $y_0 = -\dfrac{a\lambda e^{\lambda x} + b\mu e^{\mu x}}{a e^{\lambda x} + b e^{\mu x} + c}$.

37. $y'_x = y^2 + 2a\lambda x e^{\lambda x^2} - a^2 e^{2\lambda x^2}$.

Particular solution: $y_0 = ae^{\lambda x^2}$.

38. $y'_x = ae^{-\lambda x^2} y^2 + \lambda xy + ab^2$.

Solution: $y = be^{\lambda x^2/2} \tan\Big(ab \int e^{-\lambda x^2/2}\,dx + C\Big)$.

39. $y'_x = ax^n y^2 + \lambda xy + ab^2 x^n e^{\lambda x^2}$.

Solution: $y = be^{\lambda x^2/2} \tan\Big(ab \int x^n e^{\lambda x^2/2}\,dx + C\Big)$.

40. $x^4(y'_x - y^2) = a + b\exp\Big(\dfrac{k}{x}\Big) + c\exp\Big(\dfrac{2k}{x}\Big)$.

The transformation $\xi = 1/x$, $w = -x^2 y - x$ leads to an equation of the form 1.2.3.3: $w'_\xi = w^2 + a + be^{k\xi} + ce^{2k\xi}$.

1.2.4. Equations Containing Hyperbolic Functions

1. $y'_x = \alpha y^2 + \beta + \gamma\cosh x$.

The transformation $x = 2t$, $\alpha y = -u'_x/u$ leads to the modified Mathieu equation 2.1.4.1:

$$u''_{tt} - (a - 2q\cosh 2t)u = 0, \qquad \text{where} \quad a = -4\alpha\beta, \quad q = 2\alpha\gamma.$$

2. $y'_x = y^2 - a^2 + a\lambda\sinh(\lambda x) - a^2\sinh^2(\lambda x)$.

Particular solution: $y_0 = a\cosh(\lambda x)$.

3. $y'_x = y^2 - \lambda^2 + a\cosh^n(\lambda x)\sinh^{-n-4}(\lambda x)$.

The transformation

$$\xi = \coth(\lambda x), \quad w = -\frac{1}{\lambda}\sinh^2(\lambda x) y - \sinh(\lambda x)\cosh(\lambda x)$$

leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + \lambda^{-2}\xi^n$.

4. $y'_x = y^2 + a\lambda - a(a+\lambda)\tanh^2(\lambda x)$.

Particular solution: $y_0 = a\tanh(\lambda x)$.

5. $y'_x = y^2 + 3a\lambda - \lambda^2 - a(a+\lambda)\tanh^2(\lambda x)$.

Particular solution: $y_0 = a\tanh(\lambda x) - \lambda\coth(\lambda x)$.

6. $y'_x = y^2 + a\lambda - a(a+\lambda)\coth^2(\lambda x)$.

Particular solution: $y_0 = a\coth(\lambda x)$.

7. $y'_x = y^2 - \lambda^2 + 3a\lambda - a(a+\lambda)\coth^2(\lambda x)$.

Particular solution: $y_0 = a\coth(\lambda x) - \lambda\tanh(\lambda x)$.

8. $y'_x = y^2 - 2\lambda^2\tanh^2(\lambda x) - 2\lambda^2\coth^2(\lambda x)$.

Particular solution: $y_0 = \lambda\tanh(\lambda x) + \lambda\coth(\lambda x)$.

9. $y'_x = y^2 + a\lambda + b\lambda - 2ab - a(a+\lambda)\tanh^2(\lambda x) - b(b+\lambda)\coth^2(\lambda x)$.

Particular solution: $y_0 = a\tanh(\lambda x) + b\coth(\lambda x)$.

10. $y'_x = \lambda\sinh(\lambda x)y^2 - \lambda\sinh^3(\lambda x)$.

Particular solution: $y_0 = \cosh(\lambda x)$.

11. $y'_x = a\sinh(\lambda x)y^2 + b\sinh(\lambda x)\cosh^n(\lambda x)$.

The transformation $\xi = \cosh(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

12. $y'_x = a\cosh(\lambda x)y^2 + b\cosh(\lambda x)\sinh^n(\lambda x)$.

The transformation $\xi = \sinh(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

13. $y'_x = [a\sinh^2(\lambda x) - \lambda]y^2 - a\sinh^2(\lambda x) + \lambda - a$.

Particular solution: $y_0 = \coth(\lambda x)$.

14. $y'_x = [a\cosh^2(\lambda x) - \lambda]y^2 + a + \lambda - a\cosh^2(\lambda x)$.

Particular solution: $y_0 = \tanh(\lambda x)$.

15. $2y'_x = [a - \lambda + a\cosh(\lambda x)]y^2 + a + \lambda - a\cosh(\lambda x)$.

Particular solution: $y_0 = \tanh\Bigl(\dfrac{\lambda x}{2}\Bigr)$.

1.2.5. Equations Containing Logarithmic Functions

1. $y'_x = y^2 + a\ln(\beta x)y - ab\ln(\beta x) - b^2$.

Particular solution: $y_0 = b$.

2. $y'_x = y^2 + ax\ln^m(bx)y + a\ln^m(bx)$.

Particular solution: $y_0 = -1/x$.

3. $y'_x = ax^n y^2 - abx^{n+1}\ln x\,y + b\ln x + b$.

Particular solution: $y_0 = bx\ln x$.

4. $y'_x = -(n+1)x^n y^2 + a x^{n+1}(\ln x)^m y - a(\ln x)^m.$

Particular solution: $y_0 = x^{-n-1}$.

5. $y'_x = a(\ln x)^n y^2 + bm x^{m-1} - ab^2 x^{2m}(\ln x)^n.$

Particular solution: $y_0 = bx^m$.

6. $y'_x = a(\ln x)^n y^2 - abx(\ln x)^{n+1} y + b\ln x + b.$

Particular solution: $y_0 = bx\ln x$.

7. $y'_x = a(\ln x)^k (y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

8. $y'_x = a(\ln x)^n y^2 + b(\ln x)^m y + bc(\ln x)^m - ac^2(\ln x)^n.$

Particular solution: $y_0 = -c$.

9. $xy'_x = ay^2 + b\ln x + c.$

The substitution $x = e^t$ leads to an equation of the form 1.2.2.1: $y'_t = ay^2 + bt + c$.

10. $xy'_x = ay^2 + b\ln^k x + c\ln^{2k+2} x.$

The substitution $t = \ln x$ leads to an equation of the form 1.2.2.6 with $k = n - 1$: $y'_t = ay^2 + bt^k + ct^{2k+2}$.

11. $xy'_x = (ay + b\ln x)^2.$

Solution: $\displaystyle \ln x = \int \frac{dz}{az^2 + b} + C,$ where $z = ay + b\ln x$.

12. $xy'_x = xy^2 - A^2\ln^2(\beta x) + A.$

Particular solution: $y_0 = A\ln(\beta x)$.

13. $xy'_x = xy^2 - A^2 x\ln^{2k}(\beta x) + Ak\ln^{k-1}(\beta x).$

Particular solution: $y_0 = A\ln^k(\beta x)$.

14. $xy'_x = ax^n y^2 + b + ab^2 x^n \ln^2 x.$

Particular solution: $y_0 = b\ln x$.

15. $xy'_x = a\ln^m(\lambda x) y^2 + ky + ab^2 x^{2k}\ln^m(\lambda x).$

Solution: $\displaystyle y = bx^k \tan\Big[ab\int x^{k-1}\ln^m(\lambda x)\,dx + C\Big].$

16. $xy'_x = ax^n(y + b\ln x)^2 - b.$

Solution: $\displaystyle \frac{1}{y + b\ln x} + \frac{a}{n}x^n = C.$

17. $\boldsymbol{xy'_x = ax^{2n}(\ln x)y^2 + (bx^n \ln x - n)y + c\ln x}.$

Solution: $\displaystyle\int \frac{dw}{aw^2+bw+c} = \int x^{n-1}\ln x\,dx + C, \qquad \text{where} \quad w = x^n y.$

18. $\boldsymbol{x^2y'_x = x^2y^2 + a\ln^2 x + b\ln x + c}.$

The transformation $\xi = \ln x$, $w = xy + \frac{1}{2}$ leads to an equation of the form 1.2.2.3: $w'_\xi = w^2 + a\xi^2 + b\xi + c - \frac{1}{4}$.

19. $\boldsymbol{x^2y'_x = x^2y^2 + a(b\ln x + c)^n + \tfrac{1}{4}}.$

The transformation

$$\xi = b\ln x + c, \quad w = \frac{x}{b}y + \frac{1}{2b}$$

leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + ab^{-2}\xi^n$.

20. $\boldsymbol{x^2y'_x = a^2x^2y^2 - xy + b^2\ln^m x}.$

The substitution $a^2y = -u'_x/u$ leads to an equation of the form a second order linear equation of the form 2.1.5.24: $x^2u''_{xx} + xu'_x + (ab)^2\ln^m x\,u = 0$.

21. $\boldsymbol{x^2\ln(ax)(y'_x - y^2) = 1}.$

Particular solution: $y_0 = [x\ln(ax)]^{-1}$.

22. $\boldsymbol{(a\ln x + b)y'_x = y^2 + c(\ln x)^n y - \lambda^2 + \lambda c(\ln x)^n}.$

Particular solution: $y_0 = -\lambda$.

23. $\boldsymbol{(a\ln x + b)y'_x = (\ln x)^n y^2 + cy - \lambda^2(\ln x)^n + c\lambda}.$

Particular solution: $y_0 = -\lambda$.

1.2.6. Equations Containing Trigonometric Functions

1. $\boldsymbol{y'_x = \alpha y^2 + \beta + \gamma\sin(\lambda x)}.$

The substitution $2t = 2\lambda x + \pi$ leads to an equation of the form 1.2.6.2: $\lambda y'_x = \alpha y^2 + \beta + \gamma\cos t$.

2. $\boldsymbol{y'_x = \alpha y^2 + \beta + \gamma\cos x}.$

The transformation $x = 2t$, $\alpha y = -u'_x/u$ leads to the Mathieu equation 2.1.6.4:

$$u''_{xx} + (a - 2q\cos 2t)u = 0, \qquad \text{where} \quad a = 4\alpha\beta, \quad q = -2\alpha\gamma.$$

3. $\boldsymbol{y'_x = y^2 - a^2 + a\lambda\sin(\lambda x) + a^2\sin^2(\lambda x)}.$

Particular solution: $y_0 = -a\cos(\lambda x)$.

4. $\boldsymbol{y'_x = y^2 - a^2 + a\lambda\cos(\lambda x) + a^2\cos^2(\lambda x)}.$

Particular solution: $y_0 = a\sin(\lambda x)$.

5. $\boldsymbol{y'_x = y^2 + \lambda^2 + c\sin^n(\lambda x)\cos^{-n-4}(\lambda x).}$

This is a special case of equation 1.2.6.6 with $a = 0$, $b = \pi/2$.

6. $\boldsymbol{y'_x = y^2 + \lambda^2 + c\sin^n(\lambda x + a)\sin^{-n-4}(\lambda x + b).}$

The transformation

$$\xi = \frac{\sin(\lambda x + a)}{\sin(\lambda x + b)}, \quad w = \frac{\sin^2(\lambda x + b)}{\sin(b - a)}\left[\frac{y}{\lambda} + \cot(\lambda x + b)\right]$$

leads to an equation of the form 1.2.2.4:

$$w'_\xi = w^2 + A\xi^n, \qquad \text{where} \quad A = c[\lambda\sin(b - a)]^{-2}.$$

7. $\boldsymbol{y'_x = y^2 + a\sin(\beta x)y + ab\sin(\beta x) - b^2.}$

Particular solution: $y_0 = -b$.

8. $\boldsymbol{y'_x = y^2 + ax\sin^m(bx)y + a\sin^m(bx).}$

Particular solution: $y_0 = -1/x$.

9. $\boldsymbol{y'_x = y^2 + a\lambda + a(\lambda - a)\tan^2(\lambda x).}$

Particular solution: $y_0 = a\tan(\lambda x)$.

10. $\boldsymbol{y'_x = y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\tan^2(\lambda x).}$

Particular solution: $y_0 = a\tan(\lambda x) - \lambda\cot(\lambda x)$.

11. $\boldsymbol{y'_x = y^2 + a\lambda + a(\lambda - a)\cot^2(\lambda x).}$

Particular solution: $y_0 = -a\cot(\lambda x)$.

12. $\boldsymbol{y'_x = y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\cot^2(\lambda x).}$

Particular solution: $y_0 = \lambda\tan(\lambda x) - a\cot(\lambda x)$.

13. $\boldsymbol{y'_x = ay^2 + b\tan x\, y + c.}$

Having set $ay = -u'_x/u$, we obtain a second order linear equation of the form 2.1.6.29: $u''_{xx} - b\tan x\, u'_x + acu = 0$.

14. $\boldsymbol{y'_x = ay^2 + 2ab\,\tan x\, y + b(ab - 1)\tan^2 x.}$

The substitution $u = y + b\tan x$ leads to an equation of the form 1.1.2: $u'_x = au^2 + b$.

15. $\boldsymbol{y'_x = y^2 - y\tan x + a(1 - a)\cot^2 x.}$

Particular solution: $y_0 = -a\cot x$.

16. $\boldsymbol{y'_x = y^2 - my\tan x + b^2\cos^{2m} x.}$

Solution: $y = -b\cos^m x\cot\Bigl(b\displaystyle\int\cos^m x\,dx + C\Bigr)$.

17. $y'_x = y^2 - 2a\cot(ax)y + b^2 - a^2.$

Particular solution: $y_0 = a\cot(ax) - b\cot(bx)$.

18. $y'_x = y^2 + my\cot x + b^2(\sin x)^{2m}.$

Solution: $y = -b\sin^m x \cot\Bigl(b\int \sin^m x\,dx + C\Bigr)$.

19. $y'_x = y^2 - 2\lambda^2\tan^2(\lambda x) - 2\lambda^2\cot^2(\lambda x).$

Particular solution: $y_0 = \lambda\cot(\lambda x) - \lambda\tan(\lambda x)$.

20. $y'_x = y^2 + \lambda a + \lambda b + 2ab + a(\lambda - a)\tan^2(\lambda x) + b(\lambda - b)\cot^2(\lambda x).$

Particular solution: $y_0 = a\tan(\lambda x) - b\cot(\lambda x)$.

21. $y'_x = y^2 + ax\tan^m(bx)y + a\tan^m(bx).$

Particular solution: $y_0 = -1/x$.

22. $y'_x = y^2 - \frac{1}{2}\lambda^2 - \frac{3}{4}\lambda^2\tan^2(\lambda x) + a\cos^2(\lambda x)\sin^n(\lambda x).$

The transformation

$$\xi = \sin(\lambda x), \quad w = \frac{y}{\lambda\cos(\lambda x)} + \frac{\sin(\lambda x)}{2\cos^2(\lambda x)}$$

leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + a\lambda^{-2}\xi^n$.

23. $y'_x = \lambda\sin(\lambda x)y^2 + \lambda\sin^3(\lambda x).$

Particular solution: $y_0 = -\cos(\lambda x)$.

24. $y'_x = \lambda\cos(\lambda x)y^2 + \lambda\cos^3(\lambda x).$

Particular solution: $y_0 = \sin(\lambda x)$.

25. $2y'_x = [\lambda + a - a\sin(\lambda x)]y^2 + \lambda - a - a\sin(\lambda x).$

Particular solution: $y_0 = \tan\Bigl(\dfrac{\lambda x}{2} + \dfrac{\pi}{4}\Bigr)$.

26. $2y'_x = [\lambda + a + a\cos(\lambda x)]y^2 + \lambda - a + a\cos(\lambda x).$

Particular solution: $y_0 = \tan\Bigl(\dfrac{\lambda x}{2}\Bigr)$.

27. $y'_x = [\lambda + a\sin^2(\lambda x)]y^2 + \lambda - a + a\sin^2(\lambda x).$

Particular solution: $y_0 = -\cot(\lambda x)$.

28. $y'_x = [\lambda + a\cos^2(\lambda x)]y^2 + \lambda - a + a\cos^2(\lambda x).$

Particular solution: $y_0 = \tan(\lambda x)$.

29. $y'_x = a\sin(\lambda x)y^2 + b\sin(\lambda x)\cos^n(\lambda x).$

The transformation $\xi = \cos(\lambda x)$, $w = -\dfrac{a}{\lambda}y$ leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

30. $y'_x = \lambda\sin(\lambda x)y^2 + a\cos^n(\lambda x)y - a\cos^{n-1}(\lambda x).$

Particular solution: $y_0 = 1/\cos(\lambda x)$.

31. $y'_x = a\cos(\lambda x)y^2 + b\cos(\lambda x)\sin^n(\lambda x).$

The transformation $\xi = \sin(\lambda x)$, $w = \dfrac{a}{\lambda}y$ leads to an equation of the form 1.2.2.4: $w'_\xi = w^2 + ab\lambda^{-2}\xi^n$.

32. $y'_x = \lambda\sin(\lambda x)y^2 + a\sin(\lambda x)y - a\tan(\lambda x).$

Particular solution: $y_0 = 1/\cos(\lambda x)$.

33. $y'_x = \lambda\sin(\lambda x)y^2 + ax^n\cos(\lambda x)y - ax^n.$

Particular solution: $y_0 = 1/\cos(\lambda x)$.

34. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\sin x)^m y - a(\sin x)^m.$

Particular solution: $y_0 = x^{-k-1}$.

35. $y'_x = -(k+1)x^k y^2 + ax^{k+1}(\tan x)^m y - a(\tan x)^m.$

Particular solution: $y_0 = x^{-k-1}$.

36. $y'_x = a\sin^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

37. $y'_x = a\tan^n(\lambda x)y^2 - ab^2\tan^{n+2}(\lambda x) + b\lambda\tan^2(\lambda x) + b\lambda.$

Particular solution: $y_0 = b\tan(\lambda x)$.

38. $y'_x = a\tan^k(\lambda x + \mu)(y - bx^n - c)^2 + bnx^{n-1}.$

Particular solution: $y_0 = bx^n + c$.

39. $xy'_x = a\sin^m(\lambda x)y^2 + ky + ab^2x^{2k}\sin^m(\lambda x).$

Solution: $y = bx^k\tan\Big[ab\displaystyle\int x^{k-1}\sin^m(\lambda x)\,dx + C\Big]$.

40. $xy'_x = a\tan^m(\lambda x)y^2 + ky + ab^2x^{2k}\tan^m(\lambda x).$

Solution: $y = bx^k\tan\Big[ab\displaystyle\int x^{k-1}\tan^m(\lambda x)\,dx + C\Big]$.

41. $\sin^{n+1}(2x)y'_x = ay^2\sin^{2n}x + b\cos^{2n}x.$

The substitution $z = y\tan^n x$ leads to an equation of the form an equation with separation of variables: $2^n\sin(2x)z'_x = az^2 + n2^{n+1}z + b$.

42. $[a\sin(\lambda x) + b]y'_x = y^2 + c\sin(\mu x)y - d^2 + cd\sin(\mu x).$

Particular solution: $y_0 = -d$.

43. $[a\tan(\lambda x) + b]y'_x = y^2 + k\tan(\mu x)y - d^2 + kd\tan(\mu x).$

Particular solution: $y_0 = -d$.

1.2.7. Equations Containing Inverse Trigonometric Functions

► *In the equations 1–9, function* $\arccos x$ *may be substituted for* $\arcsin x$.

1. $y'_x = y^2 + \lambda(\arcsin x)^n y - a^2 + a\lambda(\arcsin x)^n.$

Particular solution: $y_0 = -a$.

2. $y'_x = y^2 + \lambda x(\arcsin x)^n y + \lambda(\arcsin x)^n.$

Particular solution: $y_0 = -1/x$.

3. $y'_x = -(k+1)x^k y^2 + \lambda(\arcsin x)^n(x^{k+1}y - 1).$

Particular solution: $y_0 = x^{-k-1}$.

4. $y'_x = \lambda(\arcsin x)^n y^2 + ay + ab - b^2\lambda(\arcsin x)^n.$

Particular solution: $y_0 = -b$.

5. $y'_x = \lambda(\arcsin x)^n y^2 - b\lambda x^m(\arcsin x)^n y + bmx^{m-1}.$

Particular solution: $y_0 = bx^m$.

6. $y'_x = \lambda(\arcsin x)^n y^2 + \beta m x^{m-1} - \lambda\beta^2 x^{2m}(\arcsin x)^n.$

Particular solution: $y_0 = \beta x^m$.

7. $y'_x = \lambda(\arcsin x)^n(y - ax^m - b)^2 + amx^{m-1}.$

Particular solution: $y_0 = ax^m + b$.

8. $xy'_x = \lambda(\arcsin x)^n y^2 + ky + \lambda b^2 x^{2k}(\arcsin x)^n.$

Solution: $y = bx^k\tan\Big[\lambda b\int x^{k-1}(\arcsin x)^n\,dx + C\Big].$

9. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\arcsin x)^m - ny.$

The substitution $z = x^n y$ leads to an equation with separation of variables: $z'_x = x^{n-1}(\arcsin x)^m(az^2 + bz + c)$.

► *In the equations 10–18, function* arccot x *may be substituted for* arctan x.

10. $y'_x = y^2 + \lambda(\arctan x)^n y - a^2 + a\lambda(\arctan x)^n.$

Particular solution: $y_0 = -a$.

11. $y'_x = y^2 + \lambda x(\arctan x)^n y + \lambda(\arctan x)^n.$

Particular solution: $y_0 = -1/x$.

12. $y'_x = -(k+1)x^k y^2 + \lambda(\arctan x)^n (x^{k+1} y - 1).$

Particular solution: $y_0 = x^{-k-1}$.

13. $y'_x = \lambda(\arctan x)^n y^2 + ay + ab - b^2\lambda(\arctan x)^n.$

Particular solution: $y_0 = -b$.

14. $y'_x = \lambda(\arctan x)^n y^2 - b\lambda x^m(\arctan x)^n y + bmx^{m-1}.$

Particular solution: $y_0 = bx^m$.

15. $y'_x = \lambda(\arctan x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\arctan x)^n.$

Particular solution: $y_0 = bx^m$.

16. $y'_x = \lambda(\arctan x)^n (y - ax^m - b)^2 + amx^{m-1}.$

Particular solution: $y_0 = ax^m + b$.

17. $xy'_x = \lambda(\arctan x)^n y^2 + ky + \lambda b^2 x^{2k}(\arctan x)^n.$

Solution: $y = bx^k \tan\Big[\lambda b \int x^{k-1}(\arctan x)^n\,dx + C\Big].$

18. $xy'_x = (ax^{2n}y^2 + bx^n y + c)(\arctan x)^m - ny.$

The substitution $z = x^n y$ leads to an equation with separation of variables: $z'_x = x^{n-1}(\arctan x)^m(az^2 + bz + c)$.

1.2.8. Equations Containing Arbitrary Functions

Notation: $f = f(x)$ *and* $g = g(x)$ *are arbitrary functions;* a, b, n, *and* λ *are arbitrary parameters.*

1. $y'_x = y^2 + fy - a^2 - af.$

Particular solution: $y_0 = a$.

2. $y'_x = fy^2 + ay - ab - b^2 f.$

Particular solution: $y_0 = b$.

3. $\boldsymbol{y'_x = y^2 + xfy + f}$.

Particular solution: $y_0 = -1/x$.

4. $\boldsymbol{y'_x = fy^2 - ax^n fy + anx^{n-1}}$.

Particular solution: $y_0 = ax^n$.

5. $\boldsymbol{y'_x = fy^2 + anx^{n-1} - a^2x^{2n} f}$.

Particular solution: $y_0 = ax^n$.

6. $\boldsymbol{y'_x = -(n+1)x^n y^2 + x^{n+1} fy - f}$.

Particular solution: $y_0 = x^{-n-1}$.

7. $\boldsymbol{xy'_x = fy^2 + ny + ax^{2n} f}$.

Solution with $a > 0$: $y = \sqrt{a}\, x^n \tan\Bigl(\sqrt{a} \int x^{n-1} f\, dx + C\Bigr)$.

Solution with $a < 0$: $y = \sqrt{|a|}\, x^n \tanh\Bigl(-\sqrt{|a|} \int x^{n-1} f\, dx + C\Bigr)$.

8. $\boldsymbol{xy'_x = x^{2n} fy^2 + (ax^n f - n)y + bf}$.

The substitution $z = x^n y$ leads to an equation with separation of variables: $z'_x = x^{n-1} f(x)(z^2 + az + b)$.

9. $\boldsymbol{y'_x = fy^2 + gy - a^2 f - ag}$.

Particular solution: $y_0 = a$.

10. $\boldsymbol{y'_x = fy^2 + gy + anx^{n-1} - ax^n g - a^2 f x^{2n}}$.

Particular solution: $y_0 = ax^n$.

11. $\boldsymbol{y'_x = fy^2 - ax^n gy + anx^{n-1} + a^2x^{2n}(g - f)}$.

Particular solution: $y_0 = ax^n$.

12. $\boldsymbol{y'_x = -f'_x y^2 + fgy - g}$.

Particular solution: $y_0 = 1/f$.

13. $\boldsymbol{y'_x = fy^2 - fgy + g'_x}$.

Particular solution: $y_0 = g$.

14. $\boldsymbol{y'_x = g(y - f)^2 + f'_x}$.

Particular solution: $y_0 = f$.

15. $\boldsymbol{y'_x = \dfrac{f'_x}{g} y^2 - \dfrac{g'_x}{f}}$.

Particular solution: $y_0 = -g/f$.

16. $f^2 y'_x - f'_x y^2 + g(y - f) = 0.$

Particular solution: $y_0 = f$.

17. $y'_x = y^2 - \dfrac{f''_{xx}}{f}.$

Particular solution: $y_0 = -f'_x/f$.

18. $y'_x = ae^{\lambda x} y^2 + ae^{\lambda x} fy + \lambda f.$

Particular solution: $y_0 = -\dfrac{\lambda}{a} e^{-\lambda x}$.

19. $y'_x = fy^2 - ae^{\lambda x} fy + a\lambda e^{\lambda x}.$

Particular solution: $y_0 = ae^{\lambda x}$.

20. $y'_x = fy^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x} f.$

Particular solution: $y_0 = ae^{\lambda x}$.

21. $y'_x = fy^2 + \lambda y + ae^{2\lambda x} f.$

Solution with $a > 0$: $y = \sqrt{a}\, e^{\lambda x} \tan\Bigl(\sqrt{a} \displaystyle\int e^{\lambda x} f\, dx + C\Bigr)$.

Solution with $a < 0$: $y = \sqrt{|a|}\, e^{\lambda x} \tanh\Bigl(-\sqrt{|a|} \displaystyle\int e^{\lambda x} f\, dx + C\Bigr)$.

22. $y'_x = fy^2 - f(ae^{\lambda x} + b)y + a\lambda e^{\lambda x}.$

Particular solution: $y_0 = ae^{\lambda x} + b$.

23. $y'_x = e^{\lambda x} fy^2 + (af - \lambda)y + be^{-\lambda x} f.$

The substitution $z = e^{\lambda x} y$ leads to an equation with separation of variables: $z'_x = f(x)(z^2 + az + b)$.

24. $y'_x = fy^2 + gy + a\lambda e^{\lambda x} - ae^{\lambda x} g - a^2 e^{2\lambda x} f.$

Particular solution: $y_0 = ae^{\lambda x}$.

25. $y'_x = fy^2 - ae^{\lambda x} gy + a\lambda e^{\lambda x} + a^2 e^{2\lambda x}(g - f).$

Particular solution: $y_0 = ae^{\lambda x}$.

26. $y'_x = fy^2 + 2a\lambda x e^{\lambda x^2} - a^2 f e^{2\lambda x^2}.$

Particular solution: $y_0 = ae^{\lambda x^2}$.

27. $y'_x = fy^2 + \lambda xy + afe^{\lambda x^2}.$

Solution with $a > 0$: $y = \sqrt{a}\, e^{\lambda x^2/2} \tan\Bigl(\sqrt{a} \displaystyle\int e^{\lambda x^2/2} f\, dx + C\Bigr)$.

Solution with $a < 0$: $y = \sqrt{|a|}\, e^{\lambda x^2/2} \tanh\Bigl(-\sqrt{|a|} \displaystyle\int e^{\lambda x^2/2} f\, dx + C\Bigr)$.

28. $\boldsymbol{y'_x = f'_x y^2 + a e^{\lambda x} f y + a e^{\lambda x}}$.

Particular solution: $y_0 = -1/f$.

29. $\boldsymbol{y'_x = f y^2 + g'_x y + a f e^{2g}}$.

Solution with $a>0$: $y = \sqrt{a}\, e^g \tan\Bigl(\sqrt{a}\int f e^g\,dx + C\Bigr)$.

Solution with $a<0$: $y = \sqrt{|a|}\, e^g \tanh\Bigl(-\sqrt{|a|}\int f e^g\,dx + C\Bigr)$.

30. $\boldsymbol{y'_x = f y^2 - a \tanh^2(\lambda x)(af + \lambda) + a\lambda}$.

Particular solution: $y_0 = a\tanh(\lambda x)$.

31. $\boldsymbol{y'_x = f y^2 - a \coth^2(\lambda x)(af + \lambda) + a\lambda}$.

Particular solution: $y_0 = a\coth(\lambda x)$.

32. $\boldsymbol{y'_x = f y^2 - a^2 f + a\lambda \sinh(\lambda x) - a^2 f \sinh^2(\lambda x)}$.

Particular solution: $y_0 = a\cosh(\lambda x)$.

33. $\boldsymbol{x y'_x = f y^2 + a + a^2 f (\ln x)^2}$.

Particular solution: $y_0 = a\ln x$.

34. $\boldsymbol{x y'_x = f (y + a\ln x)^2 - a}$.

Solution: $\dfrac{1}{y + a\ln x} + \displaystyle\int \frac{f(x)}{x}\,dx = C$.

35. $\boldsymbol{y'_x = f y^2 - a x \ln x f y + a \ln x + a}$.

Particular solution: $y_0 = a x \ln x$.

36. $\boldsymbol{y'_x = -a \ln x\, y^2 + a f (x \ln x - x) y - f}$.

Particular solution: $y_0 = \dfrac{1}{a(x\ln x - x)}$.

37. $\boldsymbol{y'_x = \lambda \sin(\lambda x) y^2 + f \cos(\lambda x) y - f}$.

Particular solution: $y_0 = \dfrac{1}{\cos(\lambda x)}$.

38. $\boldsymbol{y'_x = f y^2 - a^2 f + a\lambda \sin(\lambda x) + a^2 f \sin^2(\lambda x)}$.

Particular solution: $y_0 = -a\cos(\lambda x)$.

39. $\boldsymbol{y'_x = f y^2 - a^2 f + a\lambda \cos(\lambda x) + a^2 f \cos^2(\lambda x)}$.

Particular solution: $y_0 = a\sin(\lambda x)$.

40. $\boldsymbol{y'_x = f y^2 - a \tan^2(\lambda x)(af - \lambda) + a\lambda}$.

Particular solution: $y_0 = a\tan(\lambda x)$.

41. $\boldsymbol{y'_x = f y^2 - a \cot^2(\lambda x)(af - \lambda) + a\lambda}$.

Particular solution: $y_0 = -a\cot(\lambda x)$.

1.2.9. Some transformations

Notation: f, g, and h are arbitrary functions of a complex argument which is written in parentheses following the function name (the argument is a function of x).

1. $\boldsymbol{y'_x = y^2 + a^2 f(ax+b)}$.

The transformation $\xi = ax+b$, $u = y/a$ leads to the equation $u'_\xi = u^2 + f(\xi)$.

2. $\boldsymbol{y'_x = y^2 + x^{-4} f\left(\frac{1}{x}\right)}$.

The transformation $\xi = 1/x$, $w = -x^2 y - x$ leads to the equation $w'_\xi = w^2 + f(\xi)$.

3. $\boldsymbol{y'_x = y^2 + \dfrac{1}{(cx+d)^4} f\left(\dfrac{ax+b}{cx+d}\right)}$.

The transformation

$$\xi = \frac{ax+b}{cx+d}, \quad w = \frac{1}{\Delta}[(cx+d)^2 y + c(cx+d)], \quad \text{where} \quad \Delta = ad - bc,$$

leads to a simpler equation: $w'_\xi = w^2 + \Delta^{-2} f(\xi)$.

4. $\boldsymbol{x^2 y'_x = x^4 f(x) y^2 + 1}$.

The substitution $u = -\dfrac{1}{x^2 y} - \dfrac{1}{x}$ leads to the equation $u'_x = u^2 + f(x)$.

5. $\boldsymbol{x^2 y'_x = x^2 y^2 + \dfrac{1-n^2}{4} + x^{2n} f(ax^n + b)}$.

The transformation $\xi = ax^n + b$, $w = \dfrac{x^{1-n}}{an} y + \dfrac{1-n}{2an} x^{-n}$ leads to a simpler equation: $w'_\xi = w^2 + (an)^{-2} f(\xi)$.

6. $\boldsymbol{y'_x = f(x) y^2 + g(x) y + h(x)}$.

The substitution $y = -1/w$ leads to an equation of the same form: $w'_x = h(x) w^2 - g(x) w + f(x)$.

7. $\boldsymbol{y'_x = y^2 - \dfrac{\lambda^2}{4} + e^{2\lambda x} f(e^{\lambda x})}$.

The transformation $\xi = e^{\lambda x}$, $w = \frac{1}{\lambda} e^{-\lambda x} y - \frac{1}{2} e^{-\lambda x}$ leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

8. $\boldsymbol{y'_x = y^2 - \dfrac{\lambda^2}{4} + \dfrac{e^{2\lambda x}}{(ce^{\lambda x} + d)^4} f\left(\dfrac{ae^{\lambda x} + b}{ce^{\lambda x} + d}\right)}$.

The transformation

$$\xi = \frac{ae^{\lambda x} + b}{ce^{\lambda x} + d}, \quad w = \frac{(ce^{\lambda x} + d)^2}{\Delta \lambda e^{\lambda x}} y + \frac{c^2 e^{2\lambda x} - d^2}{2\Delta e^{\lambda x}}, \qquad \text{where} \quad \Delta = ad - bc,$$

leads to a simpler equation: $w'_\xi = w^2 + (\Delta\lambda)^{-2} f(\xi)$.

9. $\boldsymbol{y'_x = y^2 - \lambda^2 + \sinh^{-4}(\lambda x) f\big(\coth(\lambda x)\big)}$.

The transformation

$$\xi = \coth(\lambda x), \quad w = -\frac{1}{\lambda}\sinh^2(\lambda x) y - \frac{1}{2}\sinh(2\lambda x)$$

leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

10. $\boldsymbol{y'_x = y^2 - \lambda^2 + \cosh^{-4}(\lambda x) f\big(\tanh(\lambda x)\big)}$.

The transformation

$$\xi = \tanh(\lambda x), \quad w = \frac{1}{\lambda}\cosh^2(\lambda x) y + \frac{1}{2}\sinh(2\lambda x)$$

leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

11. $\boldsymbol{x^2 y'_x = x^2 y^2 + f(a \ln x + b) + \frac{1}{4}}$.

The transformation $\xi = a \ln x + b$, $w = \dfrac{x}{a} y + \dfrac{1}{2a}$ leads to a simpler equation: $w'_\xi = w^2 + a^{-2} f(\xi)$.

12. $\boldsymbol{y'_x = y^2 + \lambda^2 + \sin^{-4}(\lambda x) f(\cot(\lambda x))}$.

The transformation

$$\xi = \cot(\lambda x), \quad w = -\sin^2(\lambda x)\left[\frac{y}{\lambda} + \cot(\lambda x)\right]$$

leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

13. $\boldsymbol{y'_x = y^2 + \lambda^2 + \cos^{-4}(\lambda x) f\big(\tan(\lambda x)\big)}$.

The transformation

$$\xi = \tan(\lambda x), \quad w = \cos^2(\lambda x)\left[\frac{y}{\lambda} - \tan(\lambda x)\right]$$

leads to a simpler equation: $w'_\xi = w^2 + \lambda^{-2} f(\xi)$.

14. $\boldsymbol{y'_x = y^2 + \lambda^2 + \sin^{-4}(\lambda x + b) f\left(\dfrac{\sin(\lambda x + a)}{\sin(\lambda x + b)}\right)}$.

The transformation

$$\xi = \frac{\sin(\lambda x + a)}{\sin(\lambda x + b)}, \quad w = \frac{\sin^2(\lambda x + b)}{\sin(b - a)}\left[\frac{y}{\lambda} + \cot(\lambda x + b)\right]$$

leads to a simpler equation: $w'_\xi = w^2 + [\lambda \sin(b - a)]^{-2} f(\xi)$.

TABLE 1.1
Solvable Abel equations of the form $yy'_x - y = sx + Ax^m$,
A is an arbitrary parameter

m	s	Equation	m	s	Equation
arbitrary	$-\dfrac{2(m+1)}{(m+3)^2}$	1.3.1.10	-1	0	1.3.1.16
-7	$15/4$	1.3.1.56	$-1/2$	$-2/9$	1.3.1.26
-4	6	1.3.1.54	$-1/2$	$-4/25$	1.3.1.22
$-5/2$	12	1.3.1.47	$-1/2$	0	1.3.1.32
-2	0	1.3.1.33	$-1/2$	20	1.3.1.55
-2	2	1.3.1.19	0	arbitrary	1.3.1.2
$-5/3$	$-3/16$	1.3.1.30	0	0	1.3.1.1
$-5/3$	$-9/100$	1.3.1.23	$1/2$	$-12/49$	1.3.1.53
$-5/3$	$63/4$	1.3.1.48	2	$-6/25$	1.3.1.45
$-7/5$	$-5/36$	1.3.1.27	2	$6/25$	1.3.1.46

1.3. Abel Equations of the Second Kind

1.3.1. Equations of the Form $yy'_x - y = f(x)$

Preliminary comments. For the sake of convenience, in Tables 1.1–1.4 are listed all the Abel equations discussed in Section 1.3. Tables 1.1–1.3 classify the Abel equations wherein functions f are of the same form; Table 1.4 gives the other Abel equations. In Table 1.1, equations are arranged in accordance with the growth of parameter m. In Table 1.2, equations are arranged in accordance with the growth of parameter s. In Table 1.3, equations are arranged in accordance with the growth of parameter p. In the rightmost columns of the tables are indicated equation numbers where the corresponding solutions are written out.

Below in this section are given all the Abel equations united into groups wherein all the solutions are expressed in terms of the same functions. A notation is given before each group.

In most cases the solutions are presented in the parametric form

$$x = F_1(\tau, C), \qquad y = F_2(\tau, C),$$

where τ is a parameter, C is an arbitrary constant.

1. $\boldsymbol{yy'_x - y = A.}$

Solution: $x = y - A \ln|y + A| + C.$

2. $\boldsymbol{yy'_x - y = Ax + B, \qquad A \neq 0.}$

Solution in the parametric form:

$$x = C \exp\left(-\int \frac{\tau\, d\tau}{\tau^2 - \tau - A}\right) - \frac{B}{A}, \qquad y = C\tau \exp\left(-\int \frac{\tau\, d\tau}{\tau^2 - \tau - A}\right).$$

TABLE 1.2

Solvable Abel equations of the form $yy'_x - y = sx + \sigma A(\alpha x^{1/2} + \beta A + \gamma A^2 x^{-1/2})$, A is an arbitrary parameter

s	σ	α	β	γ	Equation
arbitrary $\neq 0$	arbitrary	0	arbitrary	0	1.3.1.2
$\dfrac{2(m-1)}{(m-3)^2}$	$\dfrac{2}{(m-3)^2}$	$m(m+3)$	$4m^2+3m+9$	$3m(m+3)$	1.3.1.12
$-1/4$	$1/4$	1	5	3	1.3.1.17
$-30/121$	$3/242$	21	35	6	1.3.1.29
$-12/49$	arbitrary	arbitrary	0	0	1.3.1.53
$-12/49$	$1/98$	25	41	10	1.3.1.25
$-12/49$	$6/49$	1	8	5	1.3.1.38
$-12/49$	$2/49$	5	34	15	1.3.1.24
$-12/49$	$4/49$	-10	27	10	1.3.1.31
$-12/49$	$1/49$	5	262	65	1.3.1.52
$-12/49$	$6/49$	-3	23	12	1.3.1.28
$-12/49$	$2/49$	1	166	55	1.3.1.58
$-12/49$	1	$3/49+3B$	$12/49-15B/2$	$15/196+75B/16$	1.3.1.64
$-6/25$	$2/25$	2	19	6	1.3.1.20
$-6/25$	$6/25$	2	7	4	1.3.1.39
$-28/121$	$2/121$	5	106	15	1.3.1.21
$-2/9$	arbitrary	0	arbitrary	arbitrary	1.3.1.3
$-2/9$	arbitrary	0	0	arbitrary	1.3.1.26
$-2/9$	6	0	1	2	1.3.1.11
$-10/49$	$2/49$	4	61	12	1.3.1.57
$-4/25$	arbitrary	0	0	arbitrary	1.3.1.22
$-4/25$	$1/50$	7	49	6	1.3.1.59
0	arbitrary	0	0	arbitrary	1.3.1.32
0	arbitrary	1	2	arbitrary	1.3.1.36
0	$n+2$	1	$2(n+2)$	$(n+1)(n+3)$	1.3.1.34
0	$n+2$	1	$2(n+2)$	$2n+3$	1.3.1.35
0	1	-1	2	0	1.3.1.37
0	2	1	4	3	1.3.1.4
0	arbitrary	0	arbitrary	0	1.3.1.1
2	2	-10	19	30	1.3.1.50
2	2	10	31	30	1.3.1.49
20	arbitrary	0	0	arbitrary	1.3.1.55

3. $\quad yy'_x - y = -\frac{2}{9} + A + Bx^{-1/2}.$

1°. Solution in the parametric form with $A > 0$:

$$x = a\left[\frac{(2k-1)C\tau^k - (k-2)\tau - k - 1}{C\tau^k + \tau + 1}\right]^2, \quad y = -6a\frac{(k-1)^2 C\tau^{k+1} + k^2 C\tau^k + \tau}{C\tau^k + \tau + 1},$$

where $A = \frac{2}{3}a(k^2 - k + 1)$, $B = \frac{2}{3}a^{3/2}(2k-1)(k-2)(k+1)$.

TABLE 1.3

Solvable Abel equations of the form $yy'_x - y = sx + \alpha A x^p + \beta A^2 x^q$, A is an arbitrary parameter

p	q	s	α	β	Equation
-1	-3	arbitrary	1	-1	1.3.1.5
-1	-3	$\dfrac{2m+1}{4m^2}$	1	-1	1.3.1.13
-1	-3	0	1	-1	1.3.1.7
$-3/5$	$-7/5$	$-5/36$	arbitrary	arbitrary	1.3.1.63
$-5/11$	$-13/11$	$-33/196$	$286A/3$	$-770A/9$	1.3.1.69
$-1/3$	$-5/3$	$-3/16$	arbitrary	arbitrary	1.3.1.61
$-1/3$	$-5/3$	$-3/16$	3	-12	1.3.1.40
$-1/3$	$-5/3$	$-3/16$	5	-12	1.3.1.15
$-1/3$	$-5/3$	$15/4$	6	-3	1.3.1.60
$-1/5$	$-4/5$	$-10/49$	$13A/5$	$-7A/20$	1.3.1.68
0	$-1/2$	$-2/9$	arbitrary	arbitrary	1.3.1.3
2	3	$4/9$	2	2	1.3.1.14

2°. Solution in the parametric form with $A < 0$:

$$x = \xi[2\lambda e^{-\lambda\tau} - (C_1\lambda - 3C_2\omega)\sin\omega\tau - (3C_1\omega + C_2\lambda)\cos\omega\tau]^2,$$
$$y = 6\xi\{(C_1^2 + C_2^2)\omega^2 - [C_1(\lambda^2 - \omega^2) - 2C_2\omega\lambda]e^{-\lambda\tau}\sin\omega\tau$$
$$- [2C_1\omega\lambda + C_2(\lambda^2 - \omega^2)]e^{-\lambda\tau}\cos\omega\tau\},$$

where $A = \frac{1}{9}a(3\omega^2 - \lambda^2)$, $B = \frac{2}{27}a\lambda(9\omega^2 + 5\lambda^2)$, $\xi = a(e^{-\lambda\tau} + C_1\sin\omega\tau + C_2\cos\omega\tau)^{-2}$.

3°. See 1.3.1.26 for the solution with $A = 0$.

4. $yy'_x - y = 2A(x^{1/2} + 4A + 3A^2x^{-1/2})$.

Solution in the parametric form:

$$x = \tfrac{1}{4}a(3 \pm 2\tau L_\pm)^2, \qquad y = \pm aL_\pm(R_\pm^2 L_\pm + \tau), \qquad A = -\tfrac{1}{2}a^{1/2},$$

where

$$L_+ = \begin{cases} \displaystyle\int \frac{d\tau}{1+\tau^2} = \arctan\tau - C, & R_+ = \sqrt{1+\tau^2}, \\ \displaystyle\int \frac{d\tau}{\tau^2 - 1} = \frac{1}{2}\ln\left|\frac{\tau-1}{\tau+1}\right| - C, & R_+ = \sqrt{\tau^2 - 1}, \end{cases}$$

$$L_- = \int \frac{d\tau}{1-\tau^2} = \frac{1}{2}\ln\left|\frac{1+\tau}{1-\tau}\right| - C, \qquad R_- = \sqrt{1-\tau^2}.$$

5. $yy'_x - y = Ax + Bx^{-1} - B^2x^{-3}$.

Solution in the parametric form:

$$x = \left(\frac{V}{W}\right)^{-1/2}, \qquad y = (\tau+1)\left(\frac{V}{W}\right)^{-1/2} - B\left(\frac{V}{W}\right)^{1/2},$$

TABLE 1.4
Other solvable Abel equations of the form $yy'_x - y = f(x)$

Function $f(x)$	Equation
$Ax^{k-1} - kBx^k + kB^2x^{2k-1}$	1.3.1.6 (particular solution)
$Ax^2 - \dfrac{9}{625\,A}$	1.3.1.44
$\dfrac{3}{4}x - \dfrac{3}{2}Ax^{1/3} + \dfrac{3}{4}A^2x^{-1/3} - \dfrac{27}{625}A^4x^{-5/3}$	1.3.1.66
$-\dfrac{6}{25}x + \dfrac{7}{5}Ax^{1/3} + \dfrac{31}{3}A^2x^{-1/3} - \dfrac{100}{3}A^4x^{-5/3}$	1.3.1.67
$-\dfrac{6}{25}x + ax^{1/3} + b + cx^{-1/3} + dx^{-2/3}$ (coefficients a, b, c, and d are related by an equality)	1.3.1.65
$-\dfrac{21}{100}x + \dfrac{7}{9}A^2\bigl(123x^{-1/7} + 280\,Ax^{-5/7} - 400\,A^2x^{-9/7}\bigr)$	1.3.1.70
$\dfrac{k}{\sqrt{Ax^2 + Bx + C}}$	1.3.1.63
$\dfrac{A}{\sqrt{x^2 + 4A}}$	1.3.1.18
$-\dfrac{3}{32}x + \dfrac{9a^2 - 6x^2}{64\sqrt{x^2 + a^2}}$	1.3.1.43
$\dfrac{3}{8}x + \dfrac{6x^2 + 5a^2}{16\sqrt{x^2 + a^2}}$	1.3.1.21
$\dfrac{3}{8}x + \dfrac{6x^2 + 9A}{16\sqrt{x^2 + A}}$	1.3.1.41
$\dfrac{9}{32}x + \dfrac{30x^2 + 33A}{64\sqrt{x^2 + A}}$	1.3.1.42
$A + B\exp\Bigl(-\dfrac{2x}{A}\Bigr)$	1.3.1.8
$A\Bigl[\exp\Bigl(\dfrac{2x}{A}\Bigr) - 1\Bigr]$	1.3.1.9
$a^2\lambda e^{2\lambda x} - a(b\lambda + 1)e^{\lambda x} + b$	1.3.1.73 (particular solution)
$a^2\lambda e^{2\lambda x} + a\lambda x e^{\lambda x} + be^{\lambda x}$	1.3.1.74 (particular solution)
$2a^2\lambda\sin(2\lambda x) + 2a\sin(\lambda x)$	1.3.1.75 (particular solution)

where

$$V = \begin{cases} (\tau^2+\tau-A)\exp\left[\dfrac{2}{\sqrt{-\Delta}}\arctan\dfrac{2\tau+1}{\sqrt{-\Delta}}\right] & \text{if } \Delta<0,\\ (\tau^2+\tau-A)\exp\left(-\dfrac{2}{2\tau+1}\right) & \text{if } \Delta=0,\\ (\tau^2+\tau-A)\left[\dfrac{2\tau+1-\sqrt{\Delta}}{2\tau+1+\sqrt{\Delta}}\right]^{\frac{1}{\sqrt{\Delta}}} & \text{if } \Delta>0, \end{cases}$$

$$W = \begin{cases} C+2B\displaystyle\int\exp\left[\dfrac{2}{\sqrt{-\Delta}}\arctan\dfrac{2\tau+1}{\sqrt{-\Delta}}\right]d\tau & \text{if } \Delta<0,\\ C+2B\displaystyle\int\exp\left(-\dfrac{2}{2\tau+1}\right)d\tau & \text{if } \Delta=0,\\ C+2B\displaystyle\int\left[\dfrac{2\tau+1-\sqrt{\Delta}}{2\tau+1+\sqrt{\Delta}}\right]^{\frac{1}{\sqrt{\Delta}}}d\tau & \text{if } \Delta>0, \end{cases}$$

where $\Delta = 4A+1$.

6. $\boldsymbol{yy'_x - y = Ax^{k-1} - kBx^k + kB^2x^{2k-1}}$.

Particular solution: $y_0 = x - Bx^k - \dfrac{A}{kB}$.

7. $\boldsymbol{yy'_x - y = Ax^{-1} - A^2x^{-3}}$.

Solution in the parametric form:

$$x = a\tau^{-1}(\tau - \ln|1+\tau| - C)^{1/2},$$
$$y = a\left[\frac{1+\tau}{\tau}(\tau - \ln|1+\tau| - C)^{1/2} - \frac{1}{2}\tau(\tau - \ln|1+\tau| - C)^{-1/2}\right],$$

where $A = a^2/2$.

8. $\boldsymbol{yy'_x - y = A + Be^{-2x/A}}$.

Solution in the parametric form:

$$x = \ln\left|\frac{\sqrt{\tau^2+AB}}{A\ln\left|\tau+\sqrt{\tau^2+AB}\right|+C}\right|, \qquad y = \tau\frac{A\ln\left|\tau+\sqrt{\tau^2+AB}\right|+C}{\sqrt{\tau^2+AB}} - A.$$

9. $\boldsymbol{yy'_x - y = A(e^{2x/A} - 1)}$.

Solution in the parametric form:

$$x = A\ln\left|\frac{\tau^2+1}{\tau}(\arctan\tau - C)\right|, \qquad y = \frac{A}{\tau}\left[\tau + (\tau^2-1)(\arctan\tau - C)\right].$$

► *In the solutions of equations 10–15, the following notation is used:*

$$E_{ml} = \int (1\pm\tau^{m+1})^{\frac{1}{l-2}}\,d\tau - C, \qquad E_m = E_{m0} = \int (1\pm\tau^{m+1})^{-1/2}\,d\tau - C,$$
$$R_m = \sqrt{1\pm\tau^{m+1}}, \qquad F_m = R_mE_m - \tau.$$

10. $\boldsymbol{yy'_x - y = -\dfrac{2(m+1)}{(m+3)^2}x + Ax^m}.$

Solution in the parametric form:

$$x = \frac{m+3}{m-1} a E_m^{\frac{2}{m-1}} \tau, \qquad y = a E_m^{\frac{2}{m-1}} \Big(R_m E_m + \frac{2}{m-1}\tau \Big),$$

where $A = \pm \dfrac{m+1}{2} \Big(\dfrac{m-1}{m+3} \Big)^{m+1} a^{1-m}$.

11. $\boldsymbol{yy'_x - y = -\tfrac{2}{9}x + 6A^2(1 + 2Ax^{-1/2}), \qquad A > 0}.$

Solution in the parametric form:

$$x = A^2 R^{-4} E^{-2} (R^2 E \pm 6\tau^{1/2})^2, \qquad y = -12 A^2 R^{-4} E^{-2} (R^2 E - 2\tau),$$

where $E = E_{-1/2,\,3/2}$, $R = R_{-1/2}$.

12. $\boldsymbol{yy'_x - y = \dfrac{2(m-1)}{(m-3)^2}x + \dfrac{2A}{(m-3)^2}[m(m+3)x^{1/2} + (4m^2+3m+9)A + 3m(m+3)A^2x^{-1/2}]}.$

Solution in the parametric form:

$$x = \frac{a}{(m-3)^2} \tau^{-2} [(m-3) R_m E_m + 3\tau]^2,$$
$$y = \frac{a}{m-3} \tau^{-2} E_m [\pm (m-1) \tau^{m+1} E_m - 2E_m + 2\tau R_m],$$

where $A = -\dfrac{a^{1/2}}{m-3}$.

13. $\boldsymbol{yy'_x - y = \dfrac{2m+1}{4m^2}x + Ax^{-1} - A^2x^{-3}}.$

Solution in the parametric form:

$$x = \frac{1}{a} \tau^{-1/2} R_m^{-2} E^{1/2}, \qquad y = \frac{1}{2ma} \tau^{-1/2} R_m^{-2} E^{-1/2} \{\tau - [1 \mp (2m+1) \tau^{m+1} R_m^2 E\},$$

where $a^2 = -2mA$, $E = E_{m,\,3/2}$.

14. $\boldsymbol{yy'_x - y = \tfrac{4}{9}x + 2Ax^2 + 2A^2x^3}.$

Solution in the parametric form:

$$x = \frac{1}{3A} \tau^{-1} F_3, \qquad y = \frac{1}{9A} \tau^{-2} E_3 (\tau R_3 - E_3 \pm \tau^4 E_3).$$

15. $\boldsymbol{yy'_x - y = -\tfrac{3}{16}x + 5Ax^{-1/3} - 12A^2x^{-5/3}}.$

Solution in the parametric form:

$$x = a\tau^{1/2} E^{-3/2} F^{3/2}, \qquad y = \frac{a}{4} \tau^{1/2} E^{-3/2} F^{-1/2} (F^2 - 2\tau F - \tau^{-2/3} E^2),$$

where $A = a^{4/3}/24$, $E = E_{-5/3}$, $F = F_{-5/3}$.

▶ *In the solutions of equations 16–18, the following notation is used:*

$$f=\int \exp(\mp\tau^2)\,d\tau - C, \qquad g=2\tau\biggl[\int \exp(\mp\tau^2)\,d\tau - C\biggr] \pm \exp(\mp\tau^2).$$

16. $\boldsymbol{yy'_x - y = Ax^{-1}}$.

Solution in the parametric form:

$$x=af^{-1}\exp(\mp\tau^2), \quad y=af^{-1}\bigl[\exp(\mp\tau^2)\pm 2\tau f\bigr],$$

where $A=\mp 2a^2$.

17. $\boldsymbol{yy'_x - y = -\frac{1}{4}x + \frac{1}{4}A(x^{1/2} + 5A + 3A^2x^{-1/2})}$.

Solution in the parametric form:

$$x=\frac{a}{16}\bigl[3\pm 8\tau f\exp(\pm\tau^2)\bigr]^2, \quad y=af\exp(\pm\tau^2)\bigl[(2\tau^2\pm 1)f\exp(\pm\tau^2)\pm\tau\bigr],$$

where $A=\frac{1}{4}\sqrt{a}$.

18. $\boldsymbol{yy'_x - y = \pm\dfrac{2a^2}{\sqrt{x^2 \pm 8a^2}}}$.

Solution in the parametric form:

$$x=\pm a(fg)^{-1}(g^2\mp 2f^2), \qquad y=a(fg)^{-1}[\exp(\mp\tau^2)g-2f^2].$$

▶ *In the solutions of equations 19–21, the following notation is used:*

$$E=\sqrt{\tau(\tau+1)}-\ln\bigl|C\bigl(\sqrt{\tau}+\sqrt{\tau+1}\,\bigr)\bigr|,$$

$$R=\sqrt{\frac{\tau+1}{\tau}}, \qquad F=1-\sqrt{\frac{\tau+1}{\tau}}\ln\bigl|C\bigl(\sqrt{\tau}+\sqrt{\tau+1}\,\bigr)\bigr|.$$

19. $\boldsymbol{yy'_x - y = 2x + Ax^{-2}}$.

Solution in the parametric form:

$$x=\frac{a}{3}E^{-2/3}\tau, \quad y=aE^{-2/3}\Bigl(\frac{2}{3}\tau-RE\Bigr), \qquad \text{where} \quad A=-\frac{243}{2}a^3.$$

20. $\boldsymbol{yy'_x - y = -\dfrac{6}{25}x + \dfrac{2A}{25}(2x^{1/2} + 19A + 6A^2x^{-1/2})}$.

Solution in the parametric form:

$$x=a\tau^{-2}(5RE-3\tau)^2, \quad y=5a\tau^{-3}E[(2\tau+3)E-2\tau^2R], \qquad \text{where} \quad A=-\sqrt{a}.$$

21. $\boldsymbol{yy'_x - y = \dfrac{3}{8}x + \dfrac{3}{8}\sqrt{x^2 + a^2} - \dfrac{a^2}{16\sqrt{x^2 + a^2}}}$.

Solution in the parametric form:

$$x=\frac{a}{2\sqrt{2}}\,\frac{E^2-2\tau^2F}{\tau E\sqrt{F}}, \qquad y=\frac{a}{4\sqrt{2}}\,\frac{4\tau F^2-E^2}{\tau E\sqrt{F}}.$$

▶ *In the solutions of equations 22–25, the following notation is used:*

$$P_2 = \pm(\tau^2 - 1), \qquad P_3 = \tau^3 - 3\tau + C, \qquad P_4 = \pm(\tau^4 - 6\tau^2 + 4C\tau - 3).$$

22. $\boldsymbol{yy'_x - y = -\tfrac{4}{25}x + Ax^{-1/2}}$.

Solution in the parametric form:

$$x = 5aP_2^2 P_3^{-4/3}, \quad y = 4aP_3^{-4/3}(P_2^2 - \tau P_3), \qquad \text{where} \quad A = \pm\frac{4a}{5}\sqrt{5a}.$$

23. $\boldsymbol{yy'_x - y = -\dfrac{9}{100}x + Ax^{-5/3}}$.

Solution in the parametric form:

$$x = 10aP_3^{3/2}P_4^{-9/8}, \quad y = 9aP_3^{-1/2}P_4^{-9/8}(P_3^2 - P_2P_4), \qquad \text{where} \quad A = \pm 9a^2(10a)^{2/3}.$$

24. $\boldsymbol{yy'_x - y = -\dfrac{12}{49}x + \dfrac{2A}{49}(5x^{1/2} + 34A + 15A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = aP_2^{-4}(14\tau P_3 - 9P_2^2)^2, \quad y = 28aP_2^{-4}P_3(4\tau^2 P_3 - 3\tau P_2^2 \mp P_2P_3),$$

where $A = -3\sqrt{a}$.

25. $\boldsymbol{yy'_x - y = -\dfrac{12}{49}x + \dfrac{A}{98}(25x^{1/2} + 41A + 10A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = aP_3^{-4}(21P_2P_4 - 16P_3^2)^2, \quad y = 21aP_3^{-4}P_4(9P_2^2P_4 \mp P_4^2 - 8P_2P_3^2),$$

where $A = -8\sqrt{a}$.

▶ *In the solutions of equations 26–29, the following notation is used:*

$$S_1 = \exp(\sqrt{3}\,\tau) + C\sin\tau, \qquad S_2 = 2\exp(\sqrt{3}\,\tau) - C\sin\tau + \sqrt{3}\,C\cos\tau,$$
$$S_3 = 2\exp(\sqrt{3}\,\tau) - C\sin\tau - \sqrt{3}\,C\cos\tau, \qquad S_4 = 4S_1S_3 - S_2^2.$$

26. $\boldsymbol{yy'_x - y = -\tfrac{2}{9}x + Ax^{-1/2}}$.

Solution in the parametric form:

$$x = 3aS_1^{-2}S_2^2, \quad y = 2aS_1^{-2}(S_2^2 - 2S_1S_3), \qquad \text{where} \quad A = 16(3a)^{3/2}.$$

27. $\boldsymbol{yy'_x - y = -\tfrac{5}{36}x + Ax^{-7/5}}$.

Solution in the parametric form:

$$x = 48aS_1^{5/2}S_4^{-5/4}, \quad y = 5aS_1^{-1/2}S_4^{-5/4}(8S_1^3 - S_2S_4), \qquad \text{where} \quad A = (48a)^{2/5}a^2.$$

28. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + \frac{6A}{49}(-3x^{1/2} + 23A + 12A^2x^{-1/2})}.$

Solution in the parametric form:

$$x = aS_2^{-4}(7S_1S_3 - 2S_2^2)^2, \quad y = -7aS_1S_2^{-4}(4S_1^2S_2 - 4S_1S_3^2 + S_2^2S_3),$$

where $A = \sqrt{a}/2$.

29. $\boldsymbol{yy'_x - y = -\frac{30}{121}x + \frac{3A}{242}(21x^{1/2} + 35A + 6A^2x^{-1/2})}.$

Solution in the parametric form:

$$x = aS_1^{-6}(11S_2S_4 - 64S_1^3)^2, \quad y = -11aS_1^{-6}S_4(S_4^2 - 5S_2^2S_4 + 32S_1^3S_2),$$

where $A = -32\sqrt{a}$.

► *In the solutions of equations 30–31, the following notation is used:*

$$T_1 = \tanh(\tau + C) + \tan\tau, \qquad T_2 = \tanh(\tau + C) - \tan\tau,$$
$$\theta_1 = \cosh\tau - \sin(\tau + C), \qquad \theta_2 = \sinh\tau + \cos(\tau + C), \qquad \theta_3 = \sinh\tau - \cos(\tau + C).$$

30. $\boldsymbol{yy'_x - y = -\frac{3}{16}x + Ax^{-5/3}}.$

Solution in the parametric form with $A < 0$:

$$x = 8aT_1^{-3/2}, \quad y = 3aT_1^{-3/2}(2 - T_1T_2), \qquad \text{where} \quad A = -12a^{8/3}.$$

Solution in the parametric form with $A > 0$:

$$x = 4a\theta_1^{3/2}\theta_2^{-3/2}, \quad y = 3a\theta_1^{-1/2}\theta_2^{-3/2}(\theta_1^2 - \theta_2\theta_3), \qquad \text{where} \quad A = 3a^2(4a)^{2/3}.$$

31. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + \frac{4A}{49}(-10x^{1/2} + 27A + 10A^2x^{-1/2})}.$

Solution in the parametric form with $A < 0$:

$$x = a(10 - 7T_1T_2)^2, \quad y = 7aT_1(T_1^3 + 3T_1T_2^2 - 4T_2), \qquad \text{where} \quad A = -2\sqrt{a}.$$

Solution in the parametric form with $A > 0$:

$$x = a\theta_1^{-4}(7\theta_2\theta_3 - 5\theta_1^2), \quad y = -7a\theta_1^{-4}\theta_2(\theta_2^3 - 3\theta_2\theta_3^2 + 2\theta_1^2\theta_3), \qquad \text{where} \quad A = \sqrt{a}.$$

► *In the solutions of equations 32–43, the following notation is used:*

$$Z_\nu = \begin{cases} C_1J_\nu(\tau) + C_2Y_\nu(\tau) & \textit{for the upper sign (Bessel functions),} \\ C_1I_\nu(\tau) + C_2K_\nu(\tau) & \textit{for the lower sign (modified Bessel functions),} \end{cases}$$

$$f_\nu = \tau(Z_\nu)'_\tau + \nu Z_\nu, \qquad Z = Z_{1/3},$$
$$U_1 = \tau Z'_\tau + \tfrac{1}{3}Z, \qquad U_2 = U_1^2 \pm \tau^2Z^2, \qquad U_3 = \pm\tfrac{2}{3}\tau^2Z^3 - 2U_1U_2.$$

Remark. *The solutions of equations 32–43 contain only the ratio Z'_τ/Z. Therefore, function Z is defined in terms of two "arbitray" constants C_1 and C_2 (instead, we may set, for instance, $C_1 = 1$, $C_2 = C$).*

32. $\boldsymbol{yy'_x - y = Ax^{-1/2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3}Z^{-2}U_1^2, \quad y = a\tau^{-4/3}Z^{-2}U_2, \qquad \text{where} \quad A = \mp\tfrac{1}{3}a^{3/2}.$$

33. $\boldsymbol{yy'_x - y = Ax^{-2}}$.

Solution in the parametric form:

$$x = 2a\tau^{4/3}Z^2U_2^{-1}, \quad y = \pm 3a\tau^{-2/3}Z^{-1}U_2^{-1}U_3, \qquad \text{where} \quad A = -36a^3.$$

34. $\boldsymbol{yy'_x - y = A(n+2)[x^{1/2} + 2(n+2)A + (n+1)(n+3)A^2x^{-1/2}]}$.

Solution in the parametric form:

$$x = aZ_\nu^{-2}[f_\nu - (\nu+1)Z_\nu]^2, \quad y = aZ_\nu^{-2}(f_\nu^2 - 2\nu Z_\nu f_\nu \pm \tau^2 Z_\nu^2),$$

where $A = \nu\sqrt{a}$, $\nu = \frac{1}{n+2}$.

35. $\boldsymbol{yy'_x - y = A(n+2)[x^{1/2} + 2(n+2)A + (2n+3)A^2x^{-1/2}]}$.

Solution in the parametric form:

$$x = af_\nu^{-2}[\tau^2 Z_\nu \pm (2-\nu)f_\nu]^2, \quad y = \pm a\tau^2 f_\nu^{-2}[f_\nu^2 + 2(1-\nu)Z_\nu f_\nu \pm \tau^2 Z_\nu^2],$$

where $A = \mp\nu\sqrt{a}$, $\nu = \frac{1}{n+2}$.

36. $\boldsymbol{yy'_x - y = Ax^{1/2} + 2A^2 + Bx^{-1/2}}$.

Solution in the parametric form:

$$x = A^2 Z_\nu^{-2}(\tau Z'_\nu - Z_\nu)^2, \quad y = A^2 Z_\nu^{-2}[\tau^2 (Z'_\nu)^2 - (\nu^2 \mp \tau^2)Z_\nu^2],$$

where $B = (1-\nu^2)A^3$, prime denotes differentiation with respect to τ.

37. $\boldsymbol{yy'_x - y = 2A^2 - Ax^{1/2}}$.

Solution in the parametric form:

$$x = a(Z'_0)^{-2}(\tau Z_0 \pm 2Z'_0)^2, \quad y = \pm a\tau(Z'_0)^{-2}[\tau(Z'_0)^2 + 2Z_0 Z'_0 \pm \tau Z_0^2],$$

where $A = \sqrt{a}$, prime denotes differentiation with respect to τ.

38. $\boldsymbol{yy'_x - y = -\dfrac{12}{49}x + \dfrac{6A}{49}(x^{1/2} + 8A + 5A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = 3aU_1^{-4}(5U_1^2 - 7\tau^2 Z^2)^2, \quad y = 28a\tau^2 Z^2 U_1^{-4}(3\tau^2 Z^2 - ZU_1 - 3U_1^2),$$

where $A = 2\sqrt{3a}$; Z and U_1 are expressed in term of modified Bessel functions.

39. $\boldsymbol{yy'_x - y = -\frac{6}{25}x + \frac{6A}{25}(2x^{1/2} + 7A + 4A^2x^{-1/2}).}$

Solution in the parametric form:

$$x = a\tau^{-4}Z^{-6}(U_1U_2 - 2U_3)^2, \quad y = 5a\tau^{-4}Z^{-6}U_2(U_2^2 - U_1U_3), \qquad \text{where} \quad A = -\sqrt{a}/2.$$

40. $\boldsymbol{yy'_x - y = -\tfrac{3}{16}x + 3Ax^{-1/3} - 12A^2x^{-5/3}.}$

Solution in the parametric form:

$$x = a\tau^{-3/2}Z_{3/2}^{-3/2}f_{3/2}^{3/2},$$
$$y = \frac{3a}{4}\tau^{-3/2}Z_{3/2}^{-3/2}f_{3/2}^{-1/2}(f_{3/2}^2 - 2Z_{3/2}f_{3/2} - \tau^2Z_{3/2}^2),$$

where $Z_{3/2}$ and $f_{3/2}$ are expressed interms of modified Bessel functions, and $A = \frac{1}{8}a^{4/3}$.

41. $\boldsymbol{yy'_x - y = \frac{3}{8}x + \frac{3}{8}\sqrt{x^2 \pm b^2} \pm \frac{3b^2}{16\sqrt{x^2 \pm b^2}}.}$

Solution in the parametric form:

$$x = -\frac{a}{4}\tau^{-1}Z^{-3/2}U_1^{-1/2}U_2^{-1}(2\tau^2Z^3U_1 \mp 3U_2^2),$$
$$y = \mp\frac{a}{8}\tau^{-1}Z^{-3/2}U_1^{-1/2}U_2^{-1}(3U_2^2 - 12U_1^2U_2 \pm 4\tau^2Z^3U_1),$$

where $b^2 = \frac{3}{2}a^2$.

42. $\boldsymbol{yy'_x - y = \frac{9}{32}x + \frac{15}{32}\sqrt{x^2 \mp b^2} \mp \frac{3b^2}{64\sqrt{x^2 \mp b^2}}.}$

Solution in the parametric form:

$$x = -\frac{a}{2}\tau^{-1}Z^{-3/2}U_2^{-3/2}U_3^{-1/2}(2\tau^2Z^3U_3 \pm 3U_2^3),$$
$$y = \pm\frac{a}{4}\tau^{-1}Z^{-3/2}U_2^{-3/2}U_3^{-1/2}(3U_2^3 \mp \tau^2Z^3U_3 - 3U_3^2),$$

where $b^2 = 6a^2$.

43. $\boldsymbol{yy'_x - y = -\frac{3}{32}x - \frac{3}{32}\sqrt{x^2 + a^2} + \frac{15a^2}{64\sqrt{x^2 + a^2}}.}$

Solution in the parametric form:

$$x = \frac{a}{2}U_2^{-3/2}U_3^{-1}(U_3^2 - U_2^3), \quad y = \frac{a}{24}U_2^{-3/2}U_3^{-1}(3U_3^2 - 12U_2^3 \pm 4\tau^2Z^3U_3).$$

▶ *In the solutions of equations 44–52, the following notation is used:*

$$E_1 = \tau^3\sqrt{\pm(4\wp^3-1)} + 3\tau^2\wp \mp 1, \qquad E_2 = \tau^2\wp \mp 1,$$
$$E_3 = \sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^2, \qquad E_4 = \tau\sqrt{\pm(4\wp^3-1)} + 2\wp.$$

Function $\wp = \wp(\tau)$ is given implicitly as follows:

$$\tau = \int \frac{d\wp}{\sqrt{\pm(4\wp^3-1)}} - C_2.$$

The upper sign in this formula corresponds to the classical elliptic Weierstrass function $\wp = \wp(\tau + C_2,\ 0,\ 1)$.

44. $\boldsymbol{yy'_x - y = Ax^2 - \frac{9}{625}A^{-1}}$.

Solution in the parametric form:

$$x = 5a\Big(\tau^2\wp \mp \frac{1}{2}\Big), \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\frac{6}{125}a^{-1}.$$

45. $\boldsymbol{yy'_x - y = -\frac{6}{25}x + Ax^2}$.

Solution in the parametric form:

$$x = 5a\tau^2\wp, \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\frac{6}{125}a^{-1}.$$

46. $\boldsymbol{yy'_x - y = \frac{6}{25}x + Ax^2}$.

Solution in the parametric form:

$$x = 5aE_2, \quad y = a\tau^2 E_4, \qquad \text{where} \quad A = \pm\frac{6}{125}a^{-1}.$$

47. $\boldsymbol{yy'_x - y = 12x + Ax^{-5/2}}$.

Solution in the parametric form:

$$x = a\wp^{-6/7}E_3^{-4/7}, \quad y = a\wp^{-6/7}E_3^{-4/7}(14\wp^2 E_4 - 3), \qquad \text{where} \quad A = \mp 147a^{7/2}.$$

48. $\boldsymbol{yy'_x - y = \frac{63}{4}x + Ax^{-5/3}}$.

Solution in the parametric form:

$$x = 2aE_3^{3/2}E_4^{-9/8}, \quad y = aE_3^{-1/2}E_4^{-9/8}(9E_3^2 \mp 16\wp E_4^2),$$

where $A = -\frac{128}{3}a^2(2a)^{2/3}$.

49. $\boldsymbol{yy'_x - y = 2x + 2A(10x^{1/2} + 31A + 30A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = a\wp^{-2}\big[\tau\sqrt{\pm(4\wp^3-1)} - 3\wp\big]^2, \quad y = -2a\tau\wp^{-2}\big[\wp\sqrt{\pm(4\wp^3-1)} \pm 2\tau\wp^3 \pm \tau\big],$$

where $A = \sqrt{a}$.

50. $\boldsymbol{yy'_x - y = 2x + 2A(-10x^{1/2} + 19A + 30A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = aE_2^{-2}(E_1 - 6E_2)^2, \quad y = -2aE_2^{-2}(\pm 6E_2^3 - E_1^2 + 7E_1E_2), \qquad \text{where} \quad A = -\sqrt{a}.$$

51. $\boldsymbol{yy'_x - y = -\frac{28}{121}x + \frac{2A}{121}(5x^{1/2} + 106A + 15A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = a(22\wp^2E_4 - 5)^2, \quad y = \pm 44a\wp^2E_3(7\wp E_3 \mp 2\tau), \qquad \text{where} \quad A = \pm 2\sqrt{a}.$$

52. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + \frac{A}{49}(5x^{1/2} + 262A + 65A^2x^{-1/2})}$.

Solution in the parametric form:

$$x = aE_3^{-4}(28\wp E_4^2 \mp 15E_3^2)^2, \quad y = 56aE_3^{-4}E_4^2(6\wp E_2 + E_4), \qquad \text{where} \quad A = \mp 3\sqrt{a}.$$

▶ *In the solutions of equations 53–60, the following notation is used:*

$$I = \int \frac{\tau\,d\tau}{\sqrt{\pm(4\tau^3 - 1)}} \quad \text{— \textit{incomplete elliptic integral of the second kind in the form of Weierstrass}}$$

$$R = \sqrt{\pm(4\tau^3 - 1)}, \quad I_1 = \tau(2I \mp \tau^{-1}R + C), \quad I_2 = \tau^{-1}(RI_1 - 1), \quad I_3 = 4\tau I_1^2 \mp I_2^2.$$

53. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + Ax^{1/2}}$.

Solution in the parametric form:

$$x = 7a\tau^2I^{-4}, \quad y = -2aI^{-4}(RI - 2\tau^2), \qquad \text{where} \quad A = \pm\frac{12}{49}\sqrt{7a}.$$

54. $\boldsymbol{yy'_x - y = 6x + Ax^{-4}}$.

Solution in the parametric form:

$$x = a\tau^{-3/5}I_1^{-2/5}, \quad y = a\tau^{-3/5}I_1^{-2/5}(5RI_1 - 2), \qquad \text{where} \quad A = \mp 150a^5.$$

55. $\boldsymbol{yy'_x - y = 20x + Ax^{-1/2}}$.

Solution in the parametric form:

$$x = aI_1^{-4/3}I_2^2, \quad y = -4aI_1^{-4/3}(I_2^2 \mp 9\tau I_1^2), \qquad \text{where} \quad A = \pm 108a^{3/2}.$$

56. $\boldsymbol{yy'_x - y = \frac{15}{4}x + Ax^{-7}}$.

Solution in the parametric form:

$$x = aI_1^{1/2}I_3^{-3/8}, \quad y = \frac{a}{2}I_1^{-3/2}I_3^{-3/8}(I_2I_3 - 3I_1^2), \qquad \text{where} \quad A = \pm\frac{3a^8}{4}.$$

57. $\boldsymbol{yy'_x - y = -\dfrac{10}{49}x + \dfrac{2A}{49}(4x^{1/2} + 61A + 12A^2x^{-1/2})}.$

Solution in the parametric form:

$$x = a(7RI_1 - 3)^2, \quad y = 14aI_1[\pm(10\tau^3 - 1)I_1 - R], \qquad \text{where} \quad A = \sqrt{a}.$$

58. $\boldsymbol{yy'_x - y = -\dfrac{12}{49}x + \dfrac{2A}{49}(x^{1/2} + 166A + 55A^2x^{-1/2})}.$

Solution in the parametric form:

$$x = aI_2^{-4}(42\tau I_1^2 \mp 5I_2^2)^2, \quad y = \mp 84aI_1^2I_2^{-4}(3\tau I_2^2 + I_2 \mp 12\tau^2 I_1^2), \qquad \text{where} \quad A = \pm\sqrt{a}.$$

59. $\boldsymbol{yy'_x - y = -\dfrac{4}{25}x + \dfrac{A}{50}(7x^{1/2} + 49A + 6A^2x^{-1/2})}.$

Solution in the parametric form:

$$x = aI_1^{-4}(5I_2I_3 - 16I_1^2)^2, \quad y = -5aI_1^{-4}I_3(\pm 3I_3^2 - I_2^2I_3 + 8I_1^2I_3), \qquad \text{where} \quad A = 8\sqrt{a}.$$

60. $\boldsymbol{yy'_x - y = \frac{15}{4}x + 6Ax^{-1/3} - 3A^2x^{-5/3}}.$

Solution in the parametric form:

$$x = 2a\tau^{3/2}I_1^{3/2}I_2^{-3/4}, \quad y = a\tau^{-1/2}I_1^{-1/2}I_2^{-3/4}(2\tau I_2^2 + I_2 - 3\tau^2 I_1^2),$$

where $A = -\frac{1}{3}a(2a)^{1/3}$.

61. $\boldsymbol{yy'_x - y = -\frac{3}{16}x + Ax^{-1/3} + Bx^{-5/3}}.$

The substitution $x = \tau^{-3/2}$ leads to the equation

$$yy'_\tau = -\frac{3}{2}\tau^{-5/2}y + \frac{9}{32}\tau^{-4} - \frac{3}{2}A\tau^{-2} - \frac{3}{2}B$$

coincident with equation 1.3.3.13 when $n = -1/2$, $c = 0$, $b = 3/4$, $d = 3A/2$, $a^2 = -3B$.

62. $\boldsymbol{yy'_x - y = -\frac{5}{36}x + Ax^{-3/5} - Bx^{-7/5}, \qquad B > 0}.$

The transformation

$$x = \left(w + \frac{A}{B} - \frac{1}{3}\sqrt{\tau}\right)^{-5/4}, \qquad y = \frac{5}{6}x + \left(\frac{5}{3}B\right)^{1/2}\sqrt{\tau}\,x^{1/5}$$

leads to an equation of the form 1.3.1.3:

$$ww'_\tau - w = -\frac{2}{9}\tau + \frac{2A}{3B} + \left(\frac{5}{27B}\right)^{1/2}\frac{1}{\sqrt{\tau}}.$$

63. $\boldsymbol{yy'_x - y = k(Ax^2 + Bx + C)^{-1/2}}.$

The transformation

$$x = \frac{4(b_2w^2 + b_1w + b_0)}{4A - w^2}, \quad y = \xi + \frac{4(b_2w^2 + b_1w + b_0)}{4A - w^2},$$

where parameters b_2, b_1, and b_0 are found from the relations $B = 4Ab_2 - b_0$ and $C = b_1^2 - 4b_0b_2$, leads to the Riccati equation:

$$\pm kw'_\xi = (-\tfrac{1}{4}\xi + b_2)w^2 + b_1w + A\xi + b_0.$$

For $C > 0$, we may set $b_2 = 0$, $b_1 = \sqrt{C}$, $b_0 = -B$.

In books by Zaitsev & Polyanin (1993, 1994) it is shown that the original equation is reducible to the degenerate hypergeometric equation.

64. $\boldsymbol{yy'_x - y = -\frac{12}{49}x + 3A(\frac{1}{49}+B)x^{1/2} + 3A^2(\frac{4}{49}-\frac{5}{2}B) + \frac{15}{4}A^3(\frac{1}{49}+\frac{5}{4}B)x^{-1/2}}$.

The substitution $x = (\xi^2 + \frac{5}{4}A)^2$ leads to an equation of the form 1.3.3.13 with $n=3$, $a = 4/7$, $c = 0$, $b = A$, $d = 12A(\frac{2}{7} - B)$:

$$yy'_\xi = (4\xi^2 + 5A)\xi y - \Big[\frac{48}{49}\xi^4 + 12A\Big(\frac{2}{7} - B\Big)\xi^2 + 3A^2\Big]\xi^3.$$

65. $\boldsymbol{yy'_x - y = -\frac{6}{25}x}$
$\boldsymbol{+\frac{4}{75}B^2\big[(2-A)x^{1/3} - \frac{3}{2}B(2A+1) + B^2(1-3A)x^{-1/3} - AB^3x^{-2/3}\big]}$.

The transformation

$$x = w^{-3}, \qquad y = \left[\xi + \frac{B(3-2Bw)}{5(w^2 + \frac{1}{B}w)}\right]\Big(w + \frac{1}{B}\Big)^2 w^{-2}$$

leads to the Riccati equation:

$$\Big(\frac{2}{B}\xi^2 - \frac{2}{5}B^2\xi + \frac{4}{25}AB^5\Big)w'_\xi = \xi w^2 + \Big(\frac{1}{B}\xi - \frac{2}{5}B^2\Big)w + \frac{3}{5}B.$$

66. $\boldsymbol{yy'_x - y = \frac{3}{4}x - \frac{3}{2}Ax^{1/3} + \frac{3}{4}A^2x^{-1/3} - \frac{27}{625}A^4x^{-5/3}}$.

The transformation

$$x = A^{3/2}f^{-3/2}, \quad y = 3A^{3/2}\Big(\xi - \frac{3}{25} - \frac{1}{2f} + \frac{1}{2f^2}\Big)\sqrt{f}, \qquad \text{where} \quad f = \frac{w}{\xi^2 - \frac{6}{25}\xi},$$

leads to an equation of the form 1.3.1.46: $ww'_\xi - w = \frac{6}{25}\xi - \xi^2$.

67. $\boldsymbol{yy'_x - y = -\frac{6}{25}x + \frac{7}{5}Ax^{1/3} + \frac{31}{3}A^2x^{-1/3} - \frac{100}{3}A^4x^{-5/3}}$.

Denote $A = \frac{7}{100}a$ and perform the transformation

$$x = \xi^{3/2}, \quad y = \frac{7}{20}\Big(w + \frac{8}{7}\xi - \frac{3}{5}a - \frac{7}{50}\frac{a^2}{\xi}\Big)\sqrt{\xi}, \qquad \text{where} \quad \xi = z - \frac{3}{10}a.$$

As a result we obtain an equation of the form 1.3.4.30 with $n = \frac{1}{7}$, $c = -\frac{3}{10}a$:

$$\Big[\Big(z - \frac{3}{10}a\Big)w + \frac{8}{7}z^2 - \frac{9}{7}az + \frac{1}{7}a^2\Big]w'_z = -\frac{1}{2}w^2 + 2zw.$$

68. $\boldsymbol{yy'_x - y = -\frac{10}{49}x + \frac{13}{5}A^2x^{-1/5} - \frac{7}{20}A^3x^{-4/5}}$.

Denote $A = 8a^{-2}$. The transformation

$$\sqrt{\tau} = a^3\Big(\frac{5}{112}x^{3/5} - \frac{1}{16}yx^{-2/5}\Big), \quad w = \frac{4}{7}\tau + x^{-3/5} - \frac{39}{42}a^2$$

leads to an equation of the form 1.3.1.64 with $B = -1/49$:

$$ww'_\tau - w = -\frac{12}{49}\tau + \frac{39}{98}a^2 - \frac{15}{784}a^3\tau^{-1/2}.$$

69. $\boldsymbol{yy'_x - y = -\frac{33}{196}x + \frac{286}{3}A^2x^{-5/11} - \frac{770}{9}A^3x^{-13/11}}.$

Denote $A = \frac{8}{5}a^{-2}$. The transformation

$$\sqrt{\tau} = \frac{15a^3}{448}\left(x^{8/11} - \frac{14}{11}yx^{-3/11}\right), \quad w = \frac{3}{7}\tau + x^{-8/11} - \frac{39}{56}a^2$$

leads to an equation of the form 1.3.1.64 with $B = -1/49$:

$$ww'_\tau - w = -\frac{12}{49}\tau + \frac{39}{98}a^2 - \frac{15}{784}a^3\tau^{-1/2}.$$

70. $\boldsymbol{yy'_x - y = -\frac{21}{100}x + \frac{7}{9}A^2(123x^{-1/7} + 280Ax^{-5/7} - 400A^2x^{-9/7})}.$

Denote $A = 1/a$. The transformation

$$x = \xi^{-7/4}, \quad y = \frac{35}{3a^2}\left(w + 4\xi + \frac{7}{5}a + \frac{3}{50}\frac{a^2}{\xi}\right)\xi^{-3/4}, \qquad \text{where} \quad \xi = z - \frac{21}{20}a,$$

leads to an equation of the form 1.3.4.30 with $n = 3$, $c = -\frac{21}{20}a$:

$$\left[\left(z - \frac{21}{20}a\right)w + 4z^2 - 7az + 3a^2\right]w'_z = \frac{3}{4}w^2 + 2zw.$$

71. $\boldsymbol{yy'_x - y = ax + bx^m}.$

1°. For $m \neq 3$, the transformation

$$\tau = B^2\left[(m-3)\frac{y}{x} + 1\right]^2, \quad w = 2(m-3)B^2\left(bx^{m-1} - \frac{y^2}{x^2} + \frac{y}{x} + a\right)$$

leads to the equation

$$ww'_\tau - w = \frac{2(m-1)}{(m-3)^2}\{\tau - mB\tau^{1/2} + [2m-3-a(m-3)^2]B^2 + [2-m+a(m-3)^2]B^3\tau^{-1/2}\}.$$

2°. Let $m \neq 1$ and $a > -1/4$. Assume

$$a = -\frac{(n+2)(n+m+1)}{(2n+m+3)^2}, \qquad \text{whence} \quad n_{1,2} = \frac{1}{2}\left(\pm\frac{m-1}{\sqrt{1+4a}} - m - 3\right).$$

Then, the transformations

$$x = \xi^{\frac{n+2}{m-1}}w, \quad y = \frac{m-1}{2n+m+3}\xi^{\frac{n+2}{m-1}}\left(\xi w'_\xi + \frac{n+2}{m-1}\right), \qquad n = n_{1,2}$$

reduce the original equation to the classical Emden—Fowler equation $w''_{\xi\xi} = A\xi^n w^m$, where $A = \left(\frac{2n+m+3}{m-1}\right)^2 b$, which is discussed below in Section 2.3.

72. $\boldsymbol{yy'_x - y = -\frac{m+1}{(m+2)^2}x + Ax^{2m+1} + Bx^{3m+1}}.$

Assume $A = -\frac{am}{2(m+2)^2b^2}$, $B = -\frac{m^2}{2(m+2)^3b^2}$. The transformation

$$\sqrt{\tau} = -\frac{(m+2)^2}{m}byx^{-m-1} + \frac{m+2}{m}bx^{-m}, \quad w = \frac{2(m+1)}{m+2}\tau + x^m + \frac{m+2}{m}a$$

leads to the equation

$$ww'_\tau - w = \frac{2m(m+1)}{(m+2)^2}\tau + a + b\tau^{-1/2}$$

(see Table 1.2 with $\alpha = 0$ in Subsection 1.3.1).

73. $yy'_x - y = a^2\lambda e^{2\lambda x} - a(b\lambda + 1)e^{\lambda x} + b.$

Particular solution: $y_0 = ae^{\lambda x} - b.$

74. $yy'_x - y = a^2\lambda e^{2\lambda x} + a\lambda x e^{\lambda x} + be^{\lambda x}.$

Particular solution: $y_0 = ae^{\lambda x} + x + \dfrac{b}{a\lambda}.$

75. $yy'_x - y = 2a^2\lambda\sin(2\lambda x) + 2a\sin(\lambda x).$

Particular solution: $y_0 = -2a\sin(\lambda x).$

76. $yy'_x - y = a^2 f'_x f''_{xx} - \dfrac{(f+b)^2}{(f'_x)^3} f''_{xx}, \qquad f = f(x).$

Particular solutions: $y_1 = af'_x + \dfrac{f+b}{f'_x}, \quad y_2 = -af'_x + \dfrac{f+b}{f'_x}.$

1.3.2. Equations of the Form $yy'_x = f(x)y + 1$

1. $yy'_x = (ax + b)y + 1.$

The substitution $\xi = y - \frac{1}{2}ax^2 - bx$ leads to the Riccati equation with respect to $x = x(\xi)$: $x'_\xi = \frac{1}{2}ax^2 + bx + \xi.$

2. $yy'_x = (ax + b)^{-2}y + 1.$

The substitution $a\xi = -(ax + b)^{-1}$ leads to an equation of the form 1.3.1.33: $yy'_\xi = y + (a\xi)^{-2}.$

3. $yy'_x = \left(a - \dfrac{1}{ax}\right)y + 1.$

The substitution $\xi = y - ax$ leads to the Bernoulli equation with respect to $x = x(\xi)$: $\xi x'_\xi + a\xi x + a^2x^2 = 0.$

4. $yy'_x = (ax + b)^{-1/2}y + 1.$

The substitution $z = \dfrac{2}{a}(ax + b)^{1/2}$ leads to an equation of the form 1.3.1.2: $yy'_z = y + \frac{1}{2}az.$

5. $yy'_x = 3(ax^{3/2} + 8x)^{-1/2}y + 1.$

The substitution $z = \dfrac{12}{a}(ax^{1/2} + 8)^{1/2}$ leads to the equation of the form 1.3.1.10 with $m = 3$: $yy'_z = y - \dfrac{2}{9}z + \dfrac{a^2}{5184}z^3.$

6. $yy'_x = (ax^{-2/3} - \frac{2}{3}a^{-1}x^{-1/3})y + 1.$

The transformation $x = a^{3/2}w^3$, $y = \xi - w^2$ leads to the Riccati equation: $3a^{3/2}\xi w'_\xi = \xi - w^2.$

7. $yy'_x = ae^{\lambda x}y + 1.$

The substitution $\xi = \dfrac{a}{\lambda}e^{\lambda x}$ leads to an equation of the form 1.3.1.16: $yy'_\xi = y + (\lambda\xi)^{-1}$.

8. $yy'_x = (ae^{\lambda x} + be^{-\lambda x})y + 1.$

The transformation

$$\xi = y + \frac{b}{\lambda}e^{-\lambda x} - \frac{a}{\lambda}e^{\lambda x}, \quad w = e^{\lambda x}$$

leads to the Riccati equation: $w'_\xi = aw^2 + \lambda\xi w - b$.

9. $yy'_x = ay\cosh x + 1.$

This is a special case of equation 1.3.3.20 with $b = 0$, $c = 1$.

10. $yy'_x = ay\sinh x + 1.$

This is a special case of equation 1.3.3.21 with $b = 0$, $c = 1$.

11. $yy'_x = a\cos(\omega x)\,y + 1.$

The transformation

$$x = -\frac{2}{\omega}\arctan\frac{4u}{\omega}, \quad y = \tau - \frac{8au}{16u^2 + \omega^2}$$

leads to the Riccati equation: $u'_\tau = -2\tau u^2 + au - \frac{1}{8}\omega^2\tau$.

12. $yy'_x = a\sin(\omega x)\,y + 1.$

The substitution $x = \xi + \dfrac{\pi}{2\omega}$ leads to an equation of the form 1.3.2.11: $yy'_\xi = a\cos(\omega\xi)\,y + 1$.

1.3.3. Equations of the Form $yy'_x = f_1(x)y + f_0(x)$

Preliminary comments. With the aid of the substitution $\xi = \int f_1(x)\,dx$, these equations are reducible to the form

$$yy'_\xi = y + f(\xi), \qquad \text{where} \quad f(\xi) = f_0(x)/f_1(x), \tag{1}$$

and by means of the substitution $z = \int f_0(x)\,dx$ they can be reduced to the form

$$yy'_z = g(z)y + 1, \qquad \text{where} \quad g(z) = f_1(x)/f_0(x). \tag{2}$$

Concrete equations of the form (1) and (2) are outlined in Subsection 1.3.1 and 1.3.2, respectively.

1. $yy'_x = (ax + 3b)y + cx^3 - abx^2 - 2b^2x.$

The substitution $y = x^2t + bx$ leads to the linear equation with respect to $x = x(t)$: $(-2t^2 + at + c)x'_t = tx + b$.

2. $yy'_x = (3ax + b)y - a^2x^3 - abx^2 + cx.$

The substitution $y = xw + ax^2$ leads to the Bernoulli equation with respect to $x = x(w)$: $(-w^2 + bw + c)x'_w = wx + ax^2$.

3. $2yy'_x = (7ax + 5b)y - 3a^2x^3 - 2cx^2 - 3b^2x.$

This is a special case of equation 1.3.3.11 with $m = 3/2$, $k = 1/2$.

4. $yy'_x = [(3 - m)x - 1]y + (m - 1)(x^3 - x^2 - ax).$

The transformation

$$x = w/z, \quad y = -z^{m-1} + x^2 - x - a$$

leads to the equation $ww'_z = w + az + z^m$ whose solvable cases are outlined in Subsection 1.3.1 (see Table 1.1).

5. $yy'_x + x(ax^2 + b)y + x = 0.$

The substitution $z = -\frac{1}{2}x^2$ leads to an equation of the form 1.3.2.1: $yy'_z = (-2az + b)y + 1$.

6. $yy'_x = 3(ax + b)^{-1/3}x^{-5/3}y + 3(ax + b)^{-2/3}x^{-7/3}.$

The substitution $w = \dfrac{1}{xy} + \dfrac{1}{3}\left(\dfrac{ax + b}{x}\right)^{1/3}$ leads to an equation with separation of variables: $w'_x = x^{-1/3}(ax + b)^{-2/3}(\frac{1}{9}a - 3w^3)$.

7. $3yy'_x = (-7Ax + 6s - 2\lambda)x^{-1/3}y$
$\qquad + 2(Ax + 5\lambda)(-Ax + 3s + 4\lambda)x^{1/3} + 6(\lambda s x - 1)x^{-2/3},$

where $A = \lambda s(3s + 4\lambda)$.

The transformation

$$x = (\xi + \lambda s)^{-1}, \quad y = (w + 4\lambda + 3s - Ax)x^{2/3}$$

leads to an equation of the form 1.3.4.12 with $a = 1/3$:

$$[(\xi + \lambda s)w + (4\lambda + 3s)\xi]w'_\xi = \tfrac{2}{3}w^2 + 2(3\lambda + s)w + 2\xi.$$

8. $yy'_x = x^{n-1}[(1 + 2n)x + an]y - nx^{2n}(x + a).$

The transformation

$$x = \frac{w}{z}, \quad y = -\frac{1}{z^n} + x^{n+1} + ax^n$$

leads to an equation with separation of variables: $w'_z = w^{-n} - a$.

9. $yy'_x = a(x - nb)x^{n-1}y + c[x^2 - (2n + 1)bx + n(n + 1)b^2]x^{2n-1}.$

The substitution $\xi = ax^n\left(\dfrac{x}{n + 1} - b\right)$ leads to an equation of the form 1.3.1.2: $yy'_\xi = y + (n + 1)ca^{-2}\xi$.

10. $\boldsymbol{yy'_x = [a(2n+k)x^k + b]x^{n-1}y + (-a^2nx^{2k} - abx^k + c)x^{2n-1}}.$

The substitution $y = x^n(w + ax^k)$ leads to the Bernoulli equation with respect to $x = x(w)$: $(-nw^2 + bw + c)x'_w = wx + ax^{k+1}$.

11. $\boldsymbol{yy'_x = [a(2m+k)x^{2k} + b(2m-k)]x^{m-k-1}y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1}}.$

The transformation $z = x^k$, $y = x^m(t + ax^k + bx^{-k})$ leads to the Riccati equation with respect to $z = z(t)$:

$$(-mt^2 + 2abm - c)z'_t = akz^2 + ktz + bk. \tag{1}$$

The substitution

$$z = \frac{mt^2 + c_0}{ak}\frac{w'_t}{w}, \qquad \text{where} \quad c_0 = c - 2abm,$$

reduces equation (1) to a second order linear equation:

$$(mt^2 + c_0)^2 w''_{tt} + (2m+k)t(mt^2 + c_0)w'_t + abk^2 w = 0. \tag{2}$$

The substitution

$$\xi = \frac{t}{\sqrt{t^2 + (c_0/m)}}, \quad u = (1 - \xi^2)^{\mu/2} w, \qquad \text{where} \quad \mu = -\frac{m+k}{2m},$$

brings equation (2) to the Legendre equation:

$$(1 - \xi^2)u''_{\xi\xi} - 2\xi u'_\xi + [\nu(\nu+1) - \mu^2(1 - \xi^2)^{-1}]u = 0,$$

where ν is a root of the quadratic equation $\nu^2 + \nu + \dfrac{m^2 - k^2}{4m^2} - \dfrac{abk^2}{mc_0} = 0.$

12. $\boldsymbol{yy'_x = [(m+2l-3)x + n - 2l + 3]x^{-l}y + [(m+l-1)x^2 + (n-m-2l+3)x - n + l - 2]x^{1-2l}}.$

The transformation

$$x = \frac{\xi}{w}w'_\xi, \qquad y = A\xi^{n-l+2}w^{m+l-1} - x^{2-l} + x^{1-l}$$

leads to the generalized Emden—Fowler equation: $w'_\xi = A\xi^n w^m (w'_\xi)^l$, which is discussed in Section 2.5.

13. $\boldsymbol{yy'_x = [a(2n+1)x^2 + cx + b(2n-1)]x^{n-2}y - (na^2x^4 + acx^3 + dx^2 + bcx + nb^2)x^{2n-3}},$

where a, b, c, d, and n are arbitrary numbers.

The substitution $y = x^n t + ax^{n+1} + bx^{n-1}$ leads to the Riccati equation with respect to $x = x(t)$:

$$(-nt^2 + ct - d + 2nab)x'_t = ax^2 + tx + b.$$

14. $yy'_x = [a(n-1)x + b(2\lambda + n)]x^{\lambda-1}(ax+b)^{-\lambda-2}y - [anx + b(\lambda+n)]x^{2\lambda-1}(ax+b)^{-2\lambda-3}.$

The substitution

$$y = \left[\frac{1}{w} + \frac{1}{x^n(ax+b)}\right] x^{\lambda+n}(ax+b)^{-\lambda}$$

leads to an equation of the form 1.3.4.5:

$$(w + ax^{n+1} + bx^n)w'_x = [anx^n + b(\lambda+n)x^{n-1}]w.$$

15. $yy'_x = (ae^x + b)y + ce^{2x} - abe^x - b^2.$

The transformation $x = \ln w$, $y = tw + b$ leads to a linear equation: $(-t^2 + at + c)w'_t = tw + b$.

16. $yy'_x = [a(2\mu + \lambda)e^{\lambda x} + b]e^{\mu x}y + (-a^2\mu e^{2\lambda x} - abe^{\lambda x} + c)e^{2\mu x}.$

The substitution $\xi = e^x$ leads to an equation of the form 1.3.3.10:

$$yy'_\xi = [a(2\mu + \lambda)\xi^\lambda + b]\xi^{\mu-1}y + (-a^2\mu\xi^{2\lambda} - ab\xi^\lambda + c)\xi^{2\mu-1}.$$

17. $yy'_x = (ae^{\lambda x} + b)y + c[a^2e^{2\lambda x} + ab(\lambda x + 1)e^{\lambda x} + b^2\lambda x].$

The substitution $\xi = \dfrac{a}{\lambda}e^{\lambda x} + bx$ leads to an equation of the form 1.3.1.2: $yy'_\xi = y + c\lambda\xi$.

18. $yy'_x = e^{\lambda x}(2a\lambda x + a + b)y - e^{2\lambda x}(a^2\lambda x^2 + abx + c).$

The substitution $y = e^{\lambda x}(\xi + ax)$ leads to a linear equation with respect to $x = x(\xi)$: $(-\lambda\xi^2 + b\xi - c)x'_\xi = ax + \xi$.

19. $yy'_x = e^{ax}(2ax^2 + 2x + b)y + e^{2ax}(-ax^4 - bx^2 + c).$

The substitution $y = e^{ax}(\xi + x^2)$ leads to the Riccati equation with respect to $x = x(\xi)$: $(-a\xi^2 + b\xi + c)x'_\xi = x^2 + \xi$.

20. $yy'_x = (a\cosh x + b)y - ab\sinh x + c.$

The transformation $t = y - a\sinh x$, $\xi = e^x$ leads to the Riccati equation: $2(bt + c)\xi'_t = a\xi^2 + 2t\xi - a$.

21. $yy'_x = (a\sinh x + b)y - ab\cosh x + c.$

The transformation $t = y - a\cosh x$, $\xi = e^x$ leads to the Riccati equation: $2(bt + c)\xi'_t = a\xi^2 + 2t\xi + a$.

22. $yy'_x = (2\ln x + a + 1)y + x(-\ln^2 x - a\ln x + b).$

The transformation $x = e^w$, $y = (\xi + w)e^w$ leads to a linear equation: $(-\xi^2 + a\xi + b)w'_\xi = w + \xi$.

23. $yy'_x = (2\ln^2 x + 2\ln x + a)y + x(-\ln^4 x - a\ln^2 x + b).$

Performing the transformation $x = e^w$, $y = (\xi + w^2)e^w$, we obtain the Riccati equation: $(-\xi^2 + a\xi + b)w'_\xi = w^2 + \xi$.

24. $\boldsymbol{yy'_x = ax\cos(\omega x^2)\,y + x}$.

The substitution $\xi = \frac{1}{2}x^2$ leads to the Abel equation of the form 1.3.2.11: $yy'_\xi = a\cos(2\omega\xi)\,y + 1$.

25. $\boldsymbol{yy'_x = ax\sin(\omega x^2)\,y + x}$.

The substitution $\xi = \frac{1}{2}x^2$ leads to the Abel equation of the form 1.3.2.12: $yy'_\xi = a\sin(2\omega\xi)\,y + 1$.

1.3.4. Equations of the Form $[g_1(x)y + g_0(x)]y'_x = f_2(x)y^2 + f_1(x)y + f_0(x)$

Preliminary comments. With the aid of the substitution

$$w = \Big(y + \frac{g_0}{g_1}\Big)E, \qquad \text{where} \quad E = \exp\Big(-\int \frac{f_2}{g_1}\,dx\Big), \tag{1}$$

these equations are reducible to a simpler form:

$$ww'_x = F_1(x)w + F_0(x), \tag{2}$$

where

$$F_1 = \left[\frac{d}{dx}\Big(\frac{g_0}{g_1}\Big) + \frac{f_1}{g_1} - 2\frac{g_0 f_2}{g_1^2}\right]E, \qquad F_0 = \Big(\frac{f_0}{g_1} - \frac{g_0 f_1}{g_1^2} + \frac{g_0^2 f_2}{g_1^3}\Big)E^2.$$

Concrete Abel equations of the form (2) are outlined in 1.3.1–1.3.3. In the degenerate cases with $F_0 \equiv 0$ or $F_1 \equiv 0$, the variables in equation (2) are separable.

1. $\boldsymbol{(Ay + Bx + a)y'_x + By + kx + b = 0}$.

Solution: $Ay^2 + kx^2 + 2(Bxy + ay + bx) = C$.

2. $\boldsymbol{(y + ax + b)y'_x = \alpha y + \beta x + \gamma}$.

The substitution $y = u - ax - b$ leads to the equation

$$uu'_x = (a+\alpha)u + (\beta - a\alpha)x + \gamma - b\alpha$$

which is separable with $a = -\alpha$.

For $a \neq -\alpha$, the substitution $u = (a+\alpha)w$ yields an equation of the form 1.3.1.1 or 1.3.1.2:

$$ww'_x = w + \Delta^{-2}(\beta - a\alpha)x + \Delta^{-2}(\gamma - b\alpha), \qquad \text{where} \quad \Delta = a + \alpha.$$

3. $\boldsymbol{(y + akx^2 + bx + c)y'_x = -ay^2 + 2akxy + my + k(k + b - m)x + s}$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[-az^2 + (m-k)z + s - ck]x'_z = akx^2 + (b+k)x + z + c.$$

4. $\boldsymbol{(y + Ax^n + a)y'_x + nAx^{n-1}y + kx^m + b = 0.}$

Solution: $\displaystyle y^2 + \frac{2k}{m+1}x^{m+1} + 2(Ax^n y + ay + bx) = C.$

5. $\boldsymbol{(y + ax^{n+1} + bx^n)y'_x = (anx^n + cx^{n-1})y.}$

The substitution $y = x^n(w - b)$ leads to the Bernoulli equation with respect to $x = x(w)$:

$$[-nw^2 + (bn + c)w - bc]x'_w = wx + ax^2.$$

6. $\boldsymbol{xyy'_x = ay^2 + by + cx^n + s.}$

The transformation $\xi = x^{-a}$, $w = -\dfrac{a}{b}x^{-a}y$ leads to the equation $ww'_\xi = w + A\xi + B\xi^m$, where $A = -ab^{-2}s$, $B = -ab^{-2}c$, $m = (a - n)/a$ (see Subsection 1.3.1).

7. $\boldsymbol{xyy'_x = -ny^2 + a(2n + 1)xy + by - a^2nx^2 - abx + c.}$

The substitution $y = w + ax$ leads to the Bernoulli equation with respect to $x = x(w)$:

$$(-nw^2 + bw + c)x'_w = wx + ax^2.$$

8. $\boldsymbol{2xyy'_x = (1 - n)y^2 + [a(2n + 1)x + 2n - 1]y - a^2nx^2 - bx - n.}$

The transformation $x = \xi^2$, $y = \xi t + a\xi^2 + 1$ leads to the Riccati equation:

$$(-nt^2 + 2an - b)\xi'_t = a\xi^2 + t\xi + 1.$$

9. $\boldsymbol{(Axy - Aky + Bx - Bk)y'_x = Cy^2 + Dxy + (B - Dk)y.}$

The transformation $x = w + k$, $y = \xi w$ leads to a linear equation with respect to $w = w(x)$:

$$[(C - A)\xi^2 + D\xi]w'_\xi = A\xi w + B.$$

10. $\boldsymbol{[(3ax + \lambda s)y + (4\lambda + 3s)x]y'_x = 2ay^2 + 2(3\lambda + s)y + 2x.}$

The substitution $w = ay^2 + (3\lambda + s)y + x$ leads to an equation of the form 1.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y, \quad \text{where } b = s + 2\lambda, \quad c = \tfrac{1}{2}a(13\lambda + 6s).$$

11. $\boldsymbol{[(4ax + \lambda s)y + (4\lambda + 3s)x]y'_x = \tfrac{3}{2}ay^2 + 2(3\lambda + s)y + 2x.}$

The substitution $w = \frac{3}{4}ay^2 + (3\lambda + s)y + x$ leads to an equation of the form 1.3.3.3:

$$2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y, \quad \text{where } b = s + 2\lambda, \quad c = \tfrac{1}{8}a(60\lambda + 25s).$$

12. $\boldsymbol{(2Axy + ay + bx + c)y'_x = Ay^2 + Ak^2x^2 + my + k(ak + b - m)x + s.}$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - ck]x'_z = 2Akx^2 + (2Az + ak + b)x + az + c.$$

13. $\displaystyle \left[2xy + (1-m)Ay - \frac{2(m+1)}{m+3}x\right]y'_x = \frac{1-m}{2}y^2 + \frac{m-1}{m+3}y + x.$

The substitution $w = \dfrac{1-m}{2}y^2 + \dfrac{m-1}{m+3}y + x$ leads to an equation of the form 1.3.3.4:

$$ww'_y = [(3-m)y-1]w + (m-1)(y^3 - y^2 - ay), \qquad \text{where} \quad a = A - \frac{2(m+1)}{(m+3)^2}.$$

14. $x(2ay + bx)y'_x = a(2-m)y^2 + b(1-m)xy + cx^2 + Ax^{m+2}.$

The transformation $z = y/x$, $w = -Ax^m + amz^2 + bmz - c$ leads to the equation with separated variables: $ww'_z = m(2az+b)(amz^2 + bmz - c)$.

15. $(xy + x^2 + a)y'_x = y^2 + xy + b.$

Solution: $(x+y)^2 + a + b = C(bx - ay)^2$.

16. $(2Axy + Bx^2 + b)y'_x = Ay^2 + k(Ak+B)x^2 + c.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(Az^2 + c - bk)x'_z = (2Ak + B)x^2 + 2Azx + b.$$

17. $(Axy + Bx^2 + kx)y'_x = Dy^2 + Exy + Fx^2 + ky.$

The substitution $y = xz$ leads to a linear equation with respect to $x = x(z)$:

$$[(D-A)z^2 + (E-B)z + F]x'_z = (Az + B)x + k$$

18. $(Axy + Bx^2 + kx)y'_x = Ay^2 + Bxy + (Ab+k)y + Bbx + bk.$

This is a special case of equation 1.3.4.22.
Solutions: $y = Cx - b$ and $Ay + Bx + k = 0$.

19. $(2Axy + Bx^2 + kx)y'_x = Ay^2 + Cxy + Dx^2 + ky - C\beta x - A\beta^2 - k\beta.$

The substitution $y = \xi x + \beta$ leads to a linear equation with respect to $x = x(\xi)$:

$$[-A\xi^2 + (C-B)\xi + D]x'_\xi = (2A\xi + B)x + 2A\beta + k.$$

20. $(Axy + Bx^2 + kx)y'_x = Ay^2 + Cxy + Dx^2 + (k - A\beta)y - C\beta x - k\beta.$

The substitution $y = \xi x + \beta$ leads to a linear equation with respect to $x = x(\xi)$:

$$[(C-B)\xi + D]x'_\xi = (A\xi + B)x + A\beta + k.$$

21. $(Axy + Aky + Bx^2 + Bkx)y'_x = Cy^2 + Dxy + k(D-B)y.$

The transformation $x = w - k$, $y = \xi w$ leads to a linear equation:

$$[(C-A)\xi^2 + (D-B)\xi]w'_\xi = (A\xi + B)w - kB.$$

22. $\boldsymbol{(Axy + Bx^2 + a_1x + b_1y + c_1)y'_x = Ay^2 + Bxy + a_2x + b_2y + c_2}$.

Jacobi equation.

1°. With the help of the transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are the parameters which are determined by solving the algebraic system

$$A\alpha\beta + B\alpha^2 + a_1\alpha + b_1\beta + c_1 = 0, \qquad A\beta^2 + B\alpha\beta + a_2\alpha + b_2\beta + c_2 = 0,$$

we obtain the equation

$$(A\bar{x}\bar{y} + B\bar{x}^2 + \bar{a}_1\bar{x} + \bar{b}_1\bar{y})\bar{y}'_{\bar{x}} = A\bar{y}^2 + B\bar{x}\bar{y} + \bar{a}_2\bar{x} + \bar{b}_2\bar{y},$$

where

$$\bar{a}_1 = 2B\alpha + A\beta + a_1, \qquad \bar{b}_1 = A\alpha + b_1,$$
$$\bar{a}_2 = B\beta + a_2, \qquad \bar{b}_2 = 2A\beta + B\alpha + b_2.$$

The transformation $z = \bar{y}/\bar{x}$, $\zeta = 1/\bar{x}$ leads to a linear equation:

$$[\bar{b}_1z^2 + (\bar{a}_1 - \bar{b}_2)z - \bar{a}_2]\zeta'_z = (\bar{b}_1z + \bar{a}_1)\zeta + Az + B.$$

2°. The original equation can be also rewritten in the form

$$(xy'_x - y)(n_3x + m_3y + k_3) - y'_x(n_1x + m_1y + k_1) + n_2x + m_2y + k_2 = 0.$$

The solution of this equation in the parametric form can be obtained from the solution of the following system of the constant-coefficient linear differential equations:

$$(x_1)'_t = n_1x_1 + m_1x_2 + k_1x_3,$$
$$(x_2)'_t = n_2x_1 + m_2x_2 + k_2x_3,$$
$$(x_3)'_t = n_3x_1 + m_3x_2 + k_3x_3,$$

using the formulae $x(t) = x_1/x_3$, $y(t) = x_2/x_3$.

23. $\boldsymbol{(Axy + Bx^2 + ay + bx + c)y'_x = kAxy + kBx^2 + my + k(ak + b - m)x + s}$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(m - ak)z + s - ck]x'_z = (Ak + B)x^2 + (Az + ak + b)x + az + c.$$

24. $\boldsymbol{(2Axy + Bx^2 + ay + bx + c)y'_x = Ay^2 + k(Ak + B)x^2 + aky + bkx + s}$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(Az^2 + s - ck)x'_z = (2Ak + B)x^2 + (2Az + ak + b)x + az + c.$$

25. $\boldsymbol{(2Axy - Akx^2 + ay + bx + c)y'_x = Ay^2 + my + k(ak + b - m)x + s}$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - ck]x'_z = Akx^2 + (2Az + ak + b)x + az + c.$$

26. $\boldsymbol{(2Axy + Bx^2 + ay - akx + b)y'_x = Ay^2 + k(Ak + B)x^2 + my - mkx + s}$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (m - ak)z + s - bk]x'_z = (2Ak + B)x^2 + 2Azx + az + c.$$

27. $(2Axy + Bx^2 + ay + bx + c)y'_x = Ay^2 + k(Ak + B)x^2 + by + ak^2x + s.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[Az^2 + (b - ak)z + s - ck]x'_z = (2Ak + B)x^2 + (2Az + ak + b)x + az + c.$$

28. $[Axy + Bx^2 + (m - 1)Aay - (Abm + Ba)x]y'_x$
$= Ay^2 + Bxy - (Ab + Bam)y + (m - 1)Bbx.$

This is a special case of equation 1.3.4.22.

Solution in the parametric form:

$$x = \frac{at + ACt^m}{t + C}, \qquad y = \frac{bt - BCt^m}{t + C}.$$

The solution can be presented in an implicit form as well:

$$C^m(Ay + Bx)^m + [A(b - y) + B(a - x)]^{m-1}(ay - bx) = 0.$$

29. $[(ax + c)y + (1 - n)x^2 + (2n - 1)x - n]y'_x = 2ay^2 + 2xy.$

The substitution $w = ay + x$ leads to an equation of the form 1.3.4.8:

$$2yww'_y = (1 - n)w^2 + [a(2n + 1)y + 2n - 1]w - a^2ny^2 - by - n,$$

where $b = (2n - 1)a - c$.

30. $[(x + c)y + (n + 1)x^2 - a(2n + 1)x + a^2n]y'_x = \dfrac{2n}{3n - 1}y^2 + 2xy.$

The transformation

$$z = \frac{3n - 1}{n - 1}\frac{1}{y}, \quad w = \frac{3n - 1}{n - 1}\frac{x}{y} + \frac{n}{n - 1}$$

leads to an equation of the form 1.3.4.8:

$$2zww'_z = (1 - n)w^2 + [a(2n + 1)z + 2n - 1]w - a^2nz^2 - bz - n,$$

where $b = \dfrac{(3n - 1)c + an(2n + 1)}{n - 1}$.

31. $x(2axy + b)y'_x = -a(m + 3)xy^2 - b(m + 2)y + cx^m.$

The transformation

$$z = xy, \quad w = -cx^{m+1} + a(m + 1)x^2y^2 + b(m + 1)xy$$

leads to the equation with separated variables: $ww'_z = (m + 1)^2(2az + b)(az^2 + bz)$.

32. $[(a_2x^2 + a_1x + a_0)y + b_2x^2 + b_1x + b_0]y'_x = c_2y^2 + c_1y + c_0.$

This is the Riccati equation with respect to $x = x(y)$.

The substitution $x = -\dfrac{c_2y^2 + c_1y + c_0}{a_2y + b_2}\dfrac{w'_y}{w}$ yields a second order linear equation:

$$f_2w''_{yy} - [(f_2)'_y + f_1f_2]w'_y + f_0f_2^2w = 0,$$

where $f_i = \dfrac{a_iy + b_i}{c_2y^2 + c_1y + c_0}$; $i = 1,\ 2,\ 3$.

33. $[(12a^2x^2 - 7ax + 1)y + 4cx^2 - 5bx]y'_x = -2x(3a^2y^2 + 2cy + 3b^2).$

The substitution $w = x(3a^2y^2 + 2cy + 3b^2)$ leads to an equation of the form 1.3.3.3: $2ww'_y = (7ay + 5b)w - 3a^2y^3 - 2cy^2 - 3b^2y$.

34. $x[(m-1)(Ax+B)y + m(Dx^2 + Ex + F)]y'_x$
$$= [A(1-n)x - Bn]y^2 + [D(2-n)x^2 + E(1-n)x - Fn]y.$$

Solution: $Axy + Dx^2 + Ex + By + F = Cx^ny^m$.

35. $x^2(2axy + b)y'_x = -4ax^2y^2 - 3bxy + cx^2 + k.$

The transformation $z = xy$, $w = 2ax^2y^2 + 2bxy - cx^2 - k$ leads to an equation with separated variables: $ww'_z = 2(2az + b)(2az^2 + 2bz - k)$.

36. $(xy + ax^n + bx^2)y'_x = y^2 + cx^n + bxy.$

The transformation $t = y/x$, $z = x^{n-2}$ leads to a linear equation: $(c - at)z'_t = (n-2)(az + t + b)$.

37. $x(2ax^ny + b)y'_x = -a(3n+m)x^ny^2 - b(2n+m)y + Ax^m + cx^{-n}.$

The transformation

$$z = x^ny, \qquad w = -Ax^{n+m} + (n+m)(az^2 + bz) - c$$

leads to an equation with separated variables:

$$ww'_z = (n+m)^2(2az+b)\Big(az^2 + bz - \frac{c}{n+m}\Big).$$

38. $yy'_x = -ny^2 + a(2n+1)e^xy + by - a^2ne^{2x} - abe^x + c.$

Performing the transformation $x = \ln\xi$, $y = w + a\xi$, we obtain the Bernoulli equation: $(-nw^2 + bw + c)\xi'_w = w\xi + a\xi^2$.

1.3.5. Some Types of First and Second Order Equations Reducible to Abel Equations of the Second Kind

► *Notation: f, g, h, p, φ, ψ, Φ, F, and G are arbitrary functions of their arguments.*

1. Quasi-homogeneous equation

$$f(x^\nu y)x^{\nu+1}y'_x + g(x^\nu y) + Ax^\lambda = 0.$$

In the particular case when $\lambda = 0$ this equation is homogeneous. The transformation

$$z = x^\nu y, \quad v = Ax^\lambda + g(z) - \nu z f(z)$$

leads to the Abel equation:

$$vv'_z = [-(\lambda+\nu)f + g'_z - \nu z f'_z]v + \lambda f(g - \nu z f).$$

2. Quasi-homogeneous equation

$$f(x^\nu y)x^{\nu+1}y'_x + g(x^\nu y) + x^\lambda[h(x^\nu y)x^{\nu+1}y'_x + p(x^\nu y)] = 0.$$

The transformation $z = x^\nu y$, $\zeta = x^{-\lambda}$ leads to the Abel equation:

$$\bigl\{[g(z) - \nu z f(z)]\zeta + p(z) - \nu z h(z)\bigr\}\zeta'_z = \lambda f(z)\zeta^2 + \lambda h(z)\zeta.$$

3. Equations of the theory of chemical reactors and the combustion theory

$$y''_{xx} - ay'_x = f(y).$$

The substitution $w(y) = y'_x/a$ leads to the Abel equation:

$$ww'_y - w = a^{-2}f(y),$$

whose solvable cases are given in Subsection 1.3.1.

4. Equations of the theory of nonlinear oscillations

$$y''_{xx} + \varphi(y)y'_x + y = 0.$$

The substitution $z(y) = y'_x$ leads to the Abel equation:

$$zz'_y + \varphi(y)z + y = 0, \qquad (1)$$

which is reduced, with the aid of the substitution $\tau = \frac{1}{2}(a - y^2)$, to the following form:

$$zz'_\tau = g(\tau)z + 1, \qquad \text{where} \quad g(\tau) = \pm\frac{\varphi\bigl(\pm\sqrt{a-2\tau}\,\bigr)}{\sqrt{a-2\tau}}. \qquad (2)$$

Concrete cases of equation (2) are outlined in Subsection 1.3.2.

5. Equations of the theory nonlinear oscillations

$$y''_{xx} + \Phi(y'_x) + y = 0.$$

The transformation $z = y'_x$, $w = -y - \Phi(y'_x)$ leads to the Abel equation of the form (1):

$$ww'_z + \Phi'_z(z)w + z = 0.$$

6. Homogeneous equation with respect to the independent variable

$$x^2y''_{xx} = xg(y)y'_x + f(y).$$

The substitution $w(y) = xy'_x$ leads to the Abel equation:

$$ww'_y = [g(y) + 1]w + f(y).$$

7. Homogeneous equation in the extended sense

$$xy''_{xx} = g(yx^k)y'_x + x^{-k-1}f(yx^k).$$

The transformation $t = yx^k$, $u = x^k(xy'_x + ky)$ leads to the Abel equation:

$$uu'_t = [g(t) + 2k + 1]u + f(t) - ktg(t) - k(k+1)t.$$

To the Emden—Fowler equation, discussed in Section 2.3, correspond $g(t) = -\sigma$, $f(t) = At^m$, $k = \frac{n+2}{m-1}$.

8. Homogeneous equation in the extended sense

$$y''_{xx} = x^\alpha y^\beta F\left(\frac{x}{y}y'_x\right) + yx^{-2}G\left(\frac{x}{y}y'_x\right).$$

The transformation

$$\eta = \frac{x}{y}y'_x, \quad v = x^{\alpha+2}y^{\beta-1}$$

leads to the Abel equation:

$$[F(\eta)v + G(\eta) + \eta - \eta^2]v'_\eta = [(\beta-1)\eta + \alpha + 2]v.$$

To the generalized Emden—Fowler equation, discussed in Section 2.5, correspond $\alpha = n - l$, $\beta = m + l$, $F(\eta) = A\eta^l$, $G(\eta) = 0$.

9. Exponential-homogeneous equation

$$y''_{xx} = x^\alpha e^{\beta y} f(xy'_x) + x^{-2}g(xy'_x).$$

The substitution $\zeta = xy'_x$, $u = x^{\alpha+2}e^{\beta y}$ leads to the Abel equation:

$$[f(\zeta)u + g(\zeta) + \zeta]u'_\zeta = (\beta\zeta + \alpha + 2)u.$$

10. Exponential-homogeneous equation

$$y''_{xx} = e^{\alpha x} y^\beta f\left(\frac{y'_x}{y}\right) + yg\left(\frac{y'_x}{y}\right).$$

The substitution $\xi = y'_x/y$, $w = e^{\alpha x}y^{\beta-1}$ leads to the Abel equation:

$$[f(\xi)w + g(\xi) - \xi^2]w'_\xi = [(\beta-1)\xi + \alpha]w.$$

1.4. Equations Containing Polinomial Functions of y

1.4.1. Abel Equations of the First Kind $y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$

Preliminary comments.

1. If $y_0 = y_0(x)$ is a particular solution of the equation in question, the substitution $y - y_0 = 1/w$ reduces it to the Abel equation of the second kind:

$$ww'_x = -(3f_3y_0^2 + 2f_2y_0 + f_1)w^2 - (3f_3y_0 + f_2)w - f_3,$$

which is discussed in Section 1.3. For $f_0(x) \equiv 0$, we may choose $y_0 \equiv 0$ as a particular solution.

2. The transformation

$$\xi = \int f_3E^2\,dx, \quad u = \left(y + \frac{f_2}{3f_3}\right)E^{-1}, \qquad \text{where} \quad E = \exp\left[\int\left(f_1 - \frac{f_2^2}{3f_3}\right)dx\right],$$

brings the original equation to the normal form:

$$u'_\xi = u^3 + \Phi(\xi),$$

where

$$\Phi = \frac{1}{f_3E^3}\left[f_0 + \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right) - \frac{f_1f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}\right].$$

1. $y'_x = ay^3 + bx^{-3/2}$.

This is a special case of equation 1.4.1.9 with $n = -1/2$.

2. $y'_x = -y^3 + 3a^2x^2y - 2a^3x^3 + a$.

The substitution $y = 1/w + ax$ leads to the Abel equation of the form 1.3.2.1: $ww'_x = 3axw + 1$.

3. $y'_x = -y^3 + (ax + b)y^2$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.1: $ww'_x = (ax + b)w + 1$.

4. $y'_x = -y^3 + (ax + b)^{-2}y^2$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.2: $ww'_x = (ax + b)^{-2}w + 1$.

5. $y'_x = -y^3 + (ax + b)^{-1/2}y^2$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.4: $ww'_x = (ax + b)^{-1/2}w + 1$.

6. $y'_x = ay^3 + 3abxy^2 - b - 2ab^3x^3$.

This is a special case of equation 1.4.1.10 with $n = 0$, $m = 1$.

7. $y'_x = axy^3 + by^2$.

The substitution $u = xy$ leads to an equation with separation of variables: $xu'_x = au^3 + bu^2 + u$.

8. $y'_x = axy^3 + 3abx^2y^2 - b - 2ab^3x^4$.

This is a special case of equation 1.4.1.10 with $n = m = 1$.

9. $y'_x = ax^{2n+1}y^3 + bx^{-n-2}$.

The substitution $w = yx^{n+1}$ leads to an equation with separation of variables: $xw'_x = aw^3 + (n + 1)w + b$.

For $a = -\dfrac{n+1}{3A^2}$ and $b = \frac{2}{3}A(n+1)$, the solution in the parametric form is written as

$$x = \exp\Bigl(\frac{F}{n+1}\Bigr), \qquad y = -A\Bigl(1 + \frac{1}{\tau}\Bigr)\exp(-F),$$

where $F = \tau - \frac{1}{3}\ln|\tau + \frac{1}{3}| + C$.

10. $y'_x = ax^ny^3 + 3abx^{n+m}y^2 - bmx^{m-1} - 2ab^3x^{n+3m}$.

The substitution $w = y + bx^m$ leads to the Bernoulli equation: $w'_x = ax^nw^3 - 3ab^2x^{n+2m}w$.

11. $y'_x = ax^ny^3 + 3abx^{n+m}y^2 + cx^ky - 2ab^3x^{n+3m} + bcx^{m+k} - bmx^{m-1}$.

The substitution $w = y + bx^m$ leads to the Bernoulli equation: $w'_x = ax^nw^3 + (cx^k - 3ab^2x^{n+2m})w$.

12. $\boldsymbol{9y'_x = -x^m(ax^{1-m}+b)^{2\lambda+1}y^3 - x^{-2m}(9a+2+9bmx^{m-1})(ax^{1-m}+b)^{-\lambda-2}}$.

With $\lambda = \dfrac{1}{3a(1-m)}$, the substitution

$$y = \Big(\frac{3}{w} + \frac{1}{ax+bx^m}\Big)\big(ax^{1-m}+b\big)^{-\lambda}$$

leads to the equation $ww'_x = w + ax + bx^m$ outlined in Subsection 1.3.1.

13. $\boldsymbol{xy'_x = ax^4y^3 + (bx^2-1)y + cx}$.

The substitution $w = xy$ leads to an equation with separation of variables: $w'_x = x(aw^3 + bw + c)$.

14. $\boldsymbol{xy'_x = ay^3 + 3abx^ny^2 - bnx^n - 2ab^3x^{3n}}$.

The substitution $w = y + bx^n$ leads to the Beroulli equation: $w'_x = ax^{-1}w^3 - 3ab^2x^{2n-1}w$.

15. $\boldsymbol{xy'_x = ax^{2n+1}y^3 + (bx-n)y + cx^{1-n}}$.

The substitution $w = yx^n$ leads to an equation with separated variables: $w'_x = aw^3 + bw + c$.

16. $\boldsymbol{xy'_x = ax^{n+2}y^3 + (bx^n-1)y + cx^{n-1}}$.

The substitution $w = xy$ leads to an equation with separation of variables: $w'_x = x^{n-1}(aw^3 + bw + c)$.

17. $\boldsymbol{x^2y'_x = y^3 - 3a^2x^4y + 2a^3x^6 + 2ax^3}$.

The transformation $x = \dfrac{1}{\xi}$, $y = \dfrac{1}{w} + ax^2$ leads to an equation of the form 1.3.2.2: $ww'_\xi = 3a\xi^{-2}w + 1$.

18. $\boldsymbol{y'_x = -(ax+bx^m)y^3 + y^2}$.

The substitution $y = -1/w$ leads to the equation $ww'_x = w + ax + bx^m$ outlined in Subsection 1.3.1.

19. $\boldsymbol{y'_x = (Ax^2+Bx+C)^{-1/2}y^3 + y^2}$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.1.63: $ww'_x = w - (Ax^2+Bx+C)^{-1/2}$.

20. $\boldsymbol{y'_x = -y^3 + ae^{\lambda x}y^2}$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.7: $ww'_x = ae^{\lambda x}w + 1$.

21. $\boldsymbol{y'_x = -y^3 + 3a^2e^{2\lambda x}y - 2a^3e^{3\lambda x} + a\lambda e^{\lambda x}}$.

The substitution $y = \dfrac{1}{w} + ae^{\lambda x}$ leads to the Abel equation of the form 1.3.2.7: $ww'_x = 3ae^{\lambda x}w + 1$.

22. $y'_x = -\frac{1}{3}\lambda^{-1}e^{2\lambda x}y^3 + \frac{2}{3}\lambda^2 e^{-\lambda x}$.

The solution in the parametric form is written as

$$x = \frac{F}{\lambda}, \quad y = -\lambda\Big(1 + \frac{1}{\tau}\Big)e^{-F}, \qquad \text{where} \quad F = \tau - \tfrac{1}{3}\ln|\tau + \tfrac{1}{3}| + C.$$

23. $y'_x = ae^{2\lambda x}y^3 + be^{\lambda x}y^2 + cy + de^{-\lambda x}$.

The substitution $y = we^{-\lambda x}$ leads to an equation with separated variables: $w'_x = aw^3 + bw^2 + (c+\lambda)w + d$.

24. $y'_x = ae^{\lambda x}y^3 + 3abe^{\lambda x}y^2 + cy - 2ab^3e^{\lambda x} + bc$.

The substitution $w = y + b$ yields the Bernoulli equation: $w'_x = ae^{\lambda x}w^3 + (c - 3ab^2e^{\lambda x})w$.

25. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 - 2ab^3e^{(\lambda+3\mu)x} - b\mu e^{\mu x}$.

The substitution $w = y + be^{\mu x}$ leads to the Bernoulli equation: $w'_x = ae^{\lambda x}w^3 - 3ab^2e^{(\lambda+2\mu)x}w$.

26. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + 2ab^2e^{(\lambda+2\mu)x}y - b\mu e^{\mu x}$.

The substitution $w = y + be^{\mu x}$ leads to the Bernoulli equation: $w'_x = ae^{\lambda x}w^3 - ab^2e^{(\lambda+2\mu)x}w$.

27. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + \mu y - 2ab^3e^{(\lambda+3\mu)x}$.

The substitution $w = y + be^{\mu x}$ leads to the Bernoulli equation: $w'_x = ae^{\lambda x}w^3 + \big[\mu - 3ab^2e^{(\lambda+2\mu)x}\big]w$.

28. $y'_x = ae^{\lambda x}y^3 + 3abe^{(\lambda+\mu)x}y^2 + [(3ab^2 + c)e^{(\lambda+2\mu)x} + s]y$
$+ b(ab^2 + c)e^{(\lambda+3\mu)x} + b(s - \mu)e^{\mu x}$.

The substitution $w = y + be^{\mu x}$ leads to the Bernoulli equation: $w'_x = ae^{\lambda x}w^3 + [ce^{(\lambda+2\mu)x} + s]w$.

29. $y'_x = [a + b\exp(2x/a)]y^3 + y^2$.

The substitution $y = -1/w$ leads to an equation of the form 1.3.1.9: $ww'_x = w - a - b\exp(2x/a)$.

30. $y'_x = -\frac{2}{3}ax^{-1}\exp(2ax^2)y^3 + (1 - \frac{4}{3}ax^2)\exp(-ax^2)$.

The substitution $y = \Big(\dfrac{1}{2aw} + x\Big)\exp(-ax^2)$ leads to an equation of the form 1.3.1.16: $ww'_x = w + (6ax)^{-1}$.

31. $y'_x = -a\exp(2ax^3)y^3 + (1 - 2ax^3)\exp(-ax^3)$.

The transformation $\xi = x^2$, $y = \Big(\dfrac{2}{3aw} + x\Big)\exp(-ax^3)$ leads to an equation of the form 1.3.1.32: $ww'_\xi = w \pm 2(9a)^{-1}\xi^{-1/2}$.

32. $y'_x = -ax^{-2}\exp(2ax^3)y^3 + 2x(1-ax^3)\exp(-ax^3)$.

The substitution $y = \left(\dfrac{1}{3aw} + x^2\right)\exp(-ax^3)$ leads to an equation of the form 1.3.1.33: $ww'_x = w + (9a)^{-1}x^{-2}$.

33. $y'_x = ay^3 + b\cosh(\lambda x)y^2$.

The transformation $t = \dfrac{1}{y} + \dfrac{b}{\lambda}\sinh(\lambda x)$, $\xi = e^{\lambda x}$ leads to the Riccati equation: $2a\xi'_t = b\xi^2 - 2\lambda t\xi - b$.

34. $y'_x = ay^3 + b\sinh(\lambda x)y^2$.

The transformation $t = \dfrac{1}{y} + \dfrac{b}{\lambda}\cosh(\lambda x)$, $\xi = e^{\lambda x}$ leads to the Riccati equation: $2a\xi'_t = b\xi^2 - 2\lambda t\xi + b$.

35. $y'_x = -y^3 + 3a^2\cosh^2 x\, y - 2a^3\cosh^3 x + a\sinh x$.

The substitution $y = \dfrac{1}{w} + a\cosh x$ leads to the Abel equation of the form 1.3.2.9: $ww'_x = 3a\cosh x\, w + 1$.

36. $y'_x = -y^3 + 3a^2\sinh^2 x\, y - 2a^3\sinh^3 x + a\cosh x$.

The substitution $y = \dfrac{1}{w} + a\sinh x$ leads to the Abel equation of the form 1.3.2.10: $ww'_x = 3a\sinh x\, w + 1$.

37. $y'_x = -y^3 + a\cos(\omega x)y^2$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.11: $ww'_x = a\cos(\omega x)w + 1$.

38. $y'_x = -y^3 + a\sin(\omega x)y^2$.

The substitution $y = -1/w$ leads to the Abel equation of the form 1.3.2.12: $ww'_x = a\sin(\omega x)w + 1$.

39. $y'_x = -y^3 + 3a^2\cos^2(\lambda x)y + a\lambda\sin(\lambda x) + 2a^3\cos^3(\lambda x)$.

The substitution $y = \dfrac{1}{w} - a\cos(\lambda x)$ leads to the Abel equation of the form 1.3.2.11: $ww'_x = -3a\cos(\lambda x)w + 1$.

40. $y'_x = -y^3 + 3a^2\sin^2(\lambda x)y + a\lambda\cos(\lambda x) - 2a^3\sin^3(\lambda x)$.

The substitution $y = \dfrac{1}{w} + a\sin(\lambda x)$ leads to an equation of the form the Abel equation of the form 1.3.2.12: $ww'_x = 3a\sin(\lambda x)w + 1$.

41. $y'_x = afy^3 + \left(bfg^2 + \dfrac{g'_x}{g}\right)y + cfg^3, \qquad f = f(x), \quad g = g(x)$.

The substitution $y = gw$ leads to an equation with separation of variables: $w'_x = fg^2(aw^3 + bw + c)$.

42. $y'_x = fy^3 + 3fhy^2 + (g + 3fh^2)y + fh^3 + gh - h'_x$,
where $f = f(x)$, $g = g(x)$, $h = h(x)$.

The substitution $w = y + h(x)$ leads to the Bernoulli equation: $w'_x = g(x)w + f(x)w^3$.

43. $y'_x = \dfrac{g'_x}{f^2(ag+b)^3}y^3 + \dfrac{f'_x}{f}y + fg'_x$, $\quad f = f(x)$, $g = g(x)$.

Solution: $\displaystyle\int \frac{du}{u^3 - au + 1} + C = \frac{1}{a}\ln|ag+b|$, $\quad$ where $\quad u = \dfrac{y}{f(ag+b)}$.

44. $y'_x = (y-f)(y-g)\left(y - \dfrac{af+bg}{a+b}\right)h + \dfrac{y-g}{f-g}f'_x + \dfrac{y-f}{g-f}g'_x$,
where $f = f(x)$, $g = g(x)$, $h = h(x)$.

Solution: $\displaystyle |y-f|^a|y-g|^b\left|y - \frac{af+bg}{a+b}\right|^{-a-b} = C\exp\left[\frac{ab}{a+b}\int (f-g)^2 h\,dx\right]$.

1.4.2. Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_0)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_0$

Preliminary comments.

1. For $A_{22} = 0$, this is the Abel equation (see Subsection 1.3.4). For $B_{11} = 0$, this is the Abel equation with respect to $x = x(y)$.

2. The transformation $z = y/x$, $\zeta = x^{-2}$, leads to the Abel equation of the second kind:

$$[(A_0 z - B_0)\zeta + A_{22}z^3 + (A_{12} - B_{22})z^2 + (A_{11} - B_{12})z - B_{11}]\zeta'_z = 2A_0\zeta^2 + 2(A_{22}z^2 + A_{12}z + A_{11})\zeta.$$

3. The transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are the parameters which are determined by solving the second order algebraic system

$$A_{22}\beta^2 + A_{12}\alpha\beta + A_{11}\alpha^2 + A_0 = 0, \quad B_{22}\beta^2 + B_{12}\alpha\beta + B_{11}\alpha^2 + B_0 = 0,$$

leads to the equation

$$[A_{22}\bar{y}^2 + A_{12}\bar{x}\bar{y} + A_{11}\bar{x}^2 + (2A_{22}\beta + A_{12}\alpha)\bar{y} + (2A_{11}\alpha + A_{12}\beta)\bar{x})]\bar{y}'_{\bar{x}} = B_{22}\bar{y}^2 + B_{12}\bar{x}\bar{y} + B_{11}\bar{x}^2 + (2B_{22}\beta + B_{12}\alpha)\bar{y} + (2B_{11}\alpha + B_{12}\beta)\bar{x},$$

The transformation $\xi = \bar{y}/\bar{x}$, $w = 1/\bar{x}$ reduces this equation to the Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - b_2)\xi - b_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi = (a_2\xi + a_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

where

$$a_2 = 2A_{22}\beta + A_{12}\alpha, \quad b_2 = 2B_{22}\beta + B_{12}\alpha, \quad a_1 = 2A_{11}\alpha + A_{12}\beta, \quad b_1 = 2B_{11}\alpha + B_{12}\beta.$$

4. The substitution $y = t + \varepsilon x$, where parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to the Abel equation of the second kind with recpect to $x = x(t)$:

$$[Qtx + (B_{22} - A_{22}\varepsilon)t^2 + B_0 - A_0\varepsilon]x'_t = (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})x^2 + (2A_{22}\varepsilon + A_{12})tx + A_{22}t^2 + A_0,$$

where $Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12})$.

1. $\boldsymbol{(Ay^2 + x^2)y'_x = -2xy + Bx^2 + a.}$

Solution: $Ay^3 - Bx^3 + 3(x^2y - ax) = C$.

2. $\boldsymbol{(Ay^2 + Bx^2 - a^2B)y'_x = Cy^2 + 2Bxy.}$

The transformation $x = w + a$, $y = \xi w$ leads to the linear equation:

$$(-A\xi^3 + C\xi^2 + B\xi)w'_\xi = (A\xi^2 + B)w + 2aB.$$

3. $\boldsymbol{(Ay^2 + Bxy + Cx^2)y'_x = Dy^2 + Exy + Fx^2.}$

Homogeneous equation. The substitution $z = y/x$ leads to an equation with separated variables:

$$xz'_x = (Az^2 + Bz + C)^{-1}[-Az^3 + (D - B)z^2 + (E - C)z + F].$$

4. $\boldsymbol{(Ay^2 - 2Akxy + Bkx^2)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + a.}$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[-(Ak + B)z^2 + a]x'_z = k(B - Ak)x^2 + Az^2.$$

5. $\boldsymbol{(Ay^2 + 2Bxy + Ak^2x^2)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + a.}$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z^2 + a]x'_z = 2k(Ak + B)x^2 + 2(Ak + B)zx + Az^2.$$

6. $\boldsymbol{(Ay^2 + Bxy + Cx^2 + a)y'_x = Aky^2 + Bkxy + Ckx^2 + b.}$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(b - ak)x'_z = (Ak^2 + Bk + C)x^2 + (2Ak + B)zx + Az^2 + a.$$

7. $\boldsymbol{(Ay^2 + 2Bxy + Dx^2 + a)y'_x = -By^2 - 2Dxy + Ex^2 + b.}$

Solution: $Ay^3 - Ex^3 + 3(Bxy^2 + Dx^2y + ay - bx) = C$.

8. $\boldsymbol{(Ay^2 - 2Axy + Bx^2 + A - B)y'_x = -Ay^2 + 2Bxy - Bx^2 + A - B.}$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = 1$.

9. $(Ay^2 + 2Axy + Bx^2 + A - B)y'_x = Ay^2 + 2Bxy + Bx^2 - A + B.$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = -1$.

10. $(Ay^2 - 4Axy + Bx^2 + 4A - B)y'_x = -2Ay^2 + 2Bxy - 2Bx^2 + 8A - 2B.$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = 2$.

11. $(Ay^2 + 4Axy + Bx^2 + 4A - B)y'_x = 2Ay^2 + 2Bxy + 4Bx^2 - 8A + 2B.$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = -2$.

12. $(Ay^2 - 6Axy + Bx^2 + 9A - B)y'_x = -3Ay^2 + 2Bxy - 3Bx^2 + 27A - 3B.$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = 3$.

13. $(Ay^2 + 6Axy + Bx^2 + 9A - B)y'_x = 3Ay^2 + 2Bxy + 3Bx^2 - 27A + 3B.$

This is a special case of equation 1.4.2.21 with $a = 1$, $b = -3$.

14. $2(Ay^2 - Axy + Bx^2 + A - 4B)y'_x = -Ay^2 + 4Bxy - Bx^2 + A - 4B.$

This is a special case of equation 1.4.2.21 with $a = 2$, $b = 1$.

15. $2(Ay^2 + Axy + Bx^2 + A - 4B)y'_x = Ay^2 + 4Bxy + Bx^2 - A + 4B.$

This is a special case of equation 1.4.2.21 with $a = 2$, $b = -1$.

16. $(ay^2 - 2bxy + ax^2 + ab^2 - a^3)y'_x = -by^2 + 2axy - bx^2 + b^3 - a^2b.$

This is a special case of equation 1.4.2.21 with $A = 1$, $B = 1$.

17. $(ay^2 - 2bxy - ax^2 + ab^2 + a^3)y'_x = -by^2 - 2axy + bx^2 + b^3 + a^2b.$

This is a special case of equation 1.4.2.21 with $A = 1$, $B = -1$.

18. $(ay^2 - 2bxy + 2ax^2 + ab^2A - 2a^3)y'_x = -by^2 + 4axy - 2bx^2 + b^3 - 2a^2b.$

This is a special case of equation 1.4.2.21 with $A = 1$, $B = 2$.

19. $(ay^2 - 2bxy - 2ax^2 + ab^2 + 2a^3)y'_x = -by^2 - 4axy + 2bx^2 + b^3 + 2a^2b.$

This is a special case of equation 1.4.2.21 with $A = 1$, $B = -2$.

20. $(Ay^2 + Bxy + Cx^2 + a)y'_x$
$$= Dy^2 + k(2Ak + B - 2D)xy + k(-Ak^2 + Dk + C)x^2 + b.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(D - Ak)z^2 + b - ak]x'_z = (Ak^2 + Bk + C)x^2 + (2Ak + B)zx + Az^2 + a.$$

21. $(aAy^2 - 2bAxy + aBx^2 + ab^2A - a^3B)y'_x$
$$= -bAy^2 + 2aBxy - bBx^2 + b^3A - a^2bB.$$

The transformtion $x = w + a$, $y = \xi w + b$ leads to the linear equation:

$$(-aA\xi^3 + bA\xi^2 + aB\xi - bB)w'_\xi = (aA\xi^2 - 2bA\xi + aB)w + 2a^2B - 2b^2A.$$

1.4.3. Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x$*

Preliminary comments.

1. For $A_{22} = 0$, this is the Abel equation (see Subsection 1.3.4). For $B_{11} = 0$ this is the Abel equation with respect to $x = x(y)$.

2. The transformation $\xi = y/x$, $w = 1/x$ leads to the Abel equation of the second kind:

$$\{[A_2\xi^2 + (A_1 - B_2)\xi - B_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi = (A_2\xi + A_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

3. In Paragraph 3 of Subsection 1.4.4, another thransformation is given which reduces the original equation to the Abel equation of the second kind.

4. Dynamical systems of the second order

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{1}$$

which describe the behaviour of the simplest Lagrangian and Hamiltonian systems in mechanics are often reduced to equations of the considered type when

$$\begin{aligned} P(x, y) &= f(x, y)(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x), \\ Q(x, y) &= f(x, y)(B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x), \end{aligned} \tag{2}$$

where $f = f(x, y)$ is an arbitrary function.

In particular, dynamical systems (1) with functions (2) and $f \equiv 1$ are met with in analyzing complex equillibrium states. In this case, functions P and Q are substituted by their Taylor-series expansions in the vicinity of the equallibrium state $x = y = 0$ with the first and second order terms retained.

When obtained the solution of the ordinary differential equation

$$(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x$$

in the parametric form $x = x(u, C_1)$, $y = y(u, C_1)$, the solution of the system (1), (2) is determined by the formulae

$$x = x(u, C_1), \quad y = y(u, C_1), \quad t = \int \frac{x'_u\, du}{P\big(x(u, C_1),\, y(u, C_1)\big)} + C_2.$$

The latter relation defines the implicit dependence of parameter u on t: $u = u(t, C_1, C_2)$, and makes it possible to find, with the aid of two former formulae, the dependence of x and y on t.

1. $(y^2 - x^2 + ay)y'_x = y^2 - x^2 + ax.$

Solution in the parametric form:

$$x = at + C|t|^{-1}e^{4t}, \quad y = -at + C|t|^{-1}e^{4t}.$$

* This section was written with A.I. Zhurov

2. $\boldsymbol{(y^2 - x^2 + ay)y'_x = 2y^2 - 2xy + ay}$.

Solution in the parametric form:

$$x = t + Ct^2 e^{a/t}, \quad y = Ct^2 e^{a/t}.$$

3. $\boldsymbol{(y^2 - x^2 + ay - ax)y'_x = y^2 - x^2 - ay + ax}$.

Solution in the parametric form:

$$x = at + Ce^{2t}, \quad y = -at + Ce^{2t}.$$

4. $\boldsymbol{(y^2 - x^2 + ay + 2ax)y'_x = y^2 - x^2 + 2ay + ax}$.

Solution in the parametric form:

$$x = -at + C|t|^3 e^{4t}, \quad y = at + C|t|^3 e^{4t}.$$

5. $\boldsymbol{(y^2 - x^2 + ay + 2ax)y'_x = 2xy - 2x^2 + ay + 2ax}$.

Solution in the parametric form:

$$x = t + Ct^{-2} e^{-a/t}, \quad y = -2t + Ct^{-2} e^{-a/t}.$$

6. $\boldsymbol{(y^2 - x^2 + ay - 2ax)y'_x = 4y^2 - 6xy + 2x^2 + ay - 2ax}$.

Solution in the parametric form:

$$x = \tfrac{1}{3}t + C|t|^{2/3} e^{a/t}, \quad y = \tfrac{2}{3}t + C|t|^{2/3} e^{a/t}.$$

7. $\boldsymbol{(y^2 - x^2 + ay + 3ax)y'_x = -y^2 + 4xy - 3x^2 + ay + 3ax}$.

Solution in the parametric form:

$$x = \tfrac{1}{2}t + C|t|^{-1} e^{-a/t}, \quad y = -\tfrac{3}{2}t + C|t|^{-1} e^{-a/t}.$$

8. $\boldsymbol{(y^2 - xy + ay + ax)y'_x = xy - x^2 + ay + ax}$.

Solution in the parametric form:

$$x = -t + C|t|^{-1} e^{a/t}, \quad y = t + C|t|^{-1} e^{a/t}.$$

9. $\boldsymbol{(y^2 - xy + ay + ax)y'_x = y^2 - xy + 2ay}$.

Solution in the parametric form:

$$x = -at + Ct^2 e^t, \quad y = Ct^2 e^t.$$

10. $\boldsymbol{(y^2 - xy + ay - 2ax)y'_x = 3y^2 - 5xy + 2x^2 + ay - 2ax}$.

Solution in the parametric form:

$$x = \tfrac{1}{2}t + C|t|^{1/2} e^{a/t}, \quad y = t + C|t|^{1/2} e^{a/t}.$$

11. $(y^2 + xy - 2x^2 + ay + ax)y'_x = y^2 + xy - 2x^2 + 2ax.$

Solution in the parametric form:

$$x = at + Ct^{-2}e^{9t}, \quad y = -2at + Ct^{-2}e^{9t}.$$

12. $(y^2 + xy - 2x^2 + ay + ax)y'_x = 2y^2 - xy - x^2 + ay + ax.$

Solution in the parametric form:

$$x = t + C|t|^3 e^{a/t}, \quad y = -t + C|t|^3 e^{a/t}.$$

13. $(y^2 + xy - 2x^2 + ay - ax)y'_x = y^2 + xy - 2x^2 - 2ay + 2ax.$

Solution in the parametric form:

$$x = at + Ce^{3t}, \quad y = -2at + Ce^{3t}.$$

14. $(y^2 + xy - 2x^2 + ay - 2ax)y'_x = 5y^2 - 7xy + 2x^2 + ay - 2ax.$

Solution in the parametric form:

$$x = \tfrac{1}{4}t + C|t|^{3/4} e^{a/t}, \quad y = \tfrac{1}{2}t + C|t|^{3/4} e^{a/t}.$$

15. $(y^2 - 2xy + x^2 + ay)y'_x = ay.$

Solution: $x = y + \dfrac{a}{C - \ln|y|}.$

16. $(y^2 - 2xy + x^2 + ay + ax)y'_x = -y^2 + 2xy - x^2 + ay + ax.$

Solution in the parametric form:

$$x = -\frac{a}{2\ln|t|} + Ct, \quad y = \frac{a}{2\ln|t|} + Ct.$$

17. $(y^2 - 2xy + x^2 + ay + 2ax)y'_x = -2(y^2 - 2xy + x^2) + ay + 2ax.$

Solution in the parametric form:

$$x = -\frac{a}{3\ln|t|} + Ct, \quad y = \frac{2a}{3\ln|t|} + Ct.$$

18. $(y^2 - 2xy + x^2 + ay - 2ax)y'_x = 2(y^2 - 2xy + x^2) + ay - 2ax.$

Solution in the parametric form:

$$x = \frac{a}{\ln|t|} + Ct, \quad y = \frac{2a}{\ln|t|} + Ct.$$

19. $(y^2 + 2xy + x^2 + ay + 2ax)y'_x = -y^2 - 2xy - x^2 + 2ay + ax.$

Solution in the parametric form:

$$x = C^2\left(t^{1/3} + \frac{4t^2}{5a}\right) + Ct, \quad y = -C^2\left(t^{1/3} + \frac{4t^2}{5a}\right) + Ct, \qquad a \neq 0.$$

20. $(y^2 + 2xy + x^2 + ay - ax)y'_x = -y^2 - 2xy - x^2 + ay - ax.$

Solution in the parametric form:

$$x = C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \quad y = -C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \qquad a \neq 0.$$

21. $(y^2 + 2xy + x^2 + ay - 2ax)y'_x = -y^2 - 2xy - x^2 - 2ay + ax.$

Solution in the parametric form:

$$x = C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \quad y = -C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \qquad a \neq 0.$$

22. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = 3y^2 - 2xy - 1x^2 + ay + ax.$

Solution in the parametric form:

$$x = \tfrac{1}{2}t + Ct^2 e^{a/t}, \quad x = -\tfrac{1}{2}t + Ct^2 e^{a/t}.$$

23. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = y^2 + 2xy - 3x^2 - ay + 3ax.$

Solution in the parametric form:

$$x = at + C|t|^{-1}e^{8t}, \quad y = -3at + C|t|^{-1}e^{8t}.$$

24. $(y^2 + 2xy - 3x^2 + ay + 2ax)y'_x = y^2 + 2xy - 3x^2 + 3ax.$

Solution in the parametric form:

$$x = at + C|t|^{-3}e^{16t}, \quad y = -3at + C|t|^{-3}e^{16t}.$$

25. $(y^2 - x^2 + ay + bx)y'_x = y^2 - x^2 + by + ax.$

Solution in the parametric form:

$$x = (a-b)t + C|t|^{-\frac{a+b}{a-b}}e^{4t}, \quad y = (b-a)t + C|t|^{-\frac{a+b}{a-b}}e^{4t}, \qquad a \neq b.$$

26. $(y^2 - xy + ay + bx)y'_x = y^2 - xy + (a+b)y.$

Solution in the parametric form:

$$x = -bt + C|t|^{\frac{a+b}{b}}e^{t}, \quad y = C|t|^{\frac{a+b}{b}}e^{t}, \qquad b \neq 0.$$

27. $(y^2 + xy - 2x^2 + ay + bx)y'_x = y^2 + xy - 2x^2 + (b-a)y + 2ax.$

Solution in the parametric form:

$$x = (2a-b)t + C|t|^{-\frac{a+b}{2a-b}}e^{9t}, \quad y = 2(b-2a)t + C|t|^{-\frac{a+b}{2a-b}}e^{9t}, \qquad b \neq 2a.$$

28. $(y^2 - 2xy + x^2 + ay - abx)y'_x = b(y^2 - 2xy + x^2) + ay - abx.$

Solution in the parametric form:

$$x = \frac{a}{b-1}\,\frac{1}{\ln|t|} + Ct, \quad y = \frac{ab}{b-1}\,\frac{1}{\ln|t|} + Ct, \qquad b \neq 1.$$

29. $(y^2 + 2xy - 3x^2 + ay + bx)y'_x = y^2 + 2xy - 3x^2 + (b - 2a)y + 3ax.$

Solution in the parametric form:

$$x = (3a - b)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \quad y = 3(b - 3a)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \qquad b \neq 3a.$$

30. $(y^2 - 3xy + 2x^2 + ay + bx)y'_x = y^2 - 3xy + 2x^2 + (3a + b)y - 2ax.$

Solution in the parametric form:

$$x = (2a + b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \quad y = 2(2a + b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \qquad b \neq -2a.$$

31. $(y^2 + 3xy - 4x^2 + ay + bx)y'_x = y^2 + 3xy - 4x^2 + (b - 3a)y + 4ax.$

Solution in the parametric form:

$$x = (4a - b)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \quad y = 4(b - 4a)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \qquad b \neq 4a.$$

32. $[y^2 + Axy - (A+1)x^2 + by - 2bx]y'_x = (A+4)y^2 - (A+6)xy + 2x^2 + by - 2bx.$

Solution in the parametric form:

$$x = \frac{t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \quad y = \frac{2t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \qquad A \neq -3.$$

33. $(y^2 - 2Axy + A^2x^2 + by - bx)y'_x = Ay^2 - 2A^2xy + A^3x^2 + by - bx.$

Solution in the parametric form:

$$x = C^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \quad y = AC^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \qquad b \neq 0.$$

34. $[y^2 - 2Axy + (2A-1)x^2 + by - Abx]y'_x = (2 - A)y^2 - 2xy + Ax^2 + by - Abx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + Ct^2 e^{b/t}, \quad y = \frac{At}{1-A} + Ct^2 e^{b/t}, \qquad A \neq 1.$$

35. $(y^2 - 2Axy + A^2x^2 + ay + bx)y'_x = A(y^2 - 2Axy + A^2x^2) + (aA + a + b)y - aAx.$

Solution in the parametric form:

$$x = C^2\left[t^{\frac{aA+b}{a+b}} + \frac{(1-A)^2}{(2-A)a+b}t^2\right] + Ct, \quad y = AC^2\left[t^{\frac{aA+b}{a+b}} + \frac{(1-A)^2}{(2-A)a+b}t^2\right] + Ct,$$

where $a + b \neq 0$ and $(2 - A)a + b \neq 0$.

36. $[y^2 - (A+2)xy + (A+1)x^2 + by - Abx]y'_x = -Axy + Ax^2 + by - Abx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^A e^{(A-1)b/t}, \quad y = \frac{At}{1-A} + C|t|^A e^{(A-1)b/t}, \qquad A \neq 1.$$

37. $[Ay^2 + xy - (A+1)x^2 + by + bx]y'_x = (A+1)y^2 - xy - Ax^2 + by + bx.$

Solution in the parametric form:

$$x = t + C|t|^{2A+1}e^{b/t}, \quad y = -t + C|t|^{2A+1}e^{b/t}.$$

38. $(Ay^2 + Bxy + Cx^2 + kx)y'_x = Dy^2 + Exy + Fx^2 + ky.$

The substitution $y = xz$ leads to a linear equation with respect to $x = x(z)$:

$$[-Az^3 + (D-B)z^2 + (E-C)z + F]x'_z = (Az^2 + Bz + C)x + k.$$

39. $(Ay^2 + Bxy + Cx^2 - \alpha By - \alpha Cx)y'_x = Dy^2 + Exy + \alpha(C-E)y.$

The transformation $x = w + \alpha$, $y = \xi w$ leads to a linear equation:

$$[-A\xi^3 + (D-B)\xi^2 + (E-C)\xi]w'_\xi = (A\xi^2 + B\xi + C)w + \alpha C.$$

40. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.57 with $C = Ak^2$.

41. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $C = Ak^2$.

42. $(Ay^2 + 2Bxy + Ak^2x^2 + ay - akx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $C = Ak^2$.

43. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.58 with $m = b$.

44. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $C = -Bk$.

45. $(Ay^2 + 2Bxy - Bkx^2 + ay - akx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $C = -Bk$.

46. $(Ay^2 + 2Akxy + Cx^2 + ay + bx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + by + ak^2x.$

This is a special case of equation 1.4.3.57 with $B = Ak$.

47. $(Ay^2 + 2Akxy + Cx^2 + ay + bx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $B = Ak$.

48. $(Ay^2 + 2Akxy + Cx^2 + ay - akx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $B = Ak$.

49. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.59 with $m = b$.

50. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + aky + bkx.$

This is a special case of equation 1.4.3.59 with $m = ak$.

51. $[y^2 + 2Axy + A^2x^2 + (A-1)By - 2ABx]y'_x$
$= -A(y^2 + 2Axy + A^2x^2) - (A^2+1)By + A(A-1)Bx.$

Solution in the parametric form:

$$x = C^2\left[t^A + \frac{A+1}{(A-2)B}t^2\right] + Ct, \quad y = -AC^2\left[t^A + \frac{A+1}{(A-2)B}t^2\right] + Ct, \qquad A \neq 2,\ B \neq 0.$$

52. $[y^2 - 2Axy + A^2x^2 + (B-1)ky + (A-B)kx]y'_x$
$= A(y^2 - 2Axy + A^2x^2) + (AB-1)ky - A(B-1)kx.$

Solution in the parametric form:

$$x = C^2\left[t^B - \frac{A-1}{(B-2)k}t^2\right] + Ct, \quad y = AC^2\left[t^B - \frac{A-1}{(B-2)k}t^2\right] + Ct, \qquad B \neq 2,\ k \neq 0.$$

53. $[2y^2 - (A+3)xy + (A+1)x^2 + By - ABx]y'_x$
$= (A+1)y^2 - (3A+1)xy + 2Ax^2 + By - ABx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^{-1}e^{-B/t}, \quad y = \frac{At}{1-A} + C|t|^{-1}e^{-B/t}, \qquad A \neq 1.$$

54. $[2y^2 - (3A+1)xy + (3A-1)x^2 + By - ABx]y'_x$
$= (3-A)y^2 - (A+3)xy + 2Ax^2 + By - ABx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^3e^{B/t}, \quad y = \frac{At}{1-A} + C|t|^3e^{B/t}, \qquad A \neq 1.$$

55. $[A(y^2 - 2xy + x^2) - A(A-B)y + B(A-B)x]y'_x$
$= B(y^2 - 2xy + x^2) - A(A-B)y + B(A-B)x.$

Solution in the parametric form:

$$x = \frac{A}{\ln|t|} + Ct, \quad y = \frac{B}{\ln|t|} + Ct.$$

56. $(Ay^2 + Bxy + Cx^2 + ay + bx)y'_x = Aky^2 + Bkxy + Ckx^2 + ny + (ak + b - n)x.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(n - ak)zx'_z = (Ak^2 + Bk + C)x^2 + [(2Ak + B)z + ak + b]x + Az^2 + az.$$

57. $(Ay^2 + 2Bxy + Cx^2 + ay + bx)y'_x$
$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + by + ak^2x.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + b - ak]zx'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

58. $\boldsymbol{(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x}$
$$\boldsymbol{= By^2 + 2Ak^2xy - Ak^3x^2 + my + k(ak + b - m)x.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + m - ak]zx'_z = (Ak^2 + Bk)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

59. $\boldsymbol{(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x}$
$$\boldsymbol{= -By^2 + 2Bkxy - Ak^3x^2 + my + k(ak + b - m)x.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[-(Ak + B)z + m - ak]zx'_z = k(B - Ak)x^2 + (ak + b)x + Az^2 + az.$$

60. $\boldsymbol{(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x}$
$$\boldsymbol{= By^2 + 2Ak^2xy + Bk^2x^2 + my + k(ak + b - m)x.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + m - ak]zx'_z = 2k(Ak + B)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

61. $\boldsymbol{(Ay^2 + 2Bxy + Cx^2 + ay - akx)y'_x}$
$$\boldsymbol{= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + my - mkx.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z^2 + m - ak]zx'_z = (Ak^2 + 2Bk + C)x^2 + 2(Ak + B)zx + Az^2 + az.$$

62. $\boldsymbol{(Ay^2 + 2Bxy + Cx^2 + ay + bx)y'_x}$
$$\boldsymbol{= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + aky + bkx.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(B - Ak)z^2x'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

63. $\boldsymbol{\{(A - 1)y^2 + [2 - A(k + 1)]xy + (Ak - 1)x^2 + By - Bkx\}y'_x}$
$$\boldsymbol{= (A - k)y^2 + [2k - A(k + 1)]xy + (A - 1)kx^2 + By - Bkx.}$$

Solution in the parametric form:

$$x = \frac{t}{1 - k} + C|t|^A e^{B/t}, \quad y = \frac{kt}{1 - k} + C|t|^A e^{B/t}, \qquad k \neq 1.$$

64. $\boldsymbol{[A(\alpha y^2 + \beta xy + \gamma x^2) + (2\alpha - A^2\sigma)y + (\beta - AB\sigma)x]y'_x}$
$$\boldsymbol{+ B(\alpha y^2 + \beta xy + \gamma x^2) + (\beta - AB\sigma)y + (2\gamma - B^2\sigma)x = 0.}$$

Solution: $\alpha y^2 + \beta xy + \gamma x^2 - A\sigma y - B\sigma x + \sigma = C\exp(-Ay - Bx)$.

65. $\boldsymbol{(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x}$
$$\boldsymbol{= B_{22}y^2 + k(2A_{22}k + A_{12} - 2B_{22})xy + k(-A_{22}k^2 + B_{22}k + A_{11})x^2 + B_2y + k(A_2k + A_1 - B_2)x.}$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B_{22} - A_{22}k)z + B_2 - A_2k]zx'_z = (A_{22}k^2 + A_{12}k + A_{11})x^2 + [2(A_{22}k + A_{12})z + A_2k + A_1]x + A_{22}z^2 + A_2z.$$

► *In equations 66–70, the following notation is used:*

$$\Delta = Ab - aB, \quad \delta = Ab + aB.$$

66. $(Aa^2y^2 - 2Aabxy + Ab^2x^2 - \Delta Aay + \Delta aBx)y'_x$
$= a^2By^2 - 2aBbxy + Bb^2x^2 - \Delta Aby + \Delta Bbx.$

Solution in the parametric form:

$$x = \frac{A}{\ln|t|} + aCt, \qquad y = \frac{B}{\ln|t|} + bCt.$$

67. $[kAa^2y^2 - k\delta axy + kaBbx^2 - m\Delta Aay + (maB - \Delta)\Delta x]y'_x$
$= kAaby^2 - k\delta bxy + kBb^2x^2 - (mAb + \Delta)\Delta y + m\Delta Bbx.$

Solution in the parametric form:

$$x = At + aC|t|^{m+1}e^{kt}, \qquad y = Bt + bC|t|^{m+1}e^{kt}.$$

68. $[mAa^2y^2 - a(m\delta - \Delta)xy + b(maB - \Delta)x^2 + k\Delta Aay - k\Delta aBx]y'_x$
$= a(mBb + \Delta)y^2 - b(m\delta + \Delta)xy + mBb^2x^2 + k\Delta Aby - k\Delta Bbx.$

Solution in the parametric form:

$$x = At + aC|t|^{m+1}e^{k/t}, \qquad y = Bt + bC|t|^{m+1}e^{k/t}.$$

69. $(kA^3y^2 - 2kA^2Bxy + kAB^2x^2 - 2\Delta a^2y + 2\Delta abx)y'_x$
$= kA^2By^2 - 2kAB^2xy + kB^3x^2 - 2\Delta aby + 2\Delta b^2x.$

Solution in the parametric form:

$$x = AC^3\sqrt{\tfrac{1}{3}kt^3 + 1} + aC^2t, \qquad y = BC^3\sqrt{\tfrac{1}{3}kt^3 + 1} + bC^2t.$$

70. $[kA^3y^2 - 2kA^2Bxy + kAB^2x^2 + m\Delta Aay - (mAb + \Delta)\Delta x]y'_x$
$= kA^2By^2 - 2kAB^2xy + kB^3x^2 + (maB - \Delta)\Delta y - m\Delta Bbx.$

Solution in the parametric form:

$$x = AC^2\left(t^{m+1} + \frac{k}{m-1}t^2\right) + aCt, \quad y = BC^2\left(t^{m+1} + \frac{k}{m-1}t^2\right) + bCt, \qquad m \neq 1.$$

1.4.4. Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x + A_0)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x + B_0$

Preliminary comments.

1. With $A_{22} = 0$, this is the Abel equation (see Subsection 1.3.4). With $B_{11} = 0$, this is the Abel equation with respect to $x = x(y)$.

See Subsection 1.4.2 for the case $A_2 = A_1 = B_2 = B_1 = 0$.

See Subsection 1.4.3 for the case $A_0 = B_0 = 0$.

2. The transformation $x = \bar{x} + \alpha$, $y = \bar{y} + \beta$, where α and β are the parameters which are determined by solving the second order algebraic system

$$A_{22}\beta^2 + A_{12}\alpha\beta + A_{11}\alpha^2 + A_2\beta + A_1\alpha + A_0 = 0,$$
$$B_{22}\beta^2 + B_{12}\alpha\beta + B_{11}\alpha^2 + B_2\beta + B_1\alpha + B_0 = 0,$$

leads to the equation

$$(A_{22}\bar{y}^2 + A_{12}\bar{x}\bar{y} + A_{11}\bar{x}^2 + a_2\bar{y} + a_1\bar{x})\bar{y}'_{\bar{x}} = B_{22}\bar{y}^2 + B_{12}\bar{x}\bar{y} + B_{11}\bar{x}^2 + b_2\bar{y} + b_1\bar{x}, \qquad (1)$$

where

$$a_2 = 2A_{22}\beta + A_{12}\alpha + A_2, \qquad a_1 = 2A_{11}\alpha + A_{12}\beta + A_1,$$
$$b_2 = 2B_{22}\beta + B_{12}\alpha + B_2, \qquad b_1 = 2B_{11}\alpha + B_{12}\beta + B_1.$$

The transformation $\xi = \bar{y}/\bar{x}$, $w = 1/\bar{x}$ reduces equation (1) to the Abel equation of the second kind:

$$\{[a_2\xi^2 + (a_1 - b_2)\xi - b_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi$$
$$= (a_2\xi + a_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

3. The substitution $y = z + \varepsilon x$, where parameter ε is determined by solving the cubic equation

$$(A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})\varepsilon - B_{22}\varepsilon^2 - B_{12}\varepsilon - B_{11} = 0,$$

leads to the Abel equation of the second kind with respect to $x = x(z)$:

$$[(Qz + R)x + (B_{22} - A_{22}\varepsilon)z^2 + (B_2 - A_2\varepsilon)z + B_0 - A_0\varepsilon]x'_z$$
$$= (A_{22}\varepsilon^2 + A_{12}\varepsilon + A_{11})x^2 + [(2A_{22}\varepsilon + A_{12})z + A_2\varepsilon + A_1]x + A_{22}z^2 + A_2 z + A_0,$$

where

$$Q = 2B_{22}\varepsilon + B_{12} - \varepsilon(2A_{22}\varepsilon + A_{12}), \quad R = B_2\varepsilon + B_1 - \varepsilon(A_2\varepsilon + A_1).$$

1. $\boldsymbol{(y + ax + b)^2 y'_x = (\alpha y + \beta x + \gamma)^2}$.

This is a special case of equation 1.7.1.6 with $f(z) = z^{-2}$.

2. $\boldsymbol{(Ay^2 + Bxy - \alpha By + kx - \alpha k)y'_x = Cy^2 + Dxy + (k - \alpha D)y}$.

The transformation $x = w + \alpha$, $y = w\xi$ leads to a linear equation with respect to $w = w(\xi)$: $[-A\xi^3 + (C - B)\xi^2 + D\xi]w'_\xi = (A\xi^2 + B\xi)w + k$.

3. $\boldsymbol{(Ay^2 + 2Axy + Bx^2 + A - B)y'_x = Ay^2 + 2Bxy + Dx^2 + 2(B - D)x + D - A}$.

The transformtion $x = w + 1$, $y = \xi w - 1$ leads to a linear equation:

$$(-A\xi^3 - A\xi^2 + B\xi + D)w'_\xi = (A\xi^2 + 2A\xi + B)w + 2(B - A).$$

4. $\boldsymbol{(Ay^2 - 2Axy + Bx^2 + A - B)y'_x = -Ay^2 + 2Bxy + Cx^2 + 2(B + C)x + A + C}$.

The transformtion $x = w - 1$, $y = \xi w - 1$ leads to a linear equation:

$$(-A\xi^3 + A\xi^2 + B\xi + C)w'_\xi = (A\xi^2 - 2A\xi + B)w + 2(A - B).$$

5. $(Ay^2+2Axy+Bx^2+A-B)y'_x = Ay^2+2Bxy+Cx^2+2(C-B)x-A+C.$

The transformtion $x = w-1$, $y = \xi w+1$ leads to a linear equation:

$$(-A\xi^3 - A\xi^2 + B\xi + C)w'_\xi = (A\xi^2 + 2A\xi + B)w + 2(A-B).$$

6. $(Ay^2-2Axy+Bx^2+A-B)y'_x = -Ay^2+2Bxy+Cx^2-2(B+C)x+A+C.$

The transformtion $x = w+1$, $y = \xi w+1$ leads to a linear equation:

$$(-A\xi^3 + A\xi^2 + B\xi + C)w'_\xi = (A\xi^2 - 2A\xi + B)w + 2(B-A).$$

7. $(Ay^2 - 2Axy + Bx^2 + A - B)y'_x$
$= Cy^2 + 2Bxy + Dx^2 - 2(A+C)y - 2(B+D)x + 2A + C + D.$

The transformtion $x = w+1$, $y = \xi w+1$ leads to a linear equation:

$$(-A\xi^3 + (2A+C)\xi^2 + B\xi + D)w'_\xi = (A\xi^2 - 2A\xi + B)w + 2(B-A).$$

8. $(2Ay^2 - 2Axy + Bx^2 + 2A - 4B)y'_x$
$= -Ay^2 + 2Bxy + Dx^2 - 2(B+2D)x + A + 4D.$

This is a special case of equation 1.4.4.34 with $\alpha = 2$, $\beta = 1$, $C = -A$.

9. $(Ay^2+4Axy+Bx^2+4A-B)y'_x = 2Ay^2+2Bxy+Cx^2-2(C-2B)x+C-8A.$

The transformtion $x = w+1$, $y = \xi w-2$ leads to a linear equation:

$$(-A\xi^3 - 2A\xi^2 + B\xi + C)w'_\xi = (A\xi^2 + 4A\xi + B)w + 2B - 8A.$$

10. $(Ay^2-4Axy+Bx^2+4A-B)y'_x = -2Ay^2+2Bxy+Cx^2-2(2B+C)x+8A+C.$

The transformtion $x = w+1$, $y = \xi w+2$ leads to a linear equation:

$$(-A\xi^3 + 2A\xi^2 + B\xi + C)w'_\xi = (A\xi^2 - 4A\xi + B)w + 2B - 8A.$$

11. $(Ay^2 + 4Axy + Bx^2 + 4A - B)y'_x$
$= Cy^2 + 2Bxy + 2Bx^2 + 4(C-2A)y + 2B + 4C - 16A.$

The transformtion $x = w+1$, $y = \xi w-2$ leads to a linear equation:

$$[-A\xi^3 + (C-4A)\xi^2 + B\xi + 2B]w'_\xi = (A\xi^2 + 4A\xi + B)w + 2B - 8A.$$

12. $(2Ay^2+2Axy+Bx^2+2A-4B)y'_x = Ay^2+2Bxy+Dx^2+2(B-2D)x+4D-A.$

This is a special case of equation 1.4.4.34 with $\alpha = 2$, $\beta = -1$, $C = A$.

13. $(2Ay^2+2Axy-Bx^2+2A+4B)y'_x = Ay^2-2Bxy-Dx^2+2(B-2D)x-A-4D.$

This is a special case of equation 1.4.4.34 with $\alpha = -2$, $\beta = 1$, $C = -A$.

14. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx + m)y'_x$
$= By^2 + 2Ak^2xy + Bk^2x^2 + by + ak^2x + s.$

This is a special case of equation 1.4.4.27 with $C = Ak^2$.

15. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx + m)y'_x$
$$= By^2 + 2Ak^2xy + Bk^2x^2 + aky + bkx + s.$$

This is a special case of equation 1.4.4.32 with $C = Ak^2$.

16. $(Ay^2 + 2Bxy + Ak^2x^2 + ay - akx + b)y'_x$
$$= By^2 + 2Ak^2xy + Bk^2x^2 + my - mkx + s.$$

This is a special case of equation 1.4.4.31 with $C = Ak^2$.

17. $(Ay^2+2Bxy-Bkx^2+ay+bx+c)y'_x = By^2+2Ak^2xy-Ak^3x^2+by+ak^2x+s.$

This is a special case of equation 1.4.4.28 with $m = b$.

18. $(Ay^2+2Bxy-Bkx^2+ay+bx+m)y'_x = By^2+2Ak^2xy-Ak^3x^2+aky+bkx+s.$

This is a special case of equation 1.4.4.32 with $C = -Bk$.

19. $(Ay^2 + 2Bxy - Bkx^2 + ay - akx + b)y'_x$
$$= By^2 + 2Ak^2xy - Ak^3x^2 + my - mkx + s.$$

This is a special case of equation 1.4.4.31 with $C = -Bk$.

20. $(Ay^2+2Akxy+Cx^2+ay+bx+m)y'_x = Aky^2+2Ak^2xy+Ckx^2+by+ak^2x+s.$

This is a special case of equation 1.4.4.27 with $B = Ak$.

21. $(Ay^2+2Akxy+Cx^2+ay+bx+m)y'_x = Aky^2+2Ak^2xy+Ckx^2+aky+bkx+s.$

This is a special case of equation 1.4.4.32 with $B = Ak$.

22. $(Ay^2 + 2Akxy + Cx^2 + ay - akx + b)y'_x$
$$= Aky^2 + 2Ak^2xy + Ckx^2 + my - mkx + s.$$

This is a special case of equation 1.4.4.31 with $B = Ak$.

23. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx + c)y'_x$
$$= -By^2 + 2Bkxy - Ak^3x^2 + by + ak^2x + s.$$

This is a special case of equation 1.4.4.29 with $m = b$.

24. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx + c)y'_x$
$$= -By^2 + 2Bkxy - Ak^3x^2 + aky + bkx + s.$$

This is a special case of equation 1.4.4.29 with $m = ak$.

25. $(Ay^2 + 2Bxy + Cx^2 - 2A\beta y + kx + A\beta^2)y'_x$
$$= By^2 + Exy + Fx^2 + ky - E\beta x - B\beta^2 - k\beta.$$

The substitution $w = y - \beta$ leads to an equation of the form 1.4.3.38:

$$(Aw^2 + 2Bxw + Cx^2 + \bar{k}x)w'_x = Bw^2 + Exw + Fx^2 + \bar{k}w,$$

where $\bar{k} = k + 2B\beta$.

26. $(Ay^2 + Bxy + Cx^2 + ay + bx + m)y'_x$
$$= Aky^2 + Bkxy + Ckx^2 + ny + (ak + b - n)x + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(n - ak)z + s - mk]x'_z = (Ak^2 + Bk + C)x^2 + [(2Ak + B)z + ak + b]x + Az^2 + az + m.$$

27. $(Ay^2 + 2Bxy + Cx^2 + ay + bx + m)y'_x$
$$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + by + ak^2x + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z^2 + (b - ak)z + s - mk]x'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az + m.$$

28. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx + c)y'_x$
$$= By^2 + 2Ak^2xy - Ak^3x^2 + my + k(ak + b - m)x + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-ck]x'_z = (Ak^2+Bk)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+c.$$

29. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx + c)y'_x$
$$= -By^2 + 2Bkxy - Ak^3x^2 + my + k(ak + b - m)x + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[-(Ak + B)z^2 + (m - ak)z + s - ck]x'_z = k(B - Ak)x^2 + (ak + b)x + Az^2 + az + c.$$

30. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx + c)y'_x$
$$= By^2 + 2Ak^2xy + Bk^2x^2 + my + k(ak + b - m)x + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-ck]x'_z = 2k(Ak+B)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+c.$$

31. $(Ay^2 + 2Bxy + Cx^2 + ay - akx + b)y'_x$
$$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + my - mkx + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B-Ak)z^2+(m-ak)z+s-bk]x'_z = (Ak^2+2Bk+C)x^2+2(Ak+B)zx+Az^2+az+b.$$

32. $(Ay^2 + 2Bxy + Cx^2 + ay + bx + m)y'_x$
$$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + aky + bkx + s.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B-Ak)z^2+s-mk]x'_z = (Ak^2+2Bk+C)x^2+[2(Ak+B)z+ak+b]x+Az^2+az+m.$$

33. $[A(\alpha y^2 + \beta xy + \gamma x^2) + (A\delta + 2\alpha)y + (A\varepsilon + \beta)x + A\sigma + \delta]y'_x$
$$+ B(\alpha y^2 + \beta xy + \gamma x^2) + (B\delta + \beta)y + (B\varepsilon + 2\gamma)x + B\sigma + \varepsilon = 0.$$

Solution: $\quad \alpha y^2 + \beta xy + \gamma x^2 + \delta y + \varepsilon x + \sigma = C\exp(-Ay - Bx).$

34. $(\alpha Ay^2 - 2\beta Axy + Bx^2 + \alpha\beta^2 A - \alpha^2 B)y'_x$
$= Cy^2 + 2Bxy + Dx^2 - 2\beta(\beta A + C)y - 2(\alpha D + \beta B)x + \alpha^2 D + \beta^2(2\beta A + C)$.

The substitution $x = w + \alpha$, $y = \xi w + \beta$ leads to a linear equation:

$$[-\alpha A\xi^3 + (2\beta A + C)\xi^2 + B\xi + D]w'_\xi = (\alpha A\xi^2 - 2\beta A\xi + B)w + 2(\alpha B - \beta^2 A).$$

35. $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2 y + A_1 x + A_0)y'_x$
$= B_{22}y^2 + k(2A_{22}k + A_{12} - 2B_{22})xy + k(-A_{22}k^2 + B_{22}k + A_{11})x^2$
$+ B_2 y + k(A_2 k + A_1 - B_2)x + B_0$.

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B_{22} - A_{22}k)z^2 + (B_2 - A_2 k)z + B_0 - A_0 k]x'_z =$$
$$(A_{22}k^2 + A_{12}k + A_{11})x^2 + [2(A_{22}k + A_{12})z + A_2 k + A_1]x + A_{22}z^2 + A_2 z + A_0.$$

36. $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2 y + A_1 x + A_0)y'_x$
$= B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2 y + B_1 x + B_0$.

Here A_{ij}, B_{ij}, and A_1 are arbitrary parameters, and the other parametes are defined by the equations

$$\begin{aligned} A_2 &= -A_{12}\alpha - 2A_{22}\beta, \\ A_0 &= -A_{11}\alpha^2 + A_{22}\beta^2 - A_1\alpha, \\ B_2 &= (2A_{11} - B_{12})\alpha + (A_{12} - 2B_{22})\beta + A_1, \\ B_1 &= -2B_{11}\alpha - B_{12}\beta, \\ B_0 &= B_{11}\alpha^2 + (B_{12} - 2A_{11})\alpha\beta + (B_{22} - A_{12})\beta^2 - A_1\beta, \end{aligned}$$

(α, β are arbitrary parameters).

The transformation $x = w + \alpha$, $y = \xi w + \beta$ leads to a linear equation:

$$[-A_{22}\xi^3 + (B_{22} - A_{12})\xi^2 + (B_{12} - A_{11})\xi + B_{11}]w'_\xi = (A_{22}\xi^2 + A_{12}\xi + A_{11})w + k,$$

where $k = 2A_{11}\alpha + A_{12}\beta + A_1$.

1.5. Nonlinear Equations of the Form $f(x,y)y'_x = g(x,y)$ Containing Arbitrary Parameters

1.5.1. Equations Containing Power Functions

1. $y'_x = A\sqrt{y} + Bx^{-1/2}$.

The substitution $w = (2/A)\sqrt{y}$ leads to the Abel equation of the form 1.3.1.32: $ww'_x = w + 2BA^{-2}x^{-1/2}$.

2. $y'_x = A\sqrt{y} + Bx^{-1}$.

Let $A = \pm 2a^{-1}\sqrt{b}$, $B = \mp 4b$ $(b > 0)$.

The solution in the parametric form is written as

$$x = af(\tau), \qquad y = b[2\tau \pm f(\tau)]^2,$$

where $f(\tau) = \exp(\mp\tau^2)\left[\int \exp(\mp\tau^2)\,d\tau + C\right]^{-1}$.

3. $\boldsymbol{y'_x = A\sqrt{y} + Bx^{-2}.}$

The substitution $w = 2A^{-1}\sqrt{y}$ leads to the Abel equation of the form 1.3.1.33: $ww'_x = w + 2BA^{-2}x^{-2}$.

4. $\boldsymbol{y'_x = a\sqrt{y} + bx + cx^m.}$

The substitution $w = 2a^{-1}\sqrt{y}$ leads to the Abel equation of the second kind: $ww'_x = w + 2a^{-2}(bx + cx^m)$, whose special cases are outlined in Subsection 1.3.1.

5. $\boldsymbol{y'_x = ay^n + bx^{\frac{n}{1-n}}.}$

Solution:

$$\int \frac{dw}{aw^n + \frac{1}{1-n}w + b} = \ln|x| + C, \qquad \text{where} \quad w = yx^{\frac{1}{n-1}}.$$

6. $\boldsymbol{y'_x = Ay^s - Bx^k.}$

The transformation $x = \left(w'_z\right)^{\frac{1}{k}}$, $y = \lambda\left(\dfrac{w}{z}\right)^{\frac{1}{s}}$, where $\lambda = \left(\dfrac{B}{A}\right)^{\frac{1}{s}}$, leads to the generalized Emden—Fowler equation:

$$w''_{zz} = -\frac{\lambda k}{sB} z^{-\frac{1}{s}} w^{\frac{1-s}{s}} \left(w'_z\right)^{\frac{k-1}{k}},$$

which is discussed in Section 2.5 (in the classification table, one should search for the equations satisfying the condition $n + m + 1 = 0$).

7. $\boldsymbol{y'_x = (ax + by + c)^n.}$

This is a special case of equation 1.7.1.1 with $f(\xi) = \xi^n$.

8. $\boldsymbol{y'_x = ax^{m-n-nm}y^n + bx^m.}$

Solution:

$$\int \frac{dw}{w^n - \lambda w + 1} + C = b\sqrt[n]{\frac{a}{b}}\ln|x|,$$

where $w = \sqrt[n]{\dfrac{a}{b}}\,yx^{-m-1}$, $\lambda = \dfrac{m+1}{b}\sqrt[n]{\dfrac{b}{a}}$.

9. $\boldsymbol{y'_x = ax^{n-1}y^{m+1} + bx^{nk-1}y^{mk+1}.}$

This is the homogeneous equation in the extended sense of the form 1.7.1.3 with $f(\xi) = a\xi + b\xi^k$.

10. $\boldsymbol{y'_x = ax^k y\sqrt{y} + bx^m y + cx^s\sqrt{y}.}$

This is a special case of equation 1.7.1.4 with $f(x) = ax^k$, $g(x) = bx^m$, $h(x) = cx^s$, $n = 1/2$.

11. $\boldsymbol{y'_x = ax^k y^{1+n} + bx^m y + cx^s y^{1-n}.}$

This is a special case of equation 1.7.1.4 with $f(x) = ax^k$, $g(x) = bx^m$, $h(x) = cx^s$.

12. $y'_x = x^{n-1}y^{1-m}(ax^n + by^m)^k.$

This is a special case of equation 1.7.1.7 with $f(\xi) = \xi^k$.

13. $xy'_x = y + ax^{n-m}y^m + bx^{n-k}y^k.$

The substitution $y = xw$ leads to an equation with separation of variables: $w'_x = x^{n-2}(aw^m + bw^k)$.

14. $(ay^n + bx)y'_x = 1.$

Solution: $x = e^{by}\left(C + a\int y^n e^{-by}\,dy\right)$.

15. $x(xy^n + a)y'_x + by = 0.$

Solution: $nb - a = x\bigl(Cy^{a/b} + y^n\bigr)$.

16. $x(ay^m + m)y'_x = y\bigl[bx^{n(\lambda-1)}y^{m\lambda} - n\bigr].$

This is a special case of equation 1.7.1.15 with $f(\xi) = a\xi$, $g(\xi) = 1$, $h(\xi) = b\xi^\lambda$, $k = n$.

17. $(ax^n + bx^2 + cxy)y'_x = kx^n + bxy + cy^2.$

The transformation $t = y/x$, $z = x^{n-2}$ leads to a linear equation: $(k - at)z'_t = (n-2)(az + b + ct)$.

18. $(ay^n + bx^2 + cxy)y'_x = ky^n + bxy + cy^2.$

The transformation $t = y/x$, $z = x^{n-2}$ leads to a linear equation: $t^n(k - at)z'_t = (n-2)(at^n z + b + ct)$.

19. $(ax^n + by^n + x)y'_x = \alpha x^k y^{n-k} + \beta x^m y^{n-m} + y.$

The transformation $t = y/x$, $z = x^{n-1}$ leads to a linear equation:

$$(\alpha t^{n-k} + \beta t^{n-m} - bt^{n+1} - at)z'_t = (n-1)(bt^n + a)z + n - 1.$$

20. $(ax^n + by^n + Ax^2 + Bxy)y'_x = \alpha x^k y^{n-k} + \beta x^m y^{n-m} + Axy + By^2.$

The transformation $t = y/x$, $z = x^{n-2}$ leads to a linear equation:

$$(\alpha t^{n-k} + \beta t^{n-m} - bt^{n+1} - at)z'_t = (n-2)(bt^n + a)z + (n-2)(Bt + A).$$

21. $[(ax + by)^n + bx]y'_x = c(ax + by)^m - ax.$

This is a special case of equation 1.7.1.13 with $f(\xi) = \xi^n$, $g(\xi) = 1$, $h(\xi) = c\xi^m$.

22. $[(ax + by)^n + by]y'_x = c(ax + by)^m - ay.$

This is a special case of equation 1.7.1.14 with $f(\xi) = \xi^n$, $g(\xi) = 1$, $h(\xi) = c\xi^m$.

23. $(\alpha x + \beta y + \gamma)^n y'_x = (ax + by + c)^n.$

This is a special case of equation 1.7.1.6 with $f(\xi) = \xi^n$.

24. $(ax^n + by^m)y'_x = x^{n-1}y^{1-m}.$

This is a special case of equation 1.7.1.7 with $f(\xi) = 1/\xi$.

25. $(ay^m + bx^n + s)y'_x + \alpha x^k + bnx^{n-1}y + \beta = 0.$

Solution:

$$a\varphi(y) + \alpha\psi(x) + bx^n y + sy + \beta x = C,$$

where

$$\varphi(y) = \begin{cases} \dfrac{y^{m+1}}{m+1} & \text{if } m \neq -1, \\ \ln|y| & \text{if } m = -1, \end{cases} \qquad \psi(x) = \begin{cases} \dfrac{x^{k+1}}{k+1} & \text{if } k \neq -1, \\ \ln|x| & \text{if } k = -1. \end{cases}$$

26. $(ax^2y^n + bxy^m + cy^k)y'_x = \alpha y^p + \beta y^q + \gamma.$

This is the Riccati equation with respect to $x = x(y)$.

27. $(ax^n y^m + x)y'_x = bx^k y^{n+m-k} + y.$

The transformation $t = y/x$, $z = x^{n+m-1}$ leads to a linear equation: $t^m(bt^{n-k} - at)z'_t = (n + m - 1)(at^m z + 1)$.

28. $x(ax^n y^m + \alpha)y'_x + y(bx^n y^m + \beta) = 0.$

Solution:

$$\frac{(y^a x^b)^A}{A} + \frac{(y^\alpha x^\beta)^B}{B} = C, \qquad \text{where} \quad A = \frac{m\beta - n\alpha}{a\beta - b\alpha}, \quad B = \frac{mb - na}{a\beta - b\alpha}.$$

29. $x(anx^k y^{n+k} + s)y'_x + y(bmx^{m+k}y^k + s) = 0.$

Solution: $aky^n + bkx^m - s(xy)^{-k} = C$.

30. $(ax^n y^m + Ax^2 + Bxy)y'_x = bx^k y^{n+m-k} + Axy + By^2.$

The transformation $t = y/x$, $z = x^{n+m-2}$ leads to a linear equation: $t^m(bt^{n-k} - at)z'_t = (n + m - 2)(at^m z + Bt + A)$.

31. $(amx^n y^{m-1} + by^k)y'_x + anx^{n-1}y^m + cx^s = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = by^k$, $g(x) = cx^s$.

32. $(ax^n y^m + bxy^k)y'_x = \alpha y^s + \beta.$

This is the Bernoulli equation with respect to $x = x(y)$ (see Subsection 1.1.5).

33. $x(ax^{n-k}y^m + m)y'_x = y(bx^{\lambda n-k}y^{\lambda m} - n).$

This is a special case of equation 1.7.1.15 with $f(\xi) = a\xi$, $g(\xi) = 1$, $h(\xi) = b\xi^\lambda$.

34. $x(ax^n y^{m-k} + m)y'_x = y(bx^{\lambda n}y^{\lambda m-k} - n).$

This is a special case of equation 1.7.1.16 with $f(\xi) = a\xi$, $g(\xi) = 1$, $h(\xi) = b\xi^\lambda$.

35. $(ax^{n+1}y^{m-1} + bx^{nk+1}y^{mk-1})y'_x = cx^{ns}y^{ms}$.

This is a special case of equation 1.7.1.3 with $f(\xi) = c\xi^s(a\xi + b\xi^k)^{-1}$.

36. $(ax^n + by^m)^k y'_x = cx^{n-1}y^{1-m}$.

This is a special case of equation 1.7.1.7 with $f(\xi) = c\xi^{-k}$.

37. $xy'_x = y + a\sqrt{y^2 + bx^2}$.

Solution: $y + \sqrt{y^2 + bx^2} = Cx^{a+1}$.

38. $$\left(e_1\frac{x+a}{r_1^3} + e_2\frac{x-a}{r_2^3}\right)y'_x - y\left(\frac{e_1}{r_1^3} + \frac{e_2}{r_2^3}\right) = 0,$$
$$\text{where} \quad r_1^2 = (x+a)^2 + y^2, \quad r_2^2 = (x-a)^2 + y^2.$$

This is the equation of force lines corresponding to the Coulomb law.

Solution: $e_1\dfrac{x+a}{r_1} + e_2\dfrac{x-a}{r_2} = C$.

1.5.2. Equations Containing Exponential Functions

1. $y'_x = ae^y + be^x$.

Solution in the parametric form:

$$x = \ln\tau, \qquad y = b\tau - \ln\left(C - a\int\frac{e^{b\tau}}{\tau}\,d\tau\right).$$

2. $y'_x = ae^y + bx^n$.

Solution in the parametric form:

with $n \neq -1$,

$$x = \tau^{\frac{1}{n+1}}, \qquad y = \frac{b}{n+1}\tau - \ln\left[C - \frac{a}{n+1}\int\tau^{-\frac{n}{n+1}}\exp\left(\frac{b\tau}{n+1}\right)d\tau\right];$$

with $n = -1,\ b \neq -1$,

$$x = e^\tau, \qquad y = -\ln\left(Ce^{-b\tau} - \frac{a}{b+1}e^\tau\right);$$

with $n = -1,\ b = -1$,

$$x = e^\tau, \qquad y = -\tau - \ln(C - a\tau).$$

3. $y'_x = ay^{-1} + be^x$.

Solution in the parametric form:

$$x = \ln(AE^{-1}) \mp \tau^2, \qquad y = B[2 \pm \exp(\mp\tau^2)E^{-1}],$$

where $a = \mp 2B^2,\ b = \pm A^{-1}B,\ E = \int\exp(\mp\tau^2)\,d\tau + C$.

4. $y'_x = Ae^{y+ax} - a.$

This is a special case of equation 1.7.1.2 with $f(\xi) = Ae^{\xi}$, $n = 1$, $b = 0$.

5. $y'_x = ae^{\nu x+\lambda y} + be^{\mu x}.$

This is a special case of equation 1.7.2.5 with $f(x) = ae^{\nu x}$, $g(x) = be^{\mu x}$.

6. $y'_x = ae^{\nu x+\lambda y} + bx^n.$

This is a special case of equation 1.7.2.5 with $f(x) = ae^{\nu x}$, $g(x) = bx^n$.

7. $y'_x = ax^n e^{\lambda y} + be^{\nu x}.$

This is a special case of equation 1.7.2.5 with $f(x) = ax^n$, $g(x) = be^{\nu x}$.

8. $y'_x = ax^n e^{\lambda y} + bx^m.$

This is a special case of equation 1.7.2.5 with $f(x) = ax^n$, $g(x) = bx^m$.

9. $y'_x = ae^{\nu x+\lambda y} + be^{\mu x-\lambda y}.$

This is a special case of equation 1.7.2.8 with $f(x) = ae^{\nu x}$, $g(x) = 0$, $h(x) = be^{\mu x}$.

10. $y'_x = ax^n e^{\lambda y} + bx^m e^{-\lambda y}.$

This is a special case of equation 1.7.2.8 with $f(x) = ax^n$, $g(x) = 0$, $h(x) = bx^m$.

11. $y'_x = (y + ae^{\lambda x})^n - a\lambda e^{\lambda x}.$

This is a special case of equation 1.7.2.10 with $f(\xi) = \xi^n$.

12. $y'_x = (ae^y + bx^{-k})^{1/k}.$

Solution in the parametric form:

$$x = \exp\Bigl\{\tau - \frac{1}{k}\bigl[f(\tau) + C\bigr]\Bigr\}, \quad y = f(\tau) + C, \qquad \text{where} \quad f(\tau) = \int \frac{k\,d\tau}{k\bigl(b + ae^{k\tau}\bigr)^{-1/k} + 1}.$$

13. $y'_x = (ay^k + be^x)^{1/k}.$

Solution in the parametric form:

$$x = f(\tau) + C, \quad y = \exp\Bigl\{\tau + \frac{1}{k}\bigl[f(\tau) + C\bigr]\Bigr\}, \qquad \text{where} \quad f(\tau) = \int \frac{k\,d\tau}{k(a + be^{-k\tau})^{1/k} - 1}.$$

14. $y'_x = ax^{n-1} e^{\lambda n y} + bx^{m-1} e^{\lambda m y}.$

This is a special case of equation 1.7.2.2 with $f(\xi) = a\xi^{n-1} + b\xi^{m-1}$.

15. $y'_x = ax^{n-1} e^{\alpha y} + bx^{nm-1} e^{\alpha m y}.$

This is a special case of equation 1.7.2.4 with $f(\xi) = a\xi + b\xi^m$.

16. $y'_x = ae^{\lambda nx}y^{n+1} + be^{-\lambda x}$.

This is a special case of equation 1.7.2.1 with $f(\xi) = a\xi^{n+1} + b$.

17. $y'_x = ae^{\alpha x}y^{m+1} + be^{\alpha nx}y^{nm+1}$.

This is a special case of equation 1.7.2.3 with $f(\xi) = a\xi + b\xi^n$.

18. $y'_x = ae^{\lambda nx}y^{n+1} + be^{\lambda mx}y^{m+1}$.

This is a special case of equation 1.7.2.1 with $f(\xi) = a\xi^{n+1} + b\xi^{m+1}$.

19. $y'_x = ae^{2\alpha x-\beta y} + be^{\alpha x} + ce^{\alpha x-\beta y}$.

This is a special case of equation 1.7.2.9 with $f(\xi) = \xi + c$.

20. $y'_x = ax^n y^k + bx^n e^{\alpha x}y^{k+1} - \alpha y$.

This is a special case of equation 1.7.2.7 with $f(\xi) = \xi^n$, $g(\xi) = a + b\xi$, $m = 1$.

21. $y'_x = ae^{\lambda x}y^{1+n} + be^{\mu x}y + ce^{\nu x}y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$, $h(x) = ce^{\nu x}$.

22. $y'_x = ae^{\lambda x}y^{1+n} + be^{\mu x}y + cx^m y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\mu x}$, $h(x) = cx^m$.

23. $y'_x = ax^k y^{1+n} + be^{\lambda x}y + cx^m y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = ax^k$, $g(x) = be^{\lambda x}$, $h(x) = cx^m$.

24. $y'_x = ae^{\lambda x}y^{1+n} + bx^m y + ce^{\mu x}y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = bx^m$, $h(x) = ce^{\mu x}$.

25. $y'_x = ae^{\lambda x}y^{1+n} + bx^m y + cx^k y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = ae^{\lambda x}$, $g(x) = bx^m$, $h(x) = cx^k$.

26. $xy'_x = ax^{n+k}e^y + bx^{nm+k}e^{my} - n$.

This is a special case of equation 1.7.2.6 with $f(x) = x^{k-1}$, $g(\xi) = a\xi + b\xi^m$.

27. $(by + \lambda)y'_x = ce^{ax+by} - ay$.

This is a special case of equation 1.7.1.14 with $f(\xi) = \lambda$, $g(\xi) = 1$, $h(\xi) = ce^{\xi}$.

28. $xyy'_x = ax^n e^y - ny$.

This is a special case of equation 1.7.2.11 with $f(\xi) = a\xi$, $\alpha = 1$.

29. $xy^2 y'_x = ax^n e^y - ny^2$.

This is a special case of equation 1.7.2.12 with $f(\xi) = a\xi$, $\alpha = 1$.

30. $(ae^y + be^x)y'_x = 1.$

Solution in the parametric form:

$$x = a\tau - \ln\left(C - b\int \frac{e^{a\tau}}{\tau}\,d\tau\right), \qquad y = \ln\tau.$$

31. $(ay^n + be^x)y'_x = 1.$

Solution in the parametric form:
with $n \neq -1$:

$$x = \frac{a}{n+1}\tau - \ln\left[C - \frac{b}{n+1}\int \tau^{-\frac{n}{n+1}} \exp\left(\frac{a\tau}{n+1}\right)d\tau\right], \qquad y = \tau^{\frac{1}{n+1}};$$

with $n = -1$, $a \neq -1$:

$$x = -\ln\left(Ce^{-a\tau} - \frac{b}{a+1}e^{\tau}\right), \qquad y = e^{\tau};$$

with $n = -1$, $a = -1$:

$$x = -\tau - \ln(C - b\tau), \qquad y = e^{\tau}.$$

32. $(ae^y + bx)y'_x = 1.$

Solution in the implicit form:

$$x = \begin{cases} Ce^{by} + \dfrac{a}{1-b}e^y & \text{if } b \neq 1, \\ e^y(C + ay) & \text{if } b = 1. \end{cases}$$

33. $(ae^y + bx^2)y'_x = 1.$

Solutions in the parametric form:

$$x = -\frac{1}{2b}\tau(\ln Z)'_\tau, \quad y = \ln\left(\frac{\tau^2}{4ab}\right), \qquad Z = C_1J_0(\tau) + C_2Y_0(\tau)$$

and

$$x = -\frac{1}{2b}\tau(\ln Z)'_\tau, \quad y = \ln\left(-\frac{\tau^2}{4ab}\right), \qquad Z = C_1I_0(\tau) + C_2K_0(\tau),$$

where J_0 and Y_0 are Bessel functions, I_0 and K_0 are modified Bessel functions.

34. $(ae^y + bx^{-1})y'_x = 1.$

Let $a = \pm A/B$, $b = \mp 2A^2$. The solution in the parametric form is written as

$$x = A\big[2\tau \pm \exp(\mp\tau^2)f(\tau)\big], \qquad y = \ln\big[Bf(\tau)\big] \mp \tau^2,$$

where $f(\tau) = \left[\int \exp(\mp\tau^2)\,d\tau + C\right]^{-1}$.

35. $(be^{\alpha y} + c)y'_x = e^{ax+by} - ae^{\alpha y}.$

This is a special case of equation 1.7.2.13 with $f(\xi) = c$, $g(\xi) = 1$, $h(\xi) = e^{\xi}$.

36. $(ae^{\alpha x} + be^{\beta y})y'_x = e^{\alpha x - \beta y}.$

This is a special case of equation 1.7.2.9 with $f(\xi) = \xi^{-1}$.

37. $(e^{\alpha x + \gamma y} + a\beta)y'_x + be^{\nu x + \beta y} + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = e^{\gamma y}$, $g(x) = be^{\nu x}$.

38. $(e^{ax+by} + bx)y'_x = ce^{ax+by} - ax.$

This is a special case of equation 1.7.1.13 with $f(\xi) = e^{\xi}$, $g(\xi) = 1$, $h(\xi) = ce^{\xi}$.

39. $(e^{ax+by} + by)y'_x = ce^{ax+by} - ay.$

This is a special case of equation 1.7.1.14 with $f(\xi) = e^{\xi}$, $g(\xi) = 1$, $h(\xi) = ce^{\xi}$.

40. $(ae^{\alpha x}y^m + b)y'_x = y.$

This is a special case of equation 1.7.2.3 with $f(\xi) = (a\xi + b)^{-1}$.

41. $(e^{\alpha x}y^n + a\beta)y'_x + be^{\nu x + \beta y} + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = y^n$, $g(x) = be^{\nu x}$.

42. $(e^{\alpha x}y^n + a\beta)y'_x + bx^m e^{\beta y} + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = y^n$, $g(x) = bx^m$.

43. $(e^{\alpha x}y^m + mx)y'_x = y(be^{\alpha n x}y^{nm} - \alpha x).$

This is a special case of equation 1.7.2.17 with $f(\xi) = \xi$, $g(\xi) = 1$, $h(\xi) = b\xi^n$.

44. $x(x^n e^{\alpha y} + \alpha y)y'_x = bx^{nm}e^{\alpha m y} - ny.$

This is a special case of equation 1.7.2.16 with $f(\xi) = \xi$, $g(\xi) = 1$, $h(\xi) = b\xi^m$.

45. $(ax^n e^{\lambda y} + bxe^{\mu y})y'_x = e^{\nu y}.$

This is the Bernoulli equation with respect to $x = x(y)$ (see 1.1.5).

46. $(ax^n e^{\lambda y} + bxy^m)y'_x = e^{\mu y}.$

This is the Bernoulli equation with respect to $x = x(y)$.

47. $(ax^n y^m + bxe^{\lambda y})y'_x = y^k.$

This is the Bernoulli equation with respect to $x = x(y)$.

48. $(ax^n y^m + bxy^k)y'_x = e^{\lambda y}.$

This is the Bernoulli equation with respect to $x = x(y)$.

49. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + ce^{\lambda x} = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = ce^{\lambda x}$.

50. $(amx^n y^{m-1} + be^{\lambda y})y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = be^{\lambda y}$, $g(x) = c$.

51. $(amx^n y^{m-1} + by^k)y'_x + anx^{n-1}y^m + ce^{\lambda x} = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = by^k$, $g(x) = ce^{\lambda x}$.

52. $[(ax + by)^n + be^{\alpha x}]y'_x = c(ax + by)^m - ae^{\alpha x}.$

This is a special case of equation 1.7.2.14 with $f(\xi) = \xi^n$, $g(x) = 1$, $h(\xi) = c\xi^m$.

53. $[(ax + by)^n + be^{\alpha y}]y'_x = c(ax + by)^m - ae^{\alpha y}.$

This is a special case of equation 1.7.2.13 with $f(\xi) = \xi^n$, $g(x) = 1$, $h(\xi) = c\xi^m$.

1.5.3. Equations Containing Hyperbolic Functions

1. $y'_x = a\cosh(\lambda y) + b\cosh(\nu x).$

This is a special case of equation 1.7.2.18 with $f(x) = 0$, $g(x) = a$, $h(x) = b\cosh(\nu x)$.

2. $y'_x = a\sinh(\lambda y) + b\sinh(\nu x).$

This is a special case of equation 1.7.2.18 with $f(x) = a$, $g(x) = 0$, $h(x) = b\sinh(\nu x)$.

3. $y'_x = ax^n\cosh(\lambda y) + bx^m.$

This is a special case of equation 1.7.2.18 with $f(x) = 0$, $g(x) = ax^n$, $h(x) = bx^m$.

4. $y'_x = ax^n\sinh(\lambda y) + bx^m.$

This is a special case of equation 1.7.2.18 with $f(x) = ax^n$, $g(x) = 0$, $h(x) = bx^m$.

5. $y'_x = ay^{1+n} + by + c\sinh(\lambda x)y^{1-n}.$

This is a special case of equation 1.7.1.4 with $f(x) = a$, $g(x) = b$, $h(x) = c\sinh(\lambda x)$.

6. $y'_x = ay^{1+n} + b\sinh(\lambda x)y + cy^{1-n}.$

This is a special case of equation 1.7.1.4 with $f(x) = a$, $g(x) = b\sinh(\lambda x)$, $h(x) = c$.

7. $y'_x = y\cosh x(ay^{nm}\sinh^{n-1}x + by^m).$

This is a special case of equation 1.7.2.20 with $f(\xi) = a\xi^n + b\xi$.

8. $y'_x = y\sinh x(ay^{nm}\cosh^{n-1}x + by^m).$

This is a special case of equation 1.7.2.22 with $f(\xi) = a\xi^n + b\xi$.

9. $xy'_x = (ax^n\cosh y + b)\coth y.$

This is a special case of equation 1.7.2.23 with $f(\xi) = a\xi + b$.

10. $xy'_x = (ax^n \sinh y + b) \tanh y.$

This is a special case of equation 1.7.2.21 with $f(\xi) = a\xi + b$.

11. $(ay^m \cosh x + b)y'_x = y^{m+1} \sinh x.$

This is a special case of equation 1.7.2.22 with $f(\xi) = \xi(a\xi + b)^{-1}$.

12. $(ay^m \sinh x + b)y'_x = y^{m+1} \cosh x.$

This is a special case of equation 1.7.2.20 with $f(\xi) = \xi(a\xi + b)^{-1}$.

13. $(ax^n + bx \cosh^m y)y'_x = y^k.$

This is the Bernoulli equation with respect to $x = x(y)$ (see Subsection 1.1.5).

14. $(ax^n + bx \tanh^m y)y'_x = y^k.$

This is the Bernoulli equation with respect to $x = x(y)$.

15. $(ax^n + bx \cosh^m y)y'_x = \cosh^k(\lambda y).$

This is the Bernoulli equation with respect to $x = x(y)$.

16. $(ax^n + bx \tanh^m y)y'_x = \tanh^k(\lambda y).$

This is the Bernoulli equation with respect to $x = x(y)$.

17. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c \sinh^k(\lambda x) = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = c \sinh^k(\lambda x)$.

18. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c \tanh^k(\lambda x) = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = c \tanh^k(\lambda x)$.

19. $(ax^n y^m + bx)y'_x = \cosh^k(\lambda y).$

This is the Bernoulli equation with respect to $x = x(y)$.

20. $(ax^n y^m + bx)y'_x = \tanh^k(\lambda y).$

This is the Bernoulli equation with respect to $x = x(y)$.

21. $(ax^n \cosh^m y + bx)y'_x = \sinh^k(\lambda y).$

This is the Bernoulli equation with respect to $x = x(y)$.

22. $(ax^n \tanh^m y + bx)y'_x = y^k.$

This is the Bernoulli equation with respect to $x = x(y)$.

23. $(amx^n y^{m-1} + b \sinh^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = b \sinh^k y$, $g(x) = c$.

24. $(amx^n y^{m-1} + b \tanh^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 1.7.1.18 with $f(y) = b \tanh^k y$, $g(x) = c$.

1.5.4. Equations Containing Logarithmic Functions

1. $y'_x = y(\alpha x + m \ln y + \beta)$.

This is a special case of equation 1.7.2.3 with $f(\xi) = \ln \xi + \beta$.

2. $y'_x = ax^{kn-1} y^{km+1} (n \ln x + m \ln y)$.

This is a special case of equation 1.7.1.3 with $f(\xi) = a\xi^k \ln \xi$.

3. $y'_x = ax^n y \ln^2 y + bx^m y \ln y + cx^k y$.

This is a special case of equation 1.7.3.1 with $f(x) = ax^n$, $g(x) = bx^m$, $h(x) = cx^k$.

4. $xy'_x = (\alpha y + n \ln x)^m + \beta$.

This is a special case of equation 1.7.2.4 with $f(\xi) = \ln^m \xi + \beta$.

5. $xy'_x = y(n \ln x + m \ln y)$.

This is a special case of equation 1.7.1.3 with $f(\xi) = \ln \xi$.

6. $mxy'_x = ax^s y^k (n \ln x + m \ln y) - ny$.

This is a special case of equation 1.7.1.5 with $f(x) = \frac{a}{m} x^{s-1}$, $g(\xi) = \ln \xi$.

7. $(x^a + b)y'_x = yx^{a-1} + c(\ln y - \ln x)$.

This is a special case of equation 1.7.1.12 with $f(\xi) = b$, $g(\xi) = c \ln \xi$, $h(\xi) = 1$.

8. $x(\alpha y + \beta)y'_x = n \ln x + (\alpha - n)y$.

This is a special case of equation 1.7.2.16 with $f(\xi) = \beta$, $g(\xi) = 1$, $h(\xi) = \ln \xi$.

9. $x(a + mx^k)y'_x = y(bn \ln x + bm \ln y - nx^k)$.

This is a special case of equation 1.7.1.15 with $f(\xi) = a$, $g(\xi) = 1$, $h(\xi) = b \ln \xi$.

10. $x(a + my^k)y'_x = y(bn \ln x + bm \ln y - ny^k)$.

This is a special case of equation 1.7.1.16 with $f(\xi) = a$, $g(\xi) = 1$, $h(\xi) = b \ln \xi$.

11. $(amx^n y^{m-1} + b)y'_x + anx^{n-1} y^m + c \ln^k (\lambda x) = 0$.

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = c \ln^k (\lambda x)$.

12. $(a \ln y + bx)y'_x = 1$.

Solution in the parametric form:

$$x = e^{b\tau} \left(\frac{a}{b} \int \frac{e^{-b\tau}}{\tau} \, d\tau + C \right) - \frac{a}{b} \ln \tau, \qquad y = \tau.$$

13. $x(\ln y)y'_x = y(ax^{nk} y^k + bx^n y) - ny \ln y$.

This is a special case of equation 1.7.3.7 with $f(\xi) = a\xi^k + b\xi$, $m = 1$.

14. $x(a+m\ln y)y'_x = y(bx^n y^m - n\ln y + c).$

This is a special case of equation 1.7.3.9 with $f(\xi)=a$, $g(\xi)=1$, $h(\xi)=b\xi+c$.

15. $(ax^n + bx\ln^m y)y'_x = \ln^k(\lambda y).$

This is the Bernoulli equation with respect to $x=x(y)$.

16. $x(ax^n y^m + m\ln x)y'_x = y(bx^{nk}y^{mk} - n\ln x).$

This is a special case of equation 1.7.3.10 with $f(\xi)=a\xi$, $g(\xi)=1$, $h(\xi)=b\xi^k$.

17. $x(ax^n y^m + m\ln y)y'_x = y(bx^{nk}y^{mk} - n\ln y).$

This is a special case of equation 1.7.3.9 with $f(\xi)=a\xi$, $g(\xi)=1$, $h(\xi)=b\xi^k$.

18. $(amx^n y^{m-1} + b\ln^k y)y'_x + anx^{n-1}y^m + c = 0.$

This is a special case of equation 1.7.1.18 with $f(y)=b\ln^k y$, $g(x)=c$.

19. $(ax^n\ln^m y + bx)y'_x = \ln^k(\lambda y).$

This is the Bernoulli equation with respect to $x=x(y)$.

20. $(ax^n\ln^m y + bx\ln^k y)y'_x = y^s.$

This is the Bernoulli equation with respect to $x=x(y)$.

1.5.5. Equations Containing Trigonometric Functions

1. $y'_x = \alpha\cos(ay) + \beta\cos(bx).$

This is a special case of equation 1.7.4.11 with $f(x)=\alpha$, $g(x)=0$, $h(x)=\beta\cos(bx)$.

2. $y'_x = \sin(ax)\cos(by) + \cos(ax)\sin(by).$

This is a special case of equation 1.7.1.1 with $f(\xi)=\sin\xi$, $c=0$.

3. $y'_x = a\tan(bxy).$

The solution is given by the relation

$$\int_0^w \exp\left(\tfrac{1}{2}t^2\right)\cos\left(\sqrt{ab}\,xt\right)dt = c\exp\left(\tfrac{1}{2}abx^2\right), \qquad \text{where} \quad w = y\sqrt{\frac{b}{a}}.$$

4. $y'_x = bx^n\cos(ay) + cx^m.$

This is a special case of equation 1.7.4.11 with $f(x)=bx^n$, $g(x)=0$, $h(x)=cx^m$.

5. $y'_x = bx^n\sin(ay) + cx^m.$

This is a special case of equation 1.7.4.11 with $f(x)=0$, $g(x)=bx^n$, $h(x)=cx^m$.

6. $y'_x = y\cos x(ay^{nm}\sin^{n-1}x + by^m)$.

This is a special case of equation 1.7.4.4 with $f(\xi) = a\xi^n + b\xi$.

7. $y'_x = y\sin x(ay^{nm}\cos^{n-1}x + by^m)$.

This is a special case of equation 1.7.4.3 with $f(\xi) = a\xi^n + b\xi$.

8. $y'_x = a\dfrac{\sin^2 y}{\cos^2 x} + b\dfrac{\cos^2 y}{\sin^2 x}$.

This is a special case of equation 1.7.4.14 with $f(\xi) = a\xi + b\xi^{-1}$.

9. $y'_x = ay^{1+n} + by + c\sin(\lambda x)y^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = a$, $g(x) = b$, $h(x) = c\sin(\lambda x)$.

10. $y'_x = ay^{1+n} + b\sin(\lambda x)y + cy^{1-n}$.

This is a special case of equation 1.7.1.4 with $f(x) = a$, $g(x) = b\sin(\lambda x)$, $h(x) = c$.

11. $xy'_x + a\sin(bx + cy) = 0$.

The substitution $w = x\tan\dfrac{bx + cy}{2}$ leads to the Riccati equation of the form 1.2.2.22: $2xw'_x - bw^2 + 2(ac - 1)w - bx^2 = 0$.

12. $xy'_x = ax^2\tan(by) + y$.

The substitution $y = xw$ leads to an equation of the form 1.5.5.3: $w'_x = a\tan(bxw)$.

13. $xy'_x = ax^n\cos^2 y + b\cos y\sin y$.

This is a special case of equation 1.7.4.8 with $f(\xi) = \frac{1}{2}(a\xi + b)$.

14. $xy'_x = ax^n\sin^2 y + b\cos y\sin y$.

This is a special case of equation 1.7.4.7 with $f(\xi) = \frac{1}{2}(a\xi + b)$.

15. $xy'_x = ax^m\sin^k y\cos^{2-k} y - n\sin 2y$.

This is a special case of equation 1.7.4.18 with $f(x) = ax^{m-2nk-1}$, $g(\xi) = \xi^k$.

16. $(1 + \tan^2 y)y'_x = a\tan^{m+1} y + b\tan y + cx^n\tan^{1-m} y$.

This is a special case of equation 1.7.4.19 with $f(x) = a$, $g(x) = b$, $h(x) = cx^n$.

17. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c\sin^k(\lambda x) = 0$.

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = c\sin^k(\lambda x)$.

18. $(amx^n y^{m-1} + b)y'_x + anx^{n-1}y^m + c\tan^k(\lambda x) = 0$.

This is a special case of equation 1.7.1.18 with $f(y) = b$, $g(x) = c\tan^k(\lambda x)$.

19. $(ax^n y^m + bx)y'_x = \cos^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$ (see 1.1.5).

20. $(ax^n y^m + bx)y'_x = \tan^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$.

21. $(ay^m \cos x + b)y'_x = y^{m+1} \sin x$.

This is a special case of equation 1.7.4.3 with $f(\xi) = \xi(a\xi + b)^{-1}$.

22. $(ay^m \sin x + b)y'_x = y^{m+1} \cos x$.

This is a special case of equation 1.7.4.4 with $f(\xi) = \xi(a\xi + b)^{-1}$.

23. $(ax^n + bx \cos^m y)y'_x = y^k$.

This is the Bernoulli equation with respect to $x = x(y)$.

24. $(ax^n + bx \cos^m y)y'_x = \cos^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$.

25. $(amx^n y^{m-1} + b \cos^k y)y'_x + anx^{n-1}y^m + c = 0$.

This is a special case of equation 1.7.1.18 with $f(y) = b \cos^k y$, $g(x) = c$.

26. $(ax^n \cos^m y + bx)y'_x = \cos^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$.

27. $(ax^n + bx \tan^m y)y'_x = y^k$.

This is the Bernoulli equation with respect to $x = x(y)$.

28. $(ax^n + bx \tan^m y)y'_x = \tan^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$.

29. $(amx^n y^{m-1} + b \tan^k y)y'_x + anx^{n-1}y^m + c = 0$.

This is a special case of equation 1.7.1.18 with $f(y) = b \tan^k y$, $g(x) = c$.

30. $(ax^n \tan^m y + bx)y'_x = \tan^k(\lambda y)$.

This is the Bernoulli equation with respect to $x = x(y)$.

1.5.6. Equations Containing Combinations of Exponential, Hyperbolic, Logarithmic, and Trigonometric Functions

1. $y'_x = ax^n e^{\lambda y} + b \ln^m x$.

This is a special case of equation 1.7.2.5 with $f(x) = ax^n$, $g(x) = b \ln^m x$.

2. $y'_x = a\ln^n(\nu x)e^{\lambda y} + bx^m$.

This is a special case of equation 1.7.2.5 with $f(x) = a\ln^n(\nu x)$, $g(x) = bx^m$.

3. $y'_x = ae^{\lambda y}(\lambda y + \ln x)^m$.

This is a special case of equation 1.7.2.2 with $f(\xi) = a\ln^m \xi$.

4. $y'_x = ae^{-\lambda x}(\lambda x + \ln y)^m$.

This is a special case of equation 1.7.2.1 with $f(\xi) = a\ln^m \xi$.

5. $y'_x = ay\ln^2 y + by\ln y + ce^{\lambda x}y$.

This is a special case of equation 1.7.3.1 with $f(x) = a$, $g(x) = b$, $h(x) = ce^{\lambda x}$.

6. $y'_x = ay\ln^2 y + be^{\lambda x}y\ln y + cy$.

This is a special case of equation 1.7.3.1 with $f(x) = a$, $g(x) = be^{\lambda x}$, $h(x) = c$.

7. $y'_x = ae^y \sin x + b\tan x$.

This is a special case of equation 1.7.5.6 with $f(\xi) = a\xi + b$.

8. $y'_x = (ae^x \sin y + b)\tan y$.

This is a special case of equation 1.7.5.4 with $f(\xi) = a\xi + b$.

9. $y'_x = ae^x \sin^2 y + be^{-x}\cos^2 y$.

This is a special case of equation 1.7.5.8 with $f(\xi) = \frac{1}{2}(a\xi + b/\xi)$.

10. $y'_x = a\cos^n(\mu x)e^{\lambda y} + bx^m$.

This is a special case of equation 1.7.2.5 with $f(x) = a\cos^n(\mu x)$, $g(x) = bx^m$.

11. $y'_x = ax^n e^{\lambda y} + b\cos^m(\mu x)$.

This is a special case of equation 1.7.2.5 with $f(x) = ax^n$, $g(x) = b\cos^m(\mu x)$.

12. $y'_x = ax^n e^{\lambda y} + b\tan^m(\mu x)$.

This is a special case of equation 1.7.2.5 with $f(x) = ax^n$, $g(x) = b\tan^m(\mu x)$.

13. $y'_x = a\tan^n(\mu x)e^{\lambda y} + bx^m$.

This is a special case of equation 1.7.2.5 with $f(x) = a\tan^n(\mu x)$, $g(x) = bx^m$.

14. $y'_x = Ae^{\lambda x}\cos(ay) + Be^{\mu x}\sin(ay) + Ae^{\lambda x}$.

The substitution $w = \tan(\frac{1}{2}ay)$ leads to a linear equation: $w'_x = aBe^{\mu x}w + aAe^{\lambda x}$.

15. $y'_x = a\sin(\mu x)\sinh(\lambda y) + b\cos(\mu x)\cosh(\lambda y)$.

This is a special case of equation 1.7.2.18 with $f(x) = a\sin(\mu x)$, $g(x) = b\cos(\mu x)$, $h(x) = 0$.

16. $y'_x = ay \ln^2 y + by \ln y + c \sin^n(\lambda x) y.$

This is a special case of equation 1.7.3.1 with $f(x) = a$, $g(x) = b$, $h(x) = c \sin^n(\lambda x)$.

17. $(1 + \tan^2 y) y'_x = a \tan^{1+m} y + b \tan y + ce^{\lambda x} \tan^{1-m} y.$

This is a special case of equation 1.7.4.19 with $f(x) = a$, $g(x) = b$, $h(x) = ce^{\lambda x}$.

18. $(ae^x \cos y + b) y'_x = \cot y.$

This is a special case of equation 1.7.5.5 with $f(\xi) = (a\xi + b)^{-1}$.

19. $(ae^x \sin y + b) y'_x = \tan y.$

This is a special case of equation 1.7.5.4 with $f(\xi) = (a\xi + b)^{-1}$.

20. $(ae^y \cos x + b) y'_x = \tan x.$

This is a special case of equation 1.7.5.6 with $f(\xi) = (a\xi + b)^{-1}$.

21. $(ae^y \sin x + b) y'_x = \cot x.$

This is a special case of equation 1.7.5.7 with $f(\xi) = (a\xi + b)^{-1}$.

22. $(e^{\alpha x} y^n + a\beta) y'_x + be^{\beta y} \ln^m x + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = y^n$, $g(x) = b \ln^m x$.

23. $(e^{\alpha x} y^n + a\beta) y'_x + be^{\beta y} \cos^m x + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = y^n$, $g(x) = b \cos^m x$.

24. $(e^{\alpha x} \cos^n y + a\beta) y'_x + be^{\beta y} \cos^m(\lambda x) + a\alpha = 0.$

This is a special case of equation 1.7.2.15 with $f(y) = \cos^n y$, $g(x) = b \cos^m(\lambda x)$.

1.6. Equations Not Solved for Derivative

1.6.1. Equations of the Second Degree in y'_x

1. $(y'_x)^2 = ay + bx^2.$

See equation 1.6.3.43.

2. $(y'_x)^2 = y + ax^2 + bx + c.$

The substitution $w = 2\sqrt{y + ax^2 + bx + c}$ leads to an equation of the form 1.3.1.2: $ww'_x - w = 4ax + 2b$.

3. $(y'_x)^2 = ay^3 + by + c.$

Solution: $\quad x = C \pm \displaystyle\int \frac{dy}{\sqrt{ay^3 + by + c}}.$

4. $\boldsymbol{(y'_x)^2 = ay + b\sqrt{x}}$.

See equation 1.6.3.26.

5. $\boldsymbol{(y'_x)^2 = ay + b\sqrt{x} + c, \qquad a \neq 0}$.

The substitution $aw = 2\sqrt{ay + b\sqrt{x} + c}$ leads to the Abel equation of the form 1.3.1.32: $ww'_x - w = ba^{-2}x^{-1/2}$.

6. $\boldsymbol{(y'_x)^2 + ay'_x + by = 0}$.

Solution in the parametric form:

$$bx = -2t - a\ln t + C, \qquad by = -t^2 - at.$$

7. $\boldsymbol{(y'_x)^2 + ayy'_x = bx + c}$.

Differentiate the equation with respect to x, take y as the independent variable, and assume $\xi = y'_x$. As a result we obtain a linear equation with respect to $y = y(\xi)$:

$$(a\xi^2 - b)y'_\xi + a\xi y + 2\xi^2 = 0.$$

8. $\boldsymbol{(y'_x)^2 + axy'_x + by + cx^2 = 0}$.

The transformation $x = e^t$, $y = x^2 u$ leads to an autonomous equation:

$$\left(u'_t + 2u + \frac{a}{2}\right)^2 = \frac{a^2}{4} - c - bu.$$

Having extracted the root and carried over the terms $2u + \frac{1}{2}a$ from the left-hand side to the right-hand side, we obtain an equation of the form 1.1.2.

9. $\boldsymbol{y = xy'_x + ax^2 + b(y'_x)^2 + cy'_x + d, \qquad a \neq 0}$.

Differentiating with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation of the form 1.3.1.2: $ww'_t - w = -4abt - 2ac$.

10. $\boldsymbol{(y'_x)^2 + (ax + b)y'_x - ay + c = 0, \qquad a \neq 0}$.

Solutions:

$$y = (ax + b)C + aC^2 + ca^{-1} \qquad \text{and} \qquad 4ay = 4c - (ax + b)^2.$$

11. $\boldsymbol{(y'_x)^2 + (ay + bx)y'_x + abxy = 0}$.

This equation can be factorized:

$$(y'_x + ay)(y'_x + bx) = 0.$$

Therefore, the solutions are

$$y = Ce^{-ax} \qquad \text{and} \qquad y = -\tfrac{1}{2}bx^2 + C.$$

12. $\boldsymbol{(y_x')^2 + ax^2y_x' + bxy = 0.}$

The transformation $z = \ln x$, $u = yx^{-3}$ leads to an equation not depending implicitly on z:

$$(u_z')^2 + (a+6u)u_z' + (3a+b+9u)u = 0.$$

Rewriting the latter equation to solve for u_z', we obtain an equation of the form 1.1.2.

13. $\boldsymbol{(y_x')^2 = y + ax^{m+1} - \dfrac{m+1}{2(m+3)^2}x^2 + b.}$

The substitution

$$w = 2\Big[y + ax^{m+1} - \frac{m+1}{2(m+3)^2}x^2 + b\Big]^{1/2}$$

leads to the Abel equation of the form 1.3.1.10:

$$ww_x' - w = -\frac{2(m+1)}{(m+3)^2}x + 2a(m+1)x^m.$$

14. $\boldsymbol{(y_x')^2 = \lambda y + ax^2 + bx^{m+1} + c.}$

With $\lambda \neq 0$, the substitution $\lambda w = 2(\lambda y + ax^2 + bx^{m+1} + c)^{1/2}$ leads to the Abel equation:

$$ww_x' - w = 4a\lambda^{-2}x + 2b\lambda^{-2}(m+1)x^m,$$

which is outlined in Subsection 1.3.1.

The special cases of the original equation are equations 1.6.1.1, 1.6.1.2, 1.6.1.4, 1.6.1.5, and 1.6.1.13.

15. $\boldsymbol{a(y_x')^2 - yy_x' - x = 0.}$

Solution in the parametric form:

$$x = \frac{t}{\sqrt{t^2+1}}\big[C + a\ln\big(t + \sqrt{t^2+1}\,\big)\big], \qquad y = at - \frac{x}{t}.$$

16. $\boldsymbol{x(y_x')^2 = axy + b.}$

See equation 1.6.3.32.

17. $\boldsymbol{x(y_x')^2 = axy + bx + c, \qquad a \neq 0.}$

The substitution $aw = 2\sqrt{ay + b + cx^{-1}}$ leads to the Abel equation of the form 1.3.1.33: $ww_x' - w = -2ca^{-2}x^{-2}$.

18. $\boldsymbol{x(y_x')^2 - ayy_x' + b = 0.}$

With $a \neq 1$, the solution in the parametric form is written as

$$x = Ct^k + \frac{b}{2a-1}t^2, \quad aty = xt^2 + b, \qquad \text{where} \quad k = \frac{1}{a-1}.$$

With $a = 1$, the solution is $\;C(y - Cx) = b$.

19. $x(y'_x)^2 + ayy'_x + bx = 0.$

With $a \neq -1$, the solution in the parametric form is written as

$$x = Ct|(a+1)t^2 + b|^{-\frac{a+2}{2(a+1)}}, \qquad y = -\frac{x}{at}(t^2 + b).$$

In addition, there is the solution $y = \pm x\sqrt{-\dfrac{b}{a+1}}$.

With $a = -1$, the solution in the parametric form is written as

$$x = Ct\exp\Bigl(-\frac{t^2}{2b}\Bigr), \qquad y = x\Bigl(t + \frac{b}{t}\Bigr).$$

20. $x(y'_x)^2 - yy'_x + ay = 0.$

Solution in the parametric form:

$$x = C(t-a)\exp(-t/a), \qquad y = Ct^2\exp(-t/a),$$

In addition, there is the solution $y = 0$.

21. $x(y'_x)^2 - yy'_x + ax^2y'_x + by'_x + c = 0, \qquad a \neq 0.$

Divide the both sides by y'_x and differentiate with respect to x. Changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation of the form 1.3.1.33: $ww'_t - w = act^{-2}$.

22. $y(y'_x)^2 + axy'_x + by = 0.$

Solution in the parametric form is defined by the relations

$$axt + y(b + t^2) = 0, \quad Cy(t^2 + a + b)^m = t^{b/(a+b)}, \qquad \text{where} \quad m = \frac{a+2b}{2(a+b)}.$$

In addition, there is the solution $y = \pm x\sqrt{-a-b}$ corresponding to the limit $C \to \infty$.

23. $x(y'_x)^2 + (a - y)y'_x + b = 0.$

Solutions: $C(Cx - y + a) + b = 0$ and $(y - a)^2 = 4bx$.

24. $ax(y'_x)^2 + (bx - ay + k)y'_x - by = 0.$

Solution: $y = Cx + \dfrac{kC}{aC + b}$.

In addition, there is the exceptional solution which may be written in the parametric form as

$$x = -\frac{bk}{(at+b)^2}, \qquad y = xt + \frac{kt}{at+b}.$$

25. $ax(y'_x)^2 - (ay + bx - a - b)y'_x + by = 0.$

Differetiating with respect to x and factorizing, we obtain

$$(2axy'_x - ay - bx + a + b)y''_{xx} = 0.$$

Equating both factors to zero and integrating, we arrive at the solutions:

$$y = Cx + \frac{C(a+b)}{aC - b} \quad \text{and} \quad (ay + bx - a - b)^2 - 4abxy = 0.$$

26. $\boldsymbol{x(y'_x)^2 + ayy'_x + bx^n y^m = 0}$.

The substitution $x = e^t$ leads to an equation of the form 1.6.1.56: $(y'_t)^2 + ayy'_t + be^{(n+1)t}y^m = 0$.

27. $\boldsymbol{x^2(y'_x)^2 - (2xy + a)y'_x + y^2 = 0}$.

Solutions:

$$y = aC^2x + aC \quad \text{and} \quad y = -\frac{a}{4x}.$$

28. $\boldsymbol{ax^2(y'_x)^2 - 2axyy'_x + y^2 - a(a-1)x^2 = 0}$.

Solutions:

$$y \pm \sqrt{y^2 + ax^2} = Cx^{1+k}, \qquad \text{where} \quad k = \sqrt{(a-1)/a}.$$

29. $\boldsymbol{(a^2 - 1)x^2(y'_x)^2 + 2xyy'_x - y^2 + a^2x^2 = 0}$.

Solution in the parametric form:

$$x = C(t^2+1)^{-1/2}\bigl(t + \sqrt{t^2+1}\,\bigr)^{-1/a}, \qquad y = xt + ax\sqrt{t^2+1}.$$

30. $\boldsymbol{x^2(y'_x)^2 + (ax^2y^3 + b)y'_x + aby^3 = 0}$.

The equation can be factorized:

$$(y'_x + ay^3)(x^2y'_x + b) = 0.$$

Equating each of the factors to zero, we obtain the solutions:

$$y^2 = 2ax + C \quad \text{and} \quad y = b/x + C.$$

31. $\boldsymbol{axy(y'_x)^2 - (ay^2 + bx^2 + k)y'_x + bxy = 0}$.

This differential equation presents an equation of curvature lines of a surface defined by the relation

$$Ax^2 + By^2 + Cz^2 = 1,$$

where

$$a = AB(C - B), \quad b = AB(A - C), \quad k = C(B - A).$$

Solutions: $\quad (aC - b)y^2 = C(aC - b)x^2 - kC \quad$ and $\quad ay^2 = bx^2 \pm 2x\sqrt{-bk} - k.$

32. $\boldsymbol{y^2(y'_x)^2 = ax^2y^2 + b}$.

See equation 1.6.3.34.

33. $\boldsymbol{y^2(y'_x)^2 = ax^{-2/5}y^2 + b}$.

See equation 1.6.3.28.

34. $\boldsymbol{y^2(y'_x)^2 + 2axyy'_x + (1-a)y^2 + ax^2 + (a-1)b = 0}$.

Solutions:

$$y^2 + ax^2 - b = (a-1)(x+C)^2 \quad \text{and} \quad y^2 + ax^2 - b = 0.$$

35. $(a-b)y^2(y'_x)^2 - 2bxyy'_x + ay^2 - bx^2 - ab = 0.$

Solutions:

$$x^2 + y^2 = Cx + b - \frac{a-b}{4a}c^2, \quad \text{and} \quad (a-b)y^2 - bx^2 = (a-b)b.$$

36. $(x^2 - a)(y'_x)^2 - 2xyy'_x - x^2 = 0.$

Solving for y, differentiating with respect to x, and setting $w(x) = y'_x$, we obtain a factorized equation:

$$(xw'_x - w)(x^2w^2 + x^2 - aw^2) = 0.$$

Equating each of the factors to zero, we arrive at the solutions:

$$y = \frac{1}{2C}(x^2 - a - C^2) \quad \text{and} \quad y^2 + x^2 = a \quad (y \neq 0).$$

37. $(x^2 - a^2)(y'_x)^2 + 2xyy'_x + y^2 = 0.$

The equation can be factorized:

$$(xy'_x + ay'_x + y)(xy'_x - ay'_x + y) = 0.$$

Equating each of the factors to zero, we obtain the solutions:

$$(x+a)y = C \quad \text{and} \quad (x-a)y = C.$$

38. $(x^2 + a)(y'_x)^2 - 2xyy'_x + y^2 + b = 0.$

Differentiating with respect to x, we obtain a factorized equation:

$$[(x^2 + a)y'_x - xy]y''_{xx} = 0.$$

Therefore, the solutions of the original equation are

$$y = C_1x + C_2, \quad \text{where } aC_1^2 + C_2^2 + b = 0; \qquad bx^2 + ay^2 + ab = 0.$$

39. $(ay - x^2)(y'_x)^2 + 2xyy'_x - y^2 = 0.$

Solution: $(Cy + x)^2 = 4ay.$

40. $(ay^2 + bx)(y'_x)^2 = 1.$

See equation 1.6.3.44.

41. $(ax^2 + by)(y'_x)^2 = x^2y.$

See equation 1.6.3.46.

42. $(axy + b)(y'_x)^2 = y.$

See equation 1.6.3.33.

43. $(y^2 - a^2x^2)(y'_x)^2 + 2xyy'_x + (1-a^2)x^2 = 0.$

Solution in the parametric form:

$$x = \frac{Ct}{\sqrt{t^2+1}}, \qquad y = aC - \frac{C}{\sqrt{t^2+1}}.$$

44. $(ay - bx)^2[a^2(y'_x)^2 + b^2] - k^2(ay'_x + b)^2 = 0.$

Solve the equation for $ay - bx$ and differentiate with respect to x. Setting $w(x) = y'_x$, we obtain a factorized equation with respect to $w(x)$:

$$(aw - b)[(a^2w^2 + b^2)^{3/2} \pm abkw'_x] = 0.$$

Equating each of the factors to zero and integrating, we arrive at the solutions:

$$(bx - C)^2 + (ay - C)^2 = k^2 \quad \text{and} \quad ay - bx = \pm k\sqrt{2}.$$

45. $x^3(y'_x)^2 + x^2yy'_x + a = 0.$

Solutions:

$$Cxy = C^2x + a \quad \text{and} \quad xy^2 = 4a.$$

46. $xy^2(y'_x)^2 = ay^2 + bx.$

See equation 1.6.3.45.

47. $(ax^2y^2 + b)(y'_x)^2 = x^2.$

See equation 1.6.3.35.

48. $(xy'_x + a)^2 - 2ay + x^2 = 0, \qquad a \neq 0.$

The substitution $2ay - x^2 = u^2$ leads to the equation $xuu'_x - a(u-a) + x^2 = 0$. Next assuming $u - a = xw(x)$, we obtain $(xw + a)w'_x + w^2 + 1 = 0$. Taking w as the independent variable, we arrive at a linear equation whose solution is

$$x = (w^2+1)^{-1/2}\bigl[C - a\ln\bigl(w + \sqrt{w^2+1}\,\bigr)\bigr].$$

49. $(xy'_x + y + 2ax)^2 = 4(xy + ax^2 + b).$

The substitution $u = xy + ax^2 + b$ leads to an equation of the form 1.1.2: $u'_x = \pm 2\sqrt{u}$.

50. $(a\sqrt{y} + bx)(y'_x)^2 = 1.$

See equation 1.6.3.27.

51. $(ax^2y^{3/5} + by)(y'_x)^2 = x^2y.$

See equation 1.6.3.29.

52. $(a_2x+b_2y+c_2)(y'_x)^2+(a_1x+b_1y+c_1)y'_x+a_0x+b_0y+c_0=0.$

The Legendre transformation $x=u'_t$, $y=tu'_t-u$ ($y'_x=t$) leads to a linear equation:

$$[f(t)+tg(t)]u'_t=g(t)u+h(t),$$

where

$$f(t)=a_2t^2+a_1t+a_0,\quad g(t)=b_2t^2+b_1t+b_0,\quad h(t)=-c_2t^2-c_1t-c_0.$$

53. $(y'_x)^2=ae^y+b.$

See equation 1.6.3.3 with $k=2$.

54. $(y'_x)^2=a+be^x.$

See equation 1.6.3.4 with $k=2$.

55. $(y'_x)^2=ay^2+be^x.$

See equation 1.6.3.8 with $k=2$.

56. $(y'_x)^2+ayy'_x+be^{\lambda x}y^m=0.$

With $m\neq 2$, solving for y'_x and performing the substitution $w=e^{\lambda x}y^{m-2}$, we arrive at an equation with separated variables of the form 1.1.2:

$$w'_x=\lambda w+\frac{2-m}{2}\bigl(a\pm\sqrt{a^2-4bw}\bigr)w.$$

With $m=2$, solving the original equation for y'_x, we obtain an equation with separation of variables:

$$2y'_x=y\bigl(-a\pm\sqrt{a^2-4be^{\lambda x}}\bigr).$$

57. $x^2(y'_x)^2=ax^2e^y+b.$

See equation 1.6.3.9 with $k=2$.

58. $(ae^y+bx^2)(y'_x)^2=1.$

See equation 1.6.3.9 with $k=-2$.

59. $(ae^xy^2+b)(y'_x)^2=y^2.$

See equation 1.6.3.8 with $k=-2$.

60. $(y'_x)^2=ay+b\ln x.$

See equation 1.6.3.13.

61. $(y'_x)^2=\lambda y+a\ln x+b,\qquad \lambda\neq 0.$

The substitution $\lambda w=2\sqrt{\lambda y+a\ln x+b}$ leads to the Abel equation of the form 1.3.1.16: $ww'_x-w=2a\lambda^{-2}x^{-1}$.

62. $(y'_x)^2-xyy'_x+y^2\ln(ay)=0.$

Solutions:

$$ay=\exp(Cx-C^2)\quad\text{and}\quad ay=\exp(\tfrac{1}{4}x^2).$$

63. $(a\ln y+bx)(y'_x)^2=1.$

See equation 1.6.3.14.

1.6.2. Equations of the Third Degree in y'_x

1. $(y'_x)^3 + ax + by + c = 0.$

This is a special case of equation 1.8.1.13 with $f(w) = w^3$.

2. $a(y'_x)^3 + by'_x = x.$

This is a special case of equation 1.8.1.1 with $f(w) = aw^3 + bw$.

3. $a(y'_x)^3 + by'_x = y.$

This is a special case of equation 1.8.1.2 with $f(w) = aw^3 + bw$.

4. $a(y'_x)^3 + xy'_x = y.$

This is a special case of equation 1.8.1.6 with $f(w) = aw^3$.

5. $a(y'_x)^3 + bxy'_x = y.$

This is a special case of equation 1.8.1.7 with $f(w) = bw$, $g(x) = aw^3$.

6. $(y'_x)^3 - axy'_x + x^3 = 0, \qquad a \neq 0.$

Solution in the parametric form:

$$x = \frac{at}{t^3+1}, \qquad y = C + \frac{a^2}{6}\frac{4t^3+1}{(t^3+1)^2}.$$

7. $(y'_x)^3 - axyy'_x + 2ay^2 = 0.$

Differentiating with respect to x and eliminating y, we obtain a factorized equation with respect to $w(x) = y'_x$:

$$[2(w'_x)^2 - axw'_x + aw](9w - ax^2) = 0.$$

Equating each of the factors to zero and integrating, we find the solutions:

$$y = \frac{a}{4}C(x-C)^2 \qquad \text{and} \qquad y = \frac{a}{27}x^3.$$

8. $a(y'_x)^3 + b(y'_x)^2 = x.$

This is a special case of equation 1.8.1.1 with $f(w) = aw^3 + bw^2$.

9. $a(y'_x)^3 + b(y'_x)^2 = y.$

This is a special case of equation 1.8.1.2 with $f(w) = aw^3 + bw^2$.

10. $(y'_x)^3 + a(y'_x)^2 + by + abx + d = 0.$

Solution in the parametric form:

$$2bx = -3t^2 + 2at - 2a^2\ln(t+a) + C, \qquad by = -abx - t^3 - at^2 - d,$$

In addition, there is the solution $by = -abx - d$.

11. $a(y'_x)^3 + b(y'_x)^2 + cy'_x = y + d.$

Solution in the parametric form:

$$x = C + \tfrac{3}{2}at^2 + 2bt + c\ln|t|, \qquad y = at^3 + bt^2 + ct - d.$$

12. $a(y'_x)^3 + bx(y'_x)^2 = y.$

This is a special case of equation 1.8.1.7 with $f(w) = bw^2$, $g(x) = aw^3$.

13. $ax(y'_x)^3 + by'_x = y.$

This is a special case of equation 1.8.1.7 with $f(w) = aw^3$, $g(w) = bw$.

14. $ax(y'_x)^3 + b(y'_x)^2 = y.$

This is a special case of equation 1.8.1.7 with $f(w) = aw^3$, $g(w) = bw^2$.

15. $(ax + by + c)(y'_x)^3 = \alpha x + \beta y + \gamma.$

Dividing both sides by $ax + by + c$ and raising to the power $1/3$, we finally arrive at an equation of the form 1.7.1.6 with $f(w) = w^{-1/3}$.

16. $ax^{3/2}(y'_x)^3 + 2xy'_x = y.$

Solution: $\left(y - aC^{3/2}\right)^2 = 4Cx.$

17. $(x^2 - a^2)(y'_x)^3 + bx(x^2 - a^2)(y'_x)^2 + y'_x + bx = 0.$

The equation can be factorized:

$$(y'_x + bx)\left[(y'_x)^2(x^2 - a^2) + 1\right] = 0,$$

whence we find the solutions:

$$y = -\frac{1}{2}bx^2 + C \qquad \text{and} \qquad y = \pm\arcsin\frac{x}{a} + C.$$

18. $ax^n(y'_x)^3 + xy'_x = y.$

This is a special case of equation 1.8.1.8 with $f(w) = aw^n$.

19. $(xy'_x - y)^3 + ay + bx = 0.$

This is a special case of equation 1.8.1.10 with $f(w) = 1$, $g(w) = a$, $h(w) = b$, $n = 3$.

20. $(xy'_x - y)^3 + ayy'_x + bx = 0.$

This is a special case of equation 1.8.1.10 with $f(w) = 1$, $g(w) = aw$, $h(w) = b$, $n = 3$.

21. $(xy'_x - y)^3 + axy'_x + by = 0.$

This is a special case of equation 1.8.1.10 with $f(w) = 1$, $g(w) = b$, $h(w) = aw$, $n = 3$.

1.6.3. Equations of the Form $(y'_x)^k = f(y) + g(x)$

Preliminary comments.

1. In the general case, the equation

$$(y'_x)^k = f(y) + g(x) \tag{1}$$

is reduced, with the aid of the transformation

$$t = \int [g(x)]^{1/k}\,dx, \qquad u = \int [f(y)]^{-1/k}\,dy,$$

to the same form

$$(u'_t)^k = F(u) + G(t),$$

where functions $F = F(u)$ and $G = G(t)$ are defined parametrically by the following formulae:

$$F(u) = \frac{1}{f(y)}, \qquad u = \int [f(y)]^{-1/k}\,dy,$$
$$G(t) = \frac{1}{g(x)}, \qquad t = \int [g(x)]^{1/k}\,dx.$$

2. Taking y as the independent variable, we obtain from equation (1) an equation of the same class for $x = x(y)$:

$$(x'_y)^{-1/k} = g(x) + f(y).$$

3. The equation

$$y'_x = a\sqrt{y} + g(x) \qquad \bigl(k = 1, \quad f = a\sqrt{y}\,\bigr)$$

is reduced, with the aid of the substitution $w(x) = 2a^{-1}\sqrt{y}$, to the Abel equation $ww'_x - w = 2a^{-2}g(x)$ which is outlined in Subsection 1.3.1.

4. The equation

$$y'_x = y^{-1} + g(x) \qquad (k = 1, \quad f = y^{-1})$$

is an alternative form of writing the Abel equation $yy'_x = g(x)y + 1$ which is outlined in Subsection 1.3.2.

5. The equation

$$y'_x = ay^s + g(x) \qquad \bigl(k = 1, \quad f = ay^s\,\bigr)$$

is reduced, with the aid of the substitution $w = y - \int g(x)\,dx$ followed by raising both sides of the equation to the power of $1/s$, to an equation of the class in question:

$$(w'_x)^{1/s} = w + \int g(x)\,dx.$$

6. The equation

$$(y'_x)^2 = ay + g(x) \qquad (k = 2, \quad f = ay, \quad a \neq 0)$$

is reduced, with the aid of the substitution $aw = 2\sqrt{ay + g(x)}$, to the Abel equation of the second kind:

$$ww'_x = w + \varphi(x), \qquad \text{where} \quad \varphi = 2a^{-2}g'_x(x),$$

which is outlined in Subsection 1.3.1.

7. The equation

$$(y'_x)^{1/2} = ay + g(x) \qquad (k = 1/2, \quad f = ay)$$

is reduced, by squaring both sides and performing the substitution $z = ay + g(x)$, to the Riccati equation

$$z'_x = az^2 + g'_x.$$

For some specific functions $g = g(x)$, the solutions of the latter equation are given in Section 1.2.

8. The equation

$$(y'_x)^{1/2} = ay^{1/2} + g(x) \qquad (k = 1/2, \quad f = ay^{1/2})$$

can be reduced, by squaring both sides and performing the substitution $y = \exp(a^2 x)\xi$, to the Abel equation of the second kind:

$$\xi\xi'_x = a\exp(-\tfrac{1}{2}a^2 x)g\xi + \tfrac{1}{2}\exp(-a^2 x)g^2$$

(see Subsection 1.3.3).

9. The equation

$$(y'_x)^{-1/2} = f(y) + ax \qquad (k = -1/2, \quad g = ax)$$

can be reduced, by squaring both sides and performing the substitution $v = f(y) + ax$, to the Riccati equation:

$$v'_y = av^2 + f'_y.$$

For some specific functions $g = g(x)$, the solutions of the latter equation are given in Section 1.2.

10. For the sake of convenience, in Tables 1.5–1.9 are listed all the equations outlined in Subsection 1.6.3. Five classification tables are given below which classfy the equations wherein functions f and g are of the same form. The rightmost columns of the tables present the numbers of equations where the corresponding solutions are given. After the tables follow the equations—they are combined into groups so that the solutions of the equations within each group are expressed in terms of the functions indicated before the groups as a notation list.

1. $\boldsymbol{(y'_x)^k = Ay^s + B}$.

Solution: $\displaystyle x = \int (Ay^s + B)^{-1/k}\,dy + C.$

2. $\boldsymbol{(y'_x)^k = A + Bx^r}$.

Solution: $\displaystyle y = \int (A + Bx^r)^{1/k}\,dx + C.$

3. $\boldsymbol{(y'_x)^k = Ae^y + B}$.

Solution: $\displaystyle x = \int (Ae^y + B)^{-1/k}\,dy + C.$

TABLE 1.5
Solvable equations of the form $(y'_x)^k = Ay^s + Bx^r$

k	s	r	Equation	k	s	r	Equation
arbitrary	arbitrary ($s \neq k$)	$\dfrac{ks}{k-s}$	1.6.3.7	-1	1	-1	1.6.3.23
arbitrary	arbitrary	0	1.6.3.1	-1	1	$1/2$	1.6.3.42
arbitrary ($k \neq -1,\ 1$)	$\dfrac{k}{1-k}$	$-\dfrac{k}{1+k}$	1.6.3.6	$-1/2$	arbitrary ($s \neq -1,\ 0$)	1	1.6.3.17
arbitrary	0	arbitrary	1.6.3.2	$-1/2$	-1	1	1.6.3.38
arbitrary	1	1	1.6.3.5	$1/2$	1	arbitrary ($r \neq -1,\ 0$)	1.6.3.16
-2	-1	-2	1.6.3.46	$1/2$	1	-1	1.6.3.37
-2	-1	1	1.6.3.33	1	-1	-2	1.6.3.20
-2	$-2/5$	-2	1.6.3.29	1	-1	$-1/2$	1.6.3.39
-2	$1/2$	1	1.6.3.27	1	-1	1	1.6.3.22
-2	2	-2	1.6.3.35	1	$1/2$	-2	1.6.3.30
-2	2	1	1.6.3.44	1	$1/2$	-1	1.6.3.11
-1	arbitrary ($s \neq 0$)	1	1.6.3.10	1	$1/2$	$-1/2$	1.6.3.24
-1	arbitrary ($s \neq -2,\ 0$)	2	1.6.3.15	1	$1/2$	1	1.6.3.41
-1	-2	-1	1.6.3.21	2	-2	-1	1.6.3.45
-1	-2	$1/2$	1.6.3.31	2	-2	$-2/5$	1.6.3.28
-1	-2	2	1.6.3.36	2	-2	2	1.6.3.34
-1	-1	$1/2$	1.6.3.12	2	1	-1	1.6.3.32
-1	$-1/2$	-1	1.6.3.40	2	1	$1/2$	1.6.3.26
-1	$-1/2$	$1/2$	1.6.3.25	2	1	2	1.6.3.43

4. $\boldsymbol{(y'_x)^k = A + Be^x}$.

Solution: $y = \displaystyle\int (A + Be^x)^{1/k}\,dx + C.$

5. $\boldsymbol{(y'_x)^k = Ay + Bx}$.

Solution in the parametric form:

$$x = \int (A\tau^{1/k} + B)^{-1}\,d\tau + C, \qquad y = \frac{1}{A}\Big[\tau - B\int (A\tau^{1/k} + B)^{-1}\,d\tau - BC\Big].$$

TABLE 1.6
Solvable equations of the form
$(y'_x)^k = Ae^y + Bx^r$

k	r	Equation
arbitrary	$-k$	1.6.3.9
arbitrary	0	1.6.3.3
-1	-1	1.5.2.34
-1	1	1.5.2.32
-1	2	1.5.2.33
$-1/2$	1	1.6.3.19
1	arbitrary	1.5.2.2

TABLE 1.7
Solvable equations of the form
$(y'_x)^k = Ay^s + Be^x$

k	s	Equation
arbitrary	k	1.6.3.8
arbitrary	0	1.6.3.4
-1	arbitrary	1.5.2.31
$1/2$	1	1.6.3.18
1	-1	1.5.2.3

TABLE 1.8
Solvable equations of the form
$(y'_x)^k = Ae^y + Be^x$

k	Equation
-1	1.5.2.30
1	1.5.2.1

TABLE 1.9
Solvable equations containing logarithmic function

Form of equation	Equation
$(y'_x)^{-2} = A\ln y + Bx$	1.6.3.14
$(y'_x)^{-1} = A\ln y + Bx$	1.5.4.12
$(y'_x)^{2} = Ay + B\ln x$	1.6.3.13

6. $(y'_x)^k = Ay^{\frac{k}{1-k}} + Bx^{-\frac{k}{1+k}}, \qquad |k| \neq 1.$

Solution in the parametric form:

$$x = a\left[\int (\gamma\tau^{1/k} + \beta)^{-1}\,d\tau + C\right]^{\frac{k+1}{k}}, \qquad y = b\left[\tau - \beta\int (\gamma\tau^{1/k} + \beta)^{-1}\,d\tau - \beta C\right]^{\frac{k-1}{k}},$$

where

$$A = a^{-\frac{k}{1+k}}\, b^{-\frac{k}{1-k}}\,\beta B, \quad B = a^{\frac{k}{1+k}}\left[\frac{b(k-1)}{a(k+1)}\gamma\right]^k.$$

7. $(y'_x)^k = Ay^s + Bx^{\frac{ks}{k-s}}, \qquad k \neq s.$

Solution in the parametric form:

$$x = C^{k-s}\exp\int\left[\frac{k-s}{s}\left(A + B\tau^{\frac{ks}{s-k}}\right)^{\frac{1}{k}} - \tau\right]^{-1} d\tau,$$

$$y = C^k\left\{\tau\exp\int\left[\frac{k-s}{s}\left(A + B\tau^{\frac{ks}{s-k}}\right)^{\frac{1}{k}} - \tau\right]^{-1} d\tau\right\}^{\frac{k}{k-s}}.$$

8. $\boldsymbol{(y'_x)^k = Ay^k + Be^x}$.

Solution in the parametric form:

$$x = \int \left[\left(A + Be^{-k\tau}\right)^{1/k} - \frac{1}{k} \right]^{-1} d\tau + C,$$

$$y = \exp\left\{ \tau + \frac{1}{k} \int \left[\left(A + Be^{-k\tau}\right)^{1/k} - \frac{1}{k} \right]^{-1} d\tau + \frac{C}{k} \right\}.$$

9. $\boldsymbol{(y'_x)^k = Ae^y + Bx^{-k}}$.

Solution in the parametric form:

$$x = \exp\left\{ \tau - \frac{1}{k} \int \left[\left(B + Ae^{k\tau}\right)^{-1/k} + \frac{1}{k} \right]^{-1} d\tau + C \right\},$$

$$y = \int \left[\left(B + Ae^{k\tau}\right)^{-1/k} + \frac{1}{k} \right]^{-1} d\tau - \frac{C}{k}.$$

10. $\boldsymbol{(y'_x)^{-1} = Ay^s + Bx}$.

Solution: $\quad x = e^{By}\left(A \int y^s e^{-By}\, dy + C \right).$

► *In the solutions of equations 11–14, the following notation is used:*

$$F = \left[\int \exp(\mp\tau^2)\, d\tau + C \right]^{-1}.$$

11. $\boldsymbol{y'_x = Ay^{1/2} + Bx^{-1}}$.

Solution in the parametric form:

$$x = aF\exp(\mp\tau^2), \qquad y = b\left[2\tau \pm F\exp(\mp\tau^2)\right]^2,$$

where $A = \pm 2a^{-1}b^{1/2}$, $B = \mp 4b$.

12. $\boldsymbol{(y'_x)^{-1} = Ay^{-1} + Bx^{1/2}}$.

Solution in the parametric form:

$$x = a\left[2\tau \pm F\exp(\mp\tau^2)\right]^2, \qquad y = bF\exp(\mp\tau^2),$$

where $A = \mp 4a$, $B = \pm 2a^{1/2}b^{-1}$.

13. $\boldsymbol{(y'_x)^2 = Ay + B\ln x}$.

Solution in the parametric form:

$$x = aF\exp(\mp\tau^2), \qquad y = b\left\{\left[2\tau \pm F\exp(\mp\tau^2)\right]^2 \pm 4\ln(aF) - 4\tau^2\right\},$$

where $A = 4a^{-2}b$, $B = \mp 4bA$.

14. $\boldsymbol{(y'_x)^{-2} = A\ln y + Bx}$.

Solution in the parametric form:

$$x = a\left\{\left[2\tau \pm F\exp(\mp\tau^2)\right]^2 \pm 4\ln(bF) - 4\tau^2\right\}, \qquad y = bF\exp(\mp\tau^2),$$

where $A = \mp 4aB$, $B = 4ab^{-2}$.

▶ *In the solutions of equations 15–19, the following notation is used:*

$$Z = \begin{cases} C_1 J_\nu(\tau) + C_2 Y_\nu(\tau) & \text{for the upper sign,} \\ C_1 I_\nu(\tau) + C_2 K_\nu(\tau) & \text{for the lower sign,} \end{cases}$$

where J_ν and Y_ν are Bessel functions, I_ν and K_ν are modified Bessel functions.

Remark. *The solutions of equations 15–19 contain only the ratio $Z'_\tau/Z = (\ln Z)'_\tau$. Therefore, for the sake of symmetric appearance, two "arbitrary" constants C_1 and C_2 are indicated in the definition of function Z (instead, we may set, for instance, $C_1 = 1$ and $C_2 = C$).*

15. $(y'_x)^{-1} = Ay^s + Bx^2, \qquad s \neq -2,\ s \neq 0.$

Solution in the parametric form:

$$x = a\tau^{-2\nu}\bigl[\tau(\ln Z)'_\tau + \nu\bigr], \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{s+2}$, $A = \mp\dfrac{s+2}{2}ab^{-1-s}$, $B = -\dfrac{s+2}{2}a^{-1}b^{-1}$.

16. $(y'_x)^{1/2} = Ay + Bx^r, \qquad r \neq -1,\ s \neq 0.$

Solution in the parametric form:

$$x = a\tau^{2\nu}, \qquad y = b\tau^{2\nu}\Bigl[\tau(\ln Z)'_\tau + \nu \pm \frac{r+1}{2r}\tau^2\Bigr],$$

where $\nu = \dfrac{1}{r+1}$, $A = b^{-1}\Bigl[-\dfrac{(r+1)b}{2a}\Bigr]^{1/2}$, $B = \mp\dfrac{r+1}{2r}a^{-r}bA$.

17. $(y'_x)^{-1/2} = Ay^s + Bx, \qquad s \neq -1,\ s \neq 0.$

Solution in the parametric form:

$$x = a\tau^{2\nu}\Bigl[\tau(\ln Z)'_\tau + \nu \pm \frac{s+1}{2s}\tau^2\Bigr], \qquad y = b\tau^{2\nu},$$

where $\nu = \dfrac{1}{s+1}$, $A = \mp\dfrac{s+1}{2s}ab^{-s}B$, $B = a^{-1}\Bigl[-\dfrac{(s+1)a}{2b}\Bigr]^{1/2}$.

18. $(y'_x)^{1/2} = Ay + Be^x.$

Solution in the parametric form:

$$x = \ln(a\tau^2), \qquad y = b\Bigl[\tau(\ln Z)'_\tau \pm \frac{1}{2}\tau^2\Bigr],$$

where $\nu = 0$, $A = b^{-1}\Bigl(-\dfrac{b}{2}\Bigr)^{1/2}$, $B = \mp\dfrac{1}{2}a^{-1}bA$.

19. $(y'_x)^{-1/2} = Ae^y + Bx.$

Solution in the parametric form:

$$x = a\Bigl[\tau(\ln Z)'_\tau \pm \frac{1}{2}\tau^2\Bigr], \qquad y = b\ln(\tau^2),$$

where $\nu = 0$, $A = \mp\dfrac{1}{2}ab^{-1}B$, $B = a^{-1}\Bigl(-\dfrac{a}{2}\Bigr)^{1/2}$.

► *In the solutions of equations 20–35, the following notation is used:*

$$Z = \begin{cases} C_1 J_{1/3}(\tau) + C_2 Y_{1/3}(\tau) & \text{for the upper sign,} \\ C_1 I_{1/3}(\tau) + C_2 K_{1/3}(\tau) & \text{for the lower sign,} \end{cases}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions, $I_{1/3}$ and $K_{1/3}$ are modified Bessel functions;

$$U_1 = \tau Z'_\tau + \tfrac{1}{3} Z, \qquad U_2 = U_1^2 \pm \tau^2 Z^2, \qquad U_3 = \pm\tfrac{2}{3}\tau^2 Z^3 - 2U_1 U_2.$$

Remark. *The solutions of equations 15–19 contain only the ratio $Z'_\tau/Z = (\ln Z)'_\tau$. Therefore, for the sake of symmetric appearance, two "arbitrary" constants C_1 and C_2 are indicated in the definition of function Z (instead, we may set, for instance, $C_1 = 1$ and $C_2 = C$).*

20. $\boldsymbol{y'_x = Ay^{-1} + Bx^{-2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_2, \quad y = b\tau^{-2/3} Z^{-1} U_2^{-1} U_3, \qquad \text{where} \quad A = 2a^{-1}b^2, \quad B = \mp\tfrac{2}{3}ab.$$

21. $\boldsymbol{(y'_x)^{-1} = Ay^{-2} + Bx^{-1}}$.

Solution in the parametric form:

$$x = a\tau^{-2/3} Z^{-1} U_2^{-1} U_3, \quad y = b\tau^{-4/3} Z^{-2} U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}ab, \quad B = 2a^2 b^{-1}.$$

22. $\boldsymbol{y'_x = Ay^{-1} + Bx}$.

Solution in the parametric form:

$$x = a\tau^{-2/3} Z^{-1} U_1, \quad y = b\tau^{-4/3} Z^{-2} U_2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}a^{-1}b^2, \quad B = 2a^{-2}b.$$

23. $\boldsymbol{(y'_x)^{-1} = Ay + Bx^{-1}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_2, \quad y = b\tau^{-2/3} Z^{-1} U_1, \qquad \text{where} \quad A = 2a^{-2}b, \quad B = \mp\tfrac{2}{3}a^2 b^{-1}.$$

24. $\boldsymbol{y'_x = Ay^{1/2} + Bx^{-1/2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_1^2, \quad y = b\tau^{-8/3} Z^{-4} U_2^2, \qquad \text{where} \quad A = 2a^{-1}b^{1/2}, \quad B = \mp\tfrac{2}{3}a^{-1/2}b.$$

25. $\boldsymbol{(y'_x)^{-1} = Ay^{-1/2} + Bx^{1/2}}$.

Solution in the parametric form:

$$x = a\tau^{-8/3} Z^{-4} U_2^2, \quad y = b\tau^{-4/3} Z^{-2} U_1^2, \qquad \text{where} \quad A = \mp\tfrac{2}{3}ab^{-1/2}, \quad B = 2a^{1/2}b^{-1}.$$

26. $\boldsymbol{(y'_x)^2 = Ay + Bx^{1/2}}$.

Solution in the parametric form:

$$x = a\tau^{-4/3} Z^{-2} U_1^2, \quad y = b\tau^{-8/3} Z^{-4} (U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1),$$

where $A = 4a^{-2}b$, $B = \mp\tfrac{4}{3}a^{-1/2}bA$.

27. $(y'_x)^{-2} = Ay^{1/2} + Bx.$

Solution in the parametric form:

$$x = a\tau^{-8/3}Z^{-4}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1), \quad y = b\tau^{-4/3}Z^{-2}U_1^2,$$

where $A = \mp\frac{4}{3}ab^{-1/2}B,\ \ B = 4ab^{-2}.$

28. $(y'_x)^2 = Ay^{-2} + Bx^{-2/5}.$

Solution in the parametric form:

$$x = a\tau^{-5/3}Z^{-5/2}U_1^{5/2}, \quad y = b\tau^{-4/3}Z^{-2}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1)^{1/2},$$

where $A = \mp\frac{4}{3}a^{-2/5}b^2 B,\ \ B = \frac{16}{25}a^{-8/5}b^2.$

29. $(y'_x)^{-2} = Ay^{-2/5} + Bx^{-2}.$

Solution in the parametric form:

$$x = a\tau^{-4/3}Z^{-2}(U_2^2 \pm \tfrac{4}{3}\tau^2 Z^3 U_1)^{1/2}, \quad y = b\tau^{-5/3}Z^{-5/2}U_1^{5/2},$$

where $A = \frac{16}{25}a^2 b^{-8/5}$, $B = \mp\frac{4}{3}a^2 b^{-2/5}A.$

30. $y'_x = Ay^{1/2} + Bx^{-2}.$

Solution in the parametric form:

$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = b\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \qquad$ where $\ A = \pm\frac{4}{3}a^{-1}b^{1/2}, \quad B = -4ab.$

31. $(y'_x)^{-1} = Ay^{-2} + Bx^{1/2}.$

Solution in the parametric form:

$x = a\tau^{-4/3}Z^{-2}U_2^{-2}U_3^2, \quad y = b\tau^{4/3}Z^2U_2^{-1}, \qquad$ where $\ A = -4ab, \quad B = \pm\frac{4}{3}a^{1/2}b^{-1}.$

32. $(y'_x)^2 = Ay + Bx^{-1}.$

Solution in the parametric form:

$x = a\tau^{4/3}Z^2U_2^{-1}, \quad y = b\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \qquad$ where $A = \frac{16}{9}a^{-2}b,\ \ B = 4abA.$

33. $(y'_x)^{-2} = Ay^{-1} + Bx.$

Solution in the parametric form:

$x = a\tau^{-4/3}Z^{-2}U_2^{-2}(U_3^2 - 4U_2^3), \quad y = b\tau^{4/3}Z^2U_2^{-1}, \qquad$ where $A = 4abB,\ \ B = \frac{16}{9}ab^{-2}.$

34. $(y'_x)^2 = Ay^{-2} + Bx^2.$

Solution in the parametric form:

$$x = a\tau^{2/3}ZU_2^{-1/2}, \quad y = b\tau^{-2/3}Z^{-1}U_2^{-1}(U_3^2 - 4U_2^3)^{1/2},$$

where $A = 4a^2b^2B,\ \ B = \frac{16}{9}a^{-4}b^2.$

35. $(y'_x)^{-2} = Ay^2 + Bx^{-2}.$

Solution in the parametric form:

$$x = a\tau^{-2/3}Z^{-1}U_2^{-1}(U_3^2 - 4U_2^3)^{1/2}, \quad y = b\tau^{2/3}ZU_2^{-1/2},$$

where $A = \frac{16}{9}a^2b^{-4},\ \ B = 4a^2b^2A.$

▶ *In the solutions of equations 36–46, the following notation is used:*

$$R = \begin{cases} C_1\tau^{\nu} + C_2\tau^{-\nu} & \textit{for the upper sign,} \\ C_1\sin(\nu\ln\tau) + C_2\cos(\nu\ln\tau) & \textit{for the lower sign,} \\ C_1\ln\tau + C_2 & \textit{for } \nu = 0. \end{cases}$$

$$Q = \begin{cases} (1+\nu)C_1\tau^{\nu} + (1-\nu)C_2\tau^{-\nu} & \textit{for the upper sign,} \\ (C_1 - \nu C_2)\sin(\nu\ln\tau) + (C_2 + \nu C_1)\cos(\nu\ln\tau) & \textit{for the lower sign,} \\ C_1\ln\tau + C_1 + C_2 & \textit{for } \nu = 0. \end{cases}$$

Remark. *The expressions for R and Q contains two "arbitrary" constants C_1 an C_2. One of them may be fixed to set it equal to any nonzero number (for example, we may set $C_2 = \pm 1$), while another constant remains arbitrary.*

36. $\boldsymbol{(y'_x)^{-1} = Ay^{-2} + Bx^2}$.

Solution in the parametric form:

$$x = a\tau^{-2}R^{-1}Q, \quad y = b\tau^2, \qquad \nu = \sqrt{|1 - 4AB|},$$

where $A = -\dfrac{1 \mp \nu^2}{2}ab$, $B = -\dfrac{1}{2}a^{-1}b^{-1}$.

37. $\boldsymbol{(y'_x)^{1/2} = Ay + Bx^{-1}}$.

Solution in the parametric form:

$$x = a\tau^2, \quad y = b\tau^{-2}\Big(R^{-1}Q - \frac{1 \mp \nu^2}{2}\Big),$$

where $A = b^{-1}\Big(-\dfrac{b}{2a}\Big)^{1/2}$, $B = \dfrac{1 \mp \nu^2}{2}abA$.

38. $\boldsymbol{(y'_x)^{-1/2} = Ay^{-1} + Bx}$.

Solution in the parametric form:

$$x = a\tau^{-2}\Big(R^{-1}Q - \frac{1 \mp \nu^2}{2}\Big), \quad y = b\tau^2,$$

where $A = \dfrac{1 \mp \nu^2}{2}abB$, $B = a^{-1}\Big(-\dfrac{a}{2b}\Big)^{1/2}$.

39. $\boldsymbol{y'_x = Ay^{-1} + Bx^{-1/2}}$.

Solution in the parametric form:

$$x = a\tau^2R^2, \quad y = b\tau Q, \qquad \text{where} \quad A = (-1 \pm \nu^2)\frac{b^2}{2a}, \quad B = a^{-1/2}b.$$

40. $\boldsymbol{(y'_x)^{-1} = Ay^{-1/2} + Bx^{-1}}$.

Solution in the parametric form:

$$x = a\tau Q, \quad y = b\tau^2R^2, \qquad \text{where} \quad A = ab^{-1/2}, \quad B = (-1 \pm \nu^2)\frac{b^2}{2a}.$$

41. $y'_x = Ay^{1/2} + Bx$.

Solution in the parametric form:

$$x = a\tau R, \quad y = b\tau^2 Q^2, \qquad \text{where} \quad A = 2(-1 \pm \nu^2)a^{-1}b^{1/2}, \quad B = 4a^{-2}b.$$

42. $(y'_x)^{-1} = Ay + Bx^{1/2}$.

Solution in the parametric form:

$$x = a\tau^2 Q^2, \quad y = b\tau R, \qquad \text{where} \quad A = 4ab^{-2}, \quad B = 2(-1 \pm \nu^2)a^{1/2}b^{-1}.$$

43. $(y'_x)^2 = Ay + Bx^2$.

Solution in the parametric form:

$$x = a\tau R, \quad y = b\tau^2[Q^2 - (-1 \pm \nu^2)R^2], \qquad \text{where} \quad A = 16a^{-2}b, \quad B = (-1 \pm \nu^2)a^{-2}bA.$$

44. $(y'_x)^{-2} = Ay^2 + Bx$.

Solution in the parametric form:

$$x = a\tau^2[Q^2 - (-1 \pm \nu^2)R^2], \quad y = b\tau R, \qquad \text{where} \quad A = (-1 \pm \nu^2)ab^{-2}B, \quad B = 16ab^{-2}.$$

45. $(y'_x)^2 = Ay^{-2} + Bx^{-1}$.

Solution in the parametric form:

$$x = a\tau^2 R^2, \quad y = b\tau[Q^2 - (-1 \pm \nu^2)R^2]^{1/2},$$

where $A = (-1 \pm \nu^2)a^{-1}b^2 B$, $B = a^{-1}b^2$.

46. $(y'_x)^{-2} = Ay^{-1} + Bx^{-2}$.

Solution in the parametric form:

$$x = a\tau[Q^2 - (-1 \pm \nu^2)R^2]^{1/2}, \quad y = b\tau^2 R^2,$$

where $A = a^2 b^{-1}$, $B = (-1 \pm \nu^2)a^2 b^{-1} A$.

1.6.4. Other equations

1. $y = xy'_x + ax^2 + b\sqrt{y'_x} + c, \qquad a \neq 0$.

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation of the form 1.3.1.32: $ww'_t - w = -abt^{-1/2}$.

2. $y = xy'_x + ax^2 + b(y'_x)^2 + c(y'_x)^{m+1} + d, \qquad a \neq 0$.

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation:

$$ww'_t - w = -4abt - 2ac(m+1)t^m,$$

whose solvable case are outlined in Subsection 1.3.1.

3. $\boldsymbol{a(y'_x)^n + b(y'_x)^m = x.}$

Solution in the parametric form:

with $n \neq -1$, $m \neq -1$,

$$x = at^n + bt^m, \qquad y = C + \frac{an}{n+1}t^{n+1} + \frac{bm}{m+1}t^{m+1};$$

with $n = -1$, $m \neq -1$,

$$x = \frac{a}{t} + bt^m, \qquad y = C + a\ln|t| + \frac{bm}{m+1}t^{m+1}.$$

4. $\boldsymbol{a(y'_x)^n + b(y'_x)^m = y.}$

Solution in the parametric form:

with $n \neq 1$, $m \neq 1$,

$$y = C + \frac{an}{n-1}t^{n-1} + \frac{bm}{m-1}t^{m-1}, \qquad y = at^n + bt^m;$$

with $n = 1$, $m \neq 1$,

$$y = C + a\ln|t| + \frac{bm}{m-1}t^{m-1}, \qquad y = at + bt^m.$$

5. $\boldsymbol{y = xy'_x + a(y'_x)^n.}$

Solution: $y = Cx + aC^n$. In addition, there is the solution $y = Ax^{\frac{n}{n-1}}$, where $aA^{n-1}n^n = -(n-1)^{n-1}$, $n \neq 1$.

6. $\boldsymbol{y = xy'_x + ax^n(y'_x)^m.}$

This is a special case of equation 1.8.1.8 with $f(w) = aw^m$.

7. $\boldsymbol{y = ax^n(y'_x)^{2n} + 2xy'_x.}$

This is a special case of equation 1.8.1.9 with $f(w) = aw^n$.

8. $\boldsymbol{y'_x = ax^n(xy'_x - y)^m.}$

The Legendre transformation $x = w'_t$, $y = tw'_t - w$ $(y'_x = t)$ leads to the equation $t = aw^m(w'_t)^n$. By integrating, we obtain the solution in the parametric form:

with $m \neq -n$, $n \neq -1$,

$$x = \left(\frac{t}{a}\right)^{\frac{1}{n}}\left[\frac{m+n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}} + C\right]^{-\frac{m}{m+n}},$$

$$y = \left[\frac{1-m}{1+n}t\left(\frac{t}{a}\right)^{\frac{1}{n}} - C\right]\left[\frac{m+n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}} + C\right]^{-\frac{m}{m+n}};$$

with $m = -n$, $n \neq -1$,

$$x = C\left(\frac{t}{a}\right)^{\frac{1}{n}}\exp\left[\frac{n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}}\right], \qquad y = C\left[t\left(\frac{t}{a}\right)^{\frac{1}{n}} - 1\right]\exp\left[\frac{n}{n+1}t\left(\frac{t}{a}\right)^{\frac{1}{n}}\right];$$

with $m \neq -n$, $n = -1$,

$$x = \frac{a}{t}\big[a(1-m)\ln|t| + C\big]^{\frac{m}{1-m}}, \qquad y = tx - \big[a(1-m)\ln|t| + C\big]^{\frac{1}{1-m}};$$

with $m = 1$, $n = -1$,

$$y = Cx^{\frac{a}{a-1}}.$$

9. $\boldsymbol{x = a\exp(\lambda y'_x) + b\exp(\mu y'_x).}$

This is a special case of equation 1.8.1.1 with $f(w) = a\exp(\lambda w) + b\exp(\mu w)$.

10. $\boldsymbol{y = a\exp(\lambda y'_x) + b\exp(\mu y'_x).}$

This is a special case of equation 1.8.1.2 with $f(w) = a\exp(\lambda w) + b\exp(\mu w)$.

11. $\boldsymbol{y = xy'_x + ax^n\exp(\lambda y'_x).}$

This is a special case of equation 1.8.1.8 with $f(w) = a\exp(\lambda w)$.

12. $\boldsymbol{y = ax\exp(\lambda y'_x) + b\exp(\mu y'_x).}$

This is a special case of equation 1.8.1.7 with $f(w) = a\exp(\lambda w)$, $g(w) = b\exp(\mu w)$.

13. $\boldsymbol{\ln y'_x + xy'_x + ay + b = 0.}$

Solution in the parametric form:

with $a \neq 0$ and $\neq -1$,

$$x = \frac{1}{at} + Ct^{-\frac{1}{a+1}}, \qquad y = -\frac{1}{a}(xt + \ln t + b);$$

with $a = 0$,

$$x = -\frac{\ln t + b}{t}, \qquad y = C + (b-1)\ln t + \frac{1}{2}(\ln t)^2;$$

with $a = -1$,

$$y = Cx + \ln C + b \quad \text{and} \quad y = \ln(-1/x) + b - 1.$$

14. $\boldsymbol{y = xy'_x + ax^2 + b\ln y'_x + c, \qquad a \neq 0.}$

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w(t) = -2ax$, we arrive at the Abel equation of the form 1.3.1.16: $ww'_t - w = -2abt^{-1}$.

15. $\boldsymbol{y = xy'_x + ax^n\ln^m(\lambda y'_x).}$

This is a special case of equation 1.8.1.8 with $f(w) = a\ln^m(\lambda w)$.

16. $\boldsymbol{y = xy'_x + ax^n\sin^m(ky'_x).}$

This is a special case of equation 1.8.1.8 with $f(w) = a\sin^m(kw)$.

17. $\boldsymbol{y = xy'_x + ax^n\cos^m(ky'_x).}$

This is a special case of equation 1.8.1.8 with $f(w) = a\cos^m(kw)$.

18. $\boldsymbol{y = xy'_x + ax^n\tan^m(ky'_x).}$

This is a special case of equation 1.8.1.8 with $f(w) = a\tan^m(kw)$.

1.7. Equations of the Form $F(x, y)y'_x = G(x, y)$ Containing Arbitrary Functions

Notation: f, g, and h are arbitrary composite functions whose argument, indicated after the function name, may depend on both x an y.

1.7.1. Equations Containing Power Functions

1. $y'_x = f(ax + by + c)$.

With $b = 0$, we have an equation of the form 1.1.1. With $b \neq 0$, the substitution $u(x) = ax + by + c$ leads to an equation of the form 1.1.2: $u'_x = bf(u)$.

2. $y'_x = f(y + ax^n + b) - anx^{n-1}$.

The substitution $u = y + ax^n + b$ leads to an equation of the form 1.1.2: $u'_x = f(u)$.

3. $y'_x = \dfrac{y}{x} f(x^n y^m)$.

Homogeneous equation in the extended sense.

The substitution $z = x^n y^m$ leads to an equation with separation of variables: $xz'_x = nz + mzf(z)$.

4. $y'_x = f(x)y^{1+n} + g(x)y + h(x)y^{1-n}$.

The substitution $w = y^n$ leads to the Riccati equation: $w'_x = nf(x)w^2 + ng(x)w + nh(x)$.

5. $y'_x = -\dfrac{n}{m}\dfrac{y}{x} + y^k f(x)g(x^n y^m)$.

The substitution $z = x^n y^m$ leads to an equation with separation of variables: $z'_x = mx^{\frac{n-nk}{m}} f(x) z^{\frac{k+m-1}{m}} g(z)$.

6. $y'_x = f\left(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\right)$.

With $\Delta = a\beta - b\alpha \neq 0$, the transformation $x = u + \dfrac{b\gamma - c\beta}{\Delta}$, $y = v(u) + \dfrac{c\alpha - a\gamma}{\Delta}$ yields

$$v'_u = f\left(\frac{au + bv}{\alpha u + \beta v}\right).$$

Dividing both the numerator and denominator of the fraction on the right-hand side by u, we obtain a homogeneous equation of the form 1.1.6.

With $\Delta = 0$, $b \neq 0$, the substitution $v(x) = ax + by + c$ leads to an equation of the form 1.1.2:

$$v'_x = a + bf\left(\frac{bv}{\beta v + b\gamma - c\beta}\right).$$

With $\Delta = 0$, $\beta \neq 0$, the substitution $v(x) = \alpha x + \beta y + \gamma$ also leads to an equation of the form 1.1.2:

$$v'_x = \alpha + \beta f\left(\frac{bv + c\beta - b\gamma}{\beta v}\right).$$

7. $\boldsymbol{y'_x = x^{n-1}y^{1-m}f(ax^n + by^m)}$.

The substitution $w = ax^n + by^m$ leads to an equation with separation of variables: $w'_x = x^{n-1}[an + bmf(w)]$.

8. $\boldsymbol{y^n y'_x + ax^n + g(x)f(y^{n+1} + ax^{n+1}) = 0}$.

The substitution $w = y^{n+1} + ax^{n+1}$ leads to an equation with separation of variables: $w'_x + (n+1)g(x)f(w) = 0$.

9. $\boldsymbol{[x^n f(y) + xg(y)]y'_x = h(y)}$.

This is the Bernoulli equation with respect to $x = x(y)$ (see 1.1.5).

10. $\boldsymbol{[x^2 + xf(y) + g(y)]y'_x = h(y)}$.

This is the Riccati equation with respect to $x = x(y)$ (see Section 1.2).

11. $\boldsymbol{y'_x = [f(x)y + g(x)]\sqrt{(y-a)(y-b)}}$.

The substitution $u^2 = (y-a)/(y-b)$ leads to the Riccati equation:

$$\pm 2u'_x = [bf(x) + g(x)]u^2 - af(x) - g(x).$$

12. $\boldsymbol{\left[f\left(\frac{y}{x}\right) + x^a h\left(\frac{y}{x}\right)\right]y'_x = g\left(\frac{y}{x}\right) + yx^{a-1}h\left(\frac{y}{x}\right)}$.

The substitution $y = xt$ leads to the Bernoulli equation with respect to $x = x(t)$: $[g(t) - tf(t)]x'_t = f(t)x + h(t)x^{a+1}$.

13. $\boldsymbol{[f(ax+by) + bxg(ax+by)]y'_x = h(ax+by) - axg(ax+by)}$.

The substitution $t = ax + by$ leads to a linear equation with respect to $x = x(t)$: $[af(t) + bh(t)]x'_t = bg(t)x + f(t)$.

14. $\boldsymbol{[f(ax+by) + byg(ax+by)]y'_x = h(ax+by) - ayg(ax+by)}$.

The substitution $t = ax + by$ leads to a linear equation with respect to $y = y(t)$: $[af(t) + bh(t)]y'_t = -ag(t)y + h(t)$.

15. $\boldsymbol{x[f(x^n y^m) + mx^k g(x^n y^m)]y'_x = y[h(x^n y^m) - nx^k g(x^n y^m)]}$.

The transformation $t = x^n y^m$, $z = x^{-k}$ leads to a linear equation: $t[nf(t) + mh(t)]z'_t = -kf(t)z - kmg(t)$.

16. $\boldsymbol{x[f(x^n y^m) + my^k g(x^n y^m)]y'_x = y[h(x^n y^m) - ny^k g(x^n y^m)]}$.

The transformation $t = x^n y^m$, $z = y^{-k}$ leads to a linear equation: $t[nf(t) + mh(t)]z'_t = -kh(t)z + kng(t)$.

17. $\boldsymbol{x[sf(x^n y^m) - mg(x^k y^s)]y'_x = y[ng(x^k y^s) - kf(x^n y^m)]}$.

The transformation $t = x^n y^m$, $w = x^k y^s$ leads to an equation with separated variables: $tf(t)w'_t = wg(w)$.

18. $[f(y) + amx^n y^{m-1}]y'_x + g(x) + anx^{n-1}y^m = 0.$

Solution: $\displaystyle\int f(y)\,dy + \int g(x)\,dx + ax^n y^m = C.$

19. $f(x,y)y'_x + g(x,y) = 0, \qquad$ **where** $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y}.$

Total differential equation.

Solution:

$$\int_{y_0}^{y} f(x_0,t)\,dt + \int_{x_0}^{x} g(t,y)\,dt = C,$$

where x_0 and y_0 are arbitrary numbres.

1.7.2. Equations Containing Exponential and Hyperbolic Functions

1. $y'_x = e^{-\lambda x} f(e^{\lambda x} y).$

The substitution $u = e^{\lambda x} y$ leads to an equation of the form 1.1.2: $u'_x = f(u) + \lambda u$.

2. $y'_x = e^{\lambda y} f(e^{\lambda y} x).$

The substitution $u = e^{\lambda y} x$ leads to an equation with separated variables: $xu'_x = \lambda u^2 f(u) + u$.

3. $y'_x = y f(e^{\alpha x} y^m).$

Exponential homogeneous equation.

The substitution $z = e^{\alpha x} y^m$ leads to an equation of the form 1.1.2: $z'_x = \alpha z + mzf(z)$.

4. $y'_x = \dfrac{1}{x} f(x^n e^{\alpha y}).$

Exponential homogeneous equation.

The substitution $z = x^n e^{\alpha y}$ leads to an equation with separated variables: $xz'_x = nz + \alpha z f(z)$.

5. $y'_x = f(x)e^{\lambda y} + g(x).$

The substitution $u = e^{-\lambda y}$ leads to a linear equation: $u'_x = -\lambda g(x)u - \lambda f(x)$.

6. $y'_x = -\dfrac{n}{x} + f(x)g(x^n e^y).$

The substitution $z = x^n e^y$ leads to an equation with separation of variables: $z'_x = f(x)zg(z)$.

7. $y'_x = -\dfrac{\alpha}{m} y + y^k f(x) g(e^{\alpha x} y^m).$

The substitution $z = e^{\alpha x} y^m$ leads to an equation with separation of variables:

$$z'_x = m \exp\Big[\frac{\alpha}{m}(1-k)x\Big] f(x) z^{\frac{k+m-1}{m}} g(z).$$

8. $y'_x = f(x)e^{\lambda y} + g(x) + h(x)e^{-\lambda y}$.

The substitution $u = e^{\lambda y}$ leads to the Riccati equation: $u'_x = \lambda f(x)u^2 + \lambda g(x)u + \lambda h(x)$.

9. $y'_x = e^{\alpha x - \beta y} f(ae^{\alpha x} + be^{\beta y})$.

The substitution $w = ae^{\alpha x} + be^{\beta y}$ leads to an equation with separated variables: $w'_x = e^{\alpha x}[a\alpha + b\beta f(w)]$.

10. $y'_x = f(y + ae^{\lambda x} + b) - a\lambda e^{\lambda x}$.

The substitution $w = y + ae^{\lambda x} + b$ leads to an equation of the form 1.1.2: $w'_x = f(w)$.

11. $y'_x = -\dfrac{n}{\alpha x} + \dfrac{f(x^n e^{\alpha y})}{xy}$.

The substitution $t = x^n e^{\alpha y}$ leads to a linear equation with respect to $y = y(t)$: $\alpha^2 t f(t) y'_t = -ny + \alpha f(t)$.

12. $y'_x = -\dfrac{n}{\alpha x} + \dfrac{f(x^n e^{\alpha y})}{xy^2}$.

The substitution $t = x^n e^{\alpha y}$ leads to the Riccati equation: $\alpha^2 t f(t) y'_t = -ny^2 + \alpha f(t)$.

13. $[f(ax + by) + be^{\alpha y} g(ax + by)]y'_x = h(ax + by) - ae^{\alpha y} g(ax + by)$.

The transformation $t = ax + by$, $z = e^{-\alpha y}$ leads to a linear equation: $[af(t) + bh(t)]z'_t = -\alpha h(t)z + \alpha a g(t)$.

14. $[f(ax + by) + be^{\alpha x} g(ax + by)]y'_x = h(ax + by) - ae^{\alpha x} g(ax + by)$.

The transformation $t = ax + by$, $z = e^{-\alpha x}$ leads to a linear equation: $[af(t) + bh(t)]z'_t = -\alpha f(t)z - \alpha b g(t)$.

15. $[e^{\alpha x} f(y) + a\beta]y'_x + e^{\beta y} g(x) + a\alpha = 0$.

Solution:

$$\int e^{-\beta y} f(y)\,dy + \int e^{-\alpha x} g(x)\,dx - ae^{-\alpha x - \beta y} = C.$$

16. $x[f(x^n e^{\alpha y}) + \alpha y g(x^n e^{\alpha y})]y'_x = h(x^n e^{\alpha y}) - nyg(x^n e^{\alpha y})$.

The substitution $t = x^n e^{\alpha y}$ leads to a linear equation with respect to $y = y(t)$: $t[nf(t) + \alpha h(t)]y'_t = -ng(t)y + h(t)$.

17. $[f(e^{\alpha x} y^m) + mxg(e^{\alpha x} y^m)]y'_x = y[h(e^{\alpha x} y^m) - \alpha x g(e^{\alpha x} y^m)]$.

The substitution $t = e^{\alpha x} y^m$ leads to a linear equation with respect to $x = x(t)$: $t[\alpha f(t) + mh(t)]x'_t = mg(t)x + f(t)$.

18. $y'_x = f(x)\sinh(\lambda y) + g(x)\cosh(\lambda y) + h(x)$.

The substitution $u = e^{\lambda y}$ leads to the Riccati equation: $2u'_x = \lambda(f + g)u^2 + 2\lambda h u + \lambda(g - f)$.

19. $y'_x = f(x)\sinh^2(\lambda y) + g(x)\cosh^2(\lambda y) + h(x)\sinh(2\lambda y) + s(x)$.

The substitution $w = \tanh(\lambda y)$ leads to the Riccati equation: $w'_x = \lambda(f+s)w^2 + 2\lambda h w + \lambda(g-s)$.

20. $y'_x = y\coth x\, f(y^m \sinh x)$.

The transformation $t = \sinh x$, $z = y^m$ leads to an equation of the form 1.7.1.3: $tz'_t = mzf(tz)$.

21. $y'_x = x^{-1}\tanh y\, f(x^n \sinh y)$.

The transformation $t = x^n$, $z = \sinh y$ leads to an equation of the form 1.7.1.3: $ntz'_t = zf(tz)$.

22. $y'_x = y\tanh x\, f(y^m \cosh x)$.

The substitution $t = \cosh x$ leads to an equation of the form 1.7.1.3: $ty'_t = yf(ty^m)$.

23. $y'_x = x^{-1}\coth y\, f(x^n \cosh y)$.

The substitution $z = \cosh y$ leads to an equation of the form 1.7.1.3: $xz'_x = zf(x^n z)$.

1.7.3. Equations Containing Logarithmic Functions

1. $y'_x = f(x)y\ln^2 y + g(x)y\ln y + h(x)y$.

The substitution $u = \ln y$ leads to the Riccati equation: $u'_x = f(x)u^2 + g(x)u + h(x)$.

2. $y'_x = x^{-1}y^{m+1}f(y^m \ln x)$.

The substitution $t = \ln x$ leads to an equation of the form 1.7.1.3: $y'_t = \dfrac{y}{t}[ty^m f(ty^m)]$.

3. $y'_x = x^{-n-1}yf(x^n \ln y)$.

The substitution $z = \ln y$ leads to an equation of the form 1.7.1.3: $z'_x = \dfrac{z}{x}\dfrac{f(x^n z)}{x^n z}$.

4. $y'_x = x^{-1}e^y f(e^y \ln x)$.

The substitution $t = \ln x$ leads to an equation of the form 1.7.2.4: $y'_t = \dfrac{1}{t}[te^y f(te^y)]$.

5. $y'_x = ye^{-x}f(e^x \ln y)$.

The substitution $z = \ln y$ leads to an equation of the form 1.7.2.3: $z'_x = z\dfrac{f(e^x z)}{e^x z}$.

6. $y'_x = -nx^{-1}y\ln y + yf(x)g(x^n \ln y)$.

The substitution $w(x) = x^n \ln y$ leads to an equation with separation of variables: $w'_x = x^n f(x)g(w)$.

7. $\boldsymbol{y'_x = -\dfrac{n}{m}\dfrac{y}{x} + \dfrac{yf(x^n y^m)}{x\ln y}}$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to a linear equation: $m^2 tf(t)z'_t = -nz + mf(t)$.

8. $\boldsymbol{y'_x = -\dfrac{n}{m}\dfrac{y}{x} + \dfrac{yf(x^n y^m)}{x(\ln y)^2}}$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to the Riccati equation: $m^2 tf(t)z'_t = -nz^2 + mf(t)$.

9. $\boldsymbol{x[f(x^n y^m) + m\ln y\, g(x^n y^m)]y'_x = y[h(x^n y^m) - n\ln y\, g(x^n y^m)]}$.

The transformation $t = x^n y^m$, $z = \ln y$ leads to a linear equation: $t[nf(t) + mh(t)]z'_t = -ng(t)z + h(t)$.

10. $\boldsymbol{x[f(x^n y^m) + m\ln x\, g(x^n y^m)]y'_x = y[h(x^n y^m) - n\ln x\, g(x^n y^m)]}$.

The transformation $t = x^n y^m$, $z = \ln x$ leads to a linear equation: $t[nf(t) + mh(t)]z'_t = mg(t)z + f(t)$.

1.7.4. Equations Containing Trigonometric Functions

1. $\boldsymbol{y'_x = y^{m+1}\sin x F(y^m\cos x)}$.

This is an equation of the form 1.7.4.3 with $f(\xi) = \xi F(\xi)$.

2. $\boldsymbol{y'_x = y^{m+1}\cos x F(y^m\sin x)}$.

This is an equation of the form 1.7.4.4 with $f(\xi) = \xi F(\xi)$.

3. $\boldsymbol{y'_x = y\tan x f(y^m\cos x)}$.

The substitution $t = \cos x$ leads to an equation of the form 1.7.1.3: $y'_t = -\dfrac{y}{t}f(ty^m)$.

4. $\boldsymbol{y'_x = y\cot x f(y^m\sin x)}$.

The substitution $t = \sin x$ leads to an equation of the form 1.7.1.3: $y'_t = \dfrac{y}{t}f(ty^m)$.

5. $\boldsymbol{y'_x = x^{-1}\tan y f(x^n\sin y)}$.

The transformation $t = x^n$, $z = \sin y$ leads to an equation of the form 1.7.1.3: $ntz'_t = zf(tz)$.

6. $\boldsymbol{y'_x = x^{-1}\cot y f(x^n\cos y)}$.

The transformation $t = x^n$, $z = \cos y$ leads to an equation of the form 1.7.1.3: $ntz'_t = -zf(tz)$.

7. $\boldsymbol{y'_x = x^{-1}\sin 2y f(x^n\tan y)}$.

The transformation $t = x^n$, $z = \tan y$ leads to an equation of the form 1.7.1.3: $ntz'_t = 2zf(tz)$.

8. $y'_x = x^{-1} \sin 2y f(x^n \cot y)$.

The transformation $t = x^n$, $z = \cot y$ leads to an equation of the form 1.7.1.3: $ntz'_t = -2zf(tz)$.

9. $y'_x = \dfrac{y}{\sin 2x} f(y^m \tan x)$.

The substitution $t = \tan x$ leads to an equation of the form 1.7.1.3: $2ty'_t = yf(ty^m)$.

10. $y'_x = \dfrac{y}{\sin 2x} f(y^m \cot x)$.

The substitution $t = \cot x$ leads to an equation of the form 1.7.1.3: $2ty'_t = -yf(ty^m)$.

11. $y'_x = f(x)\cos(ay) + g(x)\sin(ay) + h(x)$.

The substitution $u = \tan(ay/2)$ leads to the Riccati equation: $2u'_x = a(h-f)u^2 + 2agu + a(f+h)$.

12. $y'_x = f(x)\cos^2(ay) + g(x)\sin^2(ay) + h(x)\sin(2ay) + s(x)$.

The substitution $u = \tan(ay)$ leads to the Riccati equation: $u'_x = a(g+s)u^2 + 2ahu + a(f+s)$.

13. $y'_x = f(y + a\tan x) - a\tan^2 x$.

The substitution $u = y + a\tan x$ leads to an equation of the form 1.1.2: $u'_x = a + f(u)$.

14. $y'_x = \dfrac{\sin 2y}{\sin 2x} f(\tan x \tan y)$.

The transformation $t = \tan x$, $z = \tan y$ leads to an equation of the form 1.7.1.3: $tz'_t = zf(tz)$.

15. $y'_x = \cot x \tan y f(\sin x \sin y)$.

The transformation $t = \sin x$, $z = \sin y$ leads to an equation of the form 1.7.1.3: $tz'_t = zf(tz)$.

16. $y'_x = -\cot x \tan y + \dfrac{f(x)}{\cos y} g(\sin x \sin y)$.

The substitution $w(x) = \sin x \sin y$ leads to an equation with separated variables: $w'_x = \sin x f(x) g(w)$.

17. $y'_x = -\dfrac{\sin 2y}{\sin 2x} + \cos^2 y f(x) g(\tan x \tan y)$.

The substitution $w(x) = \tan x \tan y$ leads to an equation with separated variables: $w'_x = \tan x f(x) g(w)$.

18. $y'_x = -nx^{-1}\sin 2y + \cos^2 y f(x) g(x^{2n}\tan y)$.

The substitution $w(x) = x^{2n}\tan y$ leads to an equation with separated variables: $w'_x = x^{2n} f(x) g(w)$.

19. $(1 + \tan^2 y) y'_x = f(x)\tan^{m+1} y + g(x)\tan y + h(x)\tan^{1-m} y$.

The substitution $u = \tan^m y$ leads to the Riccati equation: $u'_x = mf(x)u^2 + mg(x)u + mh(x)$.

1.7.5. Equations Containing Combinations of Exponential, Logarithmic, and Trigonometric Functions

1. $y'_x = -\sin 2y + \cos^2 y\, f(x) g(e^{2x} \tan y)$.

The substitution $w(x) = e^{2x} \tan y$ leads to the equation with separated variables: $w'_x = e^{2x} f(x) g(w)$.

2. $y'_x = \dfrac{F(e^x \cos y)}{e^x \sin y}$.

This is an equation of the tipe 1.7.5.5 with $f(\xi) = F(\xi)/\xi$.

3. $y'_x = e^y \cos x\, F(e^y \sin x)$.

This is an equation of the tipe 1.7.5.7 with $f(\xi) = \xi F(\xi)$

4. $y'_x = \tan y\, f(e^x \sin y)$.

The substitution $z = \sin y$ leads to an equation of the form 1.7.2.3: $z'_x = z f(e^x z)$.

5. $y'_x = \cot y\, f(e^x \cos y)$.

The substitution $z = \cos y$ leads to an equation of the form 1.7.2.3: $z'_x = -z f(e^x z)$.

6. $y'_x = \tan x\, f(e^y \cos x)$.

The substitution $t = \cos x$ leads to an equation of the form 1.7.2.4: $t y'_t = -f(t e^y)$.

7. $y'_x = \cot x\, f(e^y \sin x)$.

The substitution $t = \sin x$ leads to an equation of the form 1.7.2.4: $t y'_t = f(t e^y)$.

8. $y'_x = \sin 2y\, f(e^x \tan y)$.

The substitution $z = \tan x$ leads to an equation of the form 1.7.2.3: $z'_x = 2z f(e^x z)$.

9. $y'_x = \sin 2y\, f(e^x \cot y)$.

The substitution $z = \cot x$ leads to an equation of the form 1.7.2.3: $z'_x = -2z f(e^x z)$.

10. $y'_x = \dfrac{F(e^x \sin y)}{e^x \cos y}$.

This is an equation of the form 1.7.5.4 with $f(\xi) = F(\xi)/\xi$.

11. $y'_x = e^y \sin x\, F(e^y \cos x)$.

This is an equation of the from 1.7.5.6 with $f(\xi) = \xi F(\xi)$.

12. $y'_x = \dfrac{f(e^y \tan x)}{\sin 2x}$.

The substitution $t = \tan x$ leads to an equation of the form 1.7.2.4: $2t y'_t = f(t e^y)$.

13. $\boldsymbol{y'_x = \dfrac{f(e^y \cot x)}{\sin 2x}}$.

The substitution $t = \cot x$ leads to an equation of the form 1.7.2.4: $2ty'_t = -f(te^y)$.

14. $\boldsymbol{y'_x = e^{-\lambda x} f(\lambda x + \ln y)}$.

The substitution $u = \lambda x + \ln y$ leads to an equation of the form 1.1.2: $u'_x = e^{-u} f(u) + \lambda$.

15. $\boldsymbol{y'_x = e^{\lambda y} f(\lambda y + \ln x)}$.

The substitution $u = \lambda y + \ln x$ leads to the equation with separated variables: $xu'_x = \lambda e^u f(u) + 1$.

1.8. Equations of the Form $F(x, y, y'_x) = 0$ Not Solved for the Derivative and Containing Arbitrary Functions

1.8.1. Some Equations

1. $\boldsymbol{x = f(y'_x)}$.

The solution is written in the parametric form:

$$x = f(t), \qquad y = \int t f'_t(t)\,dt + C.$$

2. $\boldsymbol{y = f(y'_x)}$.

The solution is written in the parametric form:

$$x = \int f'_t(t)\frac{dt}{t} + C, \qquad y = f(t).$$

3. $\boldsymbol{x^n y^m = f\left(\dfrac{xy'_x}{y}\right)}$.

This is an equation of the form 1.7.2.3. Change to a new variable $w(x) = xy'_x/y$; divide both sides of the equation by $x^n y^m$ and differentiate with respect to x. As a result we arrive at an equation with separation of variables: $xf'_w(w)w'_x = (mw + n)f(w)$.

The solution is written in the parametric form:

$$\ln|x| = \int \frac{f'_w(w)\,dw}{(mw+n)f(w)} + C, \qquad x^n y^m = f(w).$$

In addition, there are solutions $y = A_k x^{-n/m}$, where A_k are roots of the equation $A_k^m - f(-n/m) = 0$.

4. $\boldsymbol{x^n e^{\alpha y} = f(xy'_x)}$.

The substitution $y = \ln u$ leads to an equation of the form 1.8.1.3: $x^n u^\alpha = f(xu'_x/u)$.

5. $\boldsymbol{e^{\alpha x} y^n = f(y'_x/y)}$.

The substitution $x = \ln t$ leads to an equation of the form 1.8.1.3: $t^\alpha y^n = f(ty'_t/y)$.

6. $\boldsymbol{y = xy'_x + f(y'_x)}$.

The Clerot equation.

Solution: $y = Cx + f(C)$.

In addition, there is a particular solution which may be written in the parametric form as

$$x = -f'_t(t), \qquad y = -tf'_t(t) + f(t).$$

7. $\boldsymbol{y = xf(y'_x) + g(y'_x)}$.

The Lagrange—d'Alembert equation.

With $f(t) = t$, see equation 1.8.1.6. Having differentiated with respect to x, we arrive at a linear equation with respect to $x = x(t)$, where $t = y'_x$:

$$[t - f(t)]x'_t = f'_t(t)x + g'_t(t).$$

8. $\boldsymbol{y = x^n f(y'_x) + xy'_x}$.

Differentiating with respect to x and denoting $t = y'_x$, we obtain the Bernoulli equation for $x = x(t)$:

$$x'_t - \frac{1}{n}\frac{f'_t(t)}{f(t)}x - \frac{1}{nf(t)}x^{2-n} = 0.$$

9. $\boldsymbol{y = f\big(x(y'_x)^2\big) + 2xy'_x}$.

Solution: $[y - f(C)]^2 = 4Cx$.

10. $\boldsymbol{y = f\big(x(y'_x)^n\big) + \dfrac{n}{n-1}xy'_x}$.

Solution: $y = f(C^n) + \dfrac{nC}{n-1}x^{\frac{n-1}{n}}$.

11. $\boldsymbol{(xy'_x - y)^n f(y'_x) + yg(y'_x) + xh(y'_x) = 0}$.

With the aid the Legendre transformation $x = u'_t$, $y = tu'_t - u$ $(y'_x = t)$, we obtain the Bernoulli equation:

$$[tg(t) + h(t)]u'_t = g(t)u + f(t)u^n.$$

12. $\boldsymbol{(y'_x)^2 + [f(x) + g(x)]y'_x + f(x)g(x) = 0}$.

The equation can be factorized: $[y'_x + f(x)][y'_x + g(x)] = 0$, i.e., it falls into two simpler equations $y'_x + f(x) = 0$ and $y'_x + g(x) = 0$. Therefore, the solutions are

$$y + \int f(x)\,dx = C \quad \text{and} \quad y + \int g(x)\,dx = C.$$

13. $\boldsymbol{(y'_x)^2 + 2fyy'_x + gy^2 = (g - f^2)\exp\left(-2\int_a^x f\,dx\right), \qquad f = f(x),\ g = g(x)}$.

Solution:

$$y = \begin{cases} \exp\left(-\int_a^x f\,dx\right)\sin\left(\int_a^x \sqrt{g - f^2}\,dx + C\right) & \text{if } g > f^2, \\ C\exp\left(-\int_a^x f\,dx\right) & \text{if } g \equiv f^2, \\ \exp\left(-\int_a^x f\,dx\right)\cosh\left(\int_a^x \sqrt{f^2 - g}\,dx + C\right) & \text{if } g < f^2. \end{cases}$$

14. $f(y'_x) + ax + by + s = 0.$

Solution in the parametric form:

$$x = C - \int \frac{f'_t(t)\,dt}{a + bt}, \qquad by = -ax - s - f(t).$$

In addition, there is a particular solution $y = \alpha x + \beta$, where α and β are determined by solving the system of two algebraic equations $a + b\alpha = 0$ and $f(\alpha) + b\beta + s = 0$.

15. $f(yy'_x + x) = y^2\big[(y'_x)^2 + 1\big].$

Setting $u(x) = yy'_x + x$ and differentiating with respect to x, we obtain

$$u'_x[f'_u(u) - 2u + 2x] = 0. \tag{1}$$

Equating the first factor to zero, after integrating we find $y^2 = -(x - C)^2 + B$. Substituting the latter into the original equation, we have $B = f(C)$. As a result we obtain the solution: $y^2 = f(C) - (x - C)^2$.

There is also an exceptional solution that corresponds to equating the second factor of (1) to zero. The solution in the parametric form is written as

$$x = u - \tfrac{1}{2} f'_u(u), \qquad y^2 = f(u) - \tfrac{1}{4}[f'_u(u)]^2.$$

16. $y = a(y'_x)^2 + f(x - 2ay'_x).$

This is a special case of equation 1.8.1.18 with $n = 2$.

Solution:

$$y = f(C) + \frac{1}{4a}(x - C)^2.$$

In addition, there is the following solution written in the parametric form:

$$x = t + 2af'_t(t), \qquad y = f(t) + a\big[f'_t(t)\big]^2.$$

17. $y = 2a(y'_x)^3 + f\big(x - 3a(y'_x)^2\big).$

This is a special case of equation 1.8.1.18 with $n = 3$.

Solution:

$$y = f(C) + 2a\Big(\frac{x - C}{3a}\Big)^{3/2}.$$

In addition, there is the following solution written in the parametric form:

$$x = t + 3a\big[f'_t(t)\big]^2, \qquad y = f(t) + 2a\big[f'_t(t)\big]^3.$$

18. $y = a(n-1)(y'_x)^n + f\big(x - an(y'_x)^{n-1}\big).$

Differentiating with respect to x, we obtain a factorized equation:

$$\big[1 - an(n-1)(y'_x)^{n-2} y''_{xx}\big]\big[y'_x - f'_t(t)\big] = 0, \tag{1}$$

where $t = x - an(y'_x)^{n-1}$. Equate the first factor to zero and integrate the obtained equation. Substituting the expression obtained into the original equation, we find the solution:

$$y = f(C) + a(n-1)\Big(\frac{x - C}{an}\Big)^{\frac{n}{n-1}}.$$

Equating the second factor in (1) to zero, we have another solution which can be written in the parametric form as

$$x = t + an\big[f'_t(t)\big]^{n-1}, \qquad y = f(t) + a(n-1)\big[f'_t(t)\big]^n.$$

19. $f(x^2+y^2)\sqrt{(y'_x)^2+1} = xy'_x - y.$

Setting $x = r(t)\cos t$, $y = r(t)\sin t$ and integrating, we obtain the solution:

$$t = \int \frac{f(r^2)\,dr}{r\sqrt{r^2 - f^2(r^2)}} + C.$$

20. $\Phi\big(f_x + f_y y'_x,\ f - x(f_x + f_y y'_x)\big) = 0, \qquad f = f(x,y),\ f_x = \dfrac{\partial f}{\partial x},\ f_y = \dfrac{\partial f}{\partial y}.$

Differentiating with respect to x, we obtain

$$(f_x + f_y y'_x)'_x(\Phi_u - x\Phi_v) = 0,$$

where $\Phi_u = \dfrac{\partial \Phi}{\partial u}$ and $\Phi_v = \dfrac{\partial \Phi}{\partial v}$ are partial derivatives of function $\Phi(u,v)$. Equating the first factor to zero, we find the solution:

$$f(x,y) = Cx + A, \qquad \text{where} \quad \Phi(C,A) = 0.$$

It remains to be checked whether the equation $\Phi_u - x\Phi_v = 0$ possesses any solutions and which of them satisfy the original equation.

1.8.2. Some Transformations

1. $x = f(y, y'_x).$

Substituting $t = y'_x$ and differentiating both sides of the equation with respect to x, we obtain an equation with respect to $y = y(t)$:

$$[1 - t f_y(y,\ t)]y'_t = t f_t(y,\ t), \qquad \text{where} \quad f_t = \frac{\partial f}{\partial t}, \quad f_y = \frac{\partial f}{\partial y}.$$

If $y = y(t)$ is the solution of the latter equation, the solution of the original equation may be presented in the parametric form as

$$x = f(y(t),\ t), \qquad y = y(t).$$

2. $y = f(x, y'_x).$

Differentiating with respect to x and setting $t = y'_x$, we obtain an equation with respect to $x = x(t)$:

$$[t - f_x(x,\ t)]x'_t = f_t(x,\ t), \qquad \text{where} \quad f_t = \frac{\partial f}{\partial t},\ f_y = \frac{\partial f}{\partial y}.$$

If $x = x(t)$ is the solution of the latter equation, the solution of the original equation may be presented in the parametric form as

$$x = x(t), \qquad y = f\big(x(t),\ t\big).$$

3. $\boldsymbol{x^n y^m = f\left(x^k y^s,\ \dfrac{x y'_x}{y}\right)}$.

Set $z = x^k y^s$ and $w = \dfrac{x y'_x}{y}$. Divide both sides of the equation by $x^n y^m$ and differentiate with respect to x. As a result we arrive at the following equation with respect to $w = w(z)$:

$$z(sw+k)(f_z + f_w w'_z) = (mw+n)f, \qquad \text{where} \quad f = f(z,\ w),$$

which is usually is simpler than the the original equation, since it is readily solved for the derivative. If $w = w(z)$ is the solution of the equation obtained, the solution of the original equation is written in the parametric form as

$$x^k y^s = z, \qquad x^n y^m = f\bigl(z,\ w(z)\bigr).$$

4. $\boldsymbol{x^n e^{\alpha y} = f(x^m e^{\beta y},\ x y'_x)}$.

The substitution $y = \ln u$ leads to an equation of the form 1.8.2.3:

$$x^n u^\alpha = f\left(x^m u^\beta,\ \frac{x u'_x}{u}\right).$$

5. $\boldsymbol{e^{\alpha x} y^n = f(e^{\beta x} y^m,\ y'_x/y)}$.

The substitution $x = \ln t$ leads to an equation of the form 1.8.2.3:

$$t^\alpha y^n = f\left(t^\beta y^m,\ \frac{t y'_t}{y}\right).$$

6. $\boldsymbol{f(x,\ x y'_x - y,\ y'_x) = 0}$.

The Legendre transformation $x = u'_t$, $y = t u'_t - u$ $(y'_x = t)$, where $u = u(t)$, leads to the equation $f(u'_t,\ u,\ t) = 0$. The inverse transformation: $t = y'_x$, $u = x y'_x - y$, $u'_t = x$.

7. $\boldsymbol{(y'_x)^2 = \lambda y + f(x)}$.

With $\lambda \neq 0$, the substitution $\lambda w = 2\sqrt{\lambda y + f(x)}$ leads to the Abel equation of the second kind:

$$w w'_x = w + \varphi(x), \qquad \text{where} \quad \varphi = 2\lambda^{-2} f'_x(x),$$

which is outlined in Subsection 1.3.1 for specific functions φ.

8. $\boldsymbol{y = x y'_x + a x^2 + f(y'_x), \qquad a \neq 0}$.

Differentiating the equation with respect to x and changing to new variables $t = y'_x$ and $w = -2ax$, we arrive at the Abel equation of the second kind:

$$w w'_t = w + \varphi(t), \qquad \text{where} \quad \varphi = -2a f'_t(t),$$

which is outlined in Subsection 1.3.1 for specific functions φ.